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Total mutual-visibility in Hamming graphs

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Abstract

If G is a graph and $X \subseteq V(G)$, then X is a total mutual-visibility set if every pair of vertices x and y of G admits a shortest x,y-path P with $V(P) \cap X \subseteq \{x,y\}$. The cardinality of a largest total mutual-visibility set of G is the total mutual-visibility number $\mu_{\rm t}(G)$ of G. In this paper the total mutual-visibility number is studied on Hamming graphs, that is, Cartesian products of complete graphs. Different equivalent formulations for the problem are derived. The values $\mu_{\rm t}(K_{n_1}\,\square\,K_{n_2}\,\square\,K_{n_3})$ are determined. It is proved that $\mu_{\rm t}(K_{n_1}\,\square\,\cdots\,\square\,K_{n_r}) = \mathcal{O}(N^{r-2})$, where $N = n_1 + \cdots + n_r$, and that $\mu_{\rm t}(K_s^{\square,r}) = \Theta(s^{r-2})$ for every $r \geq 3$, where $K_s^{\square,r}$ denotes the Cartesian product of r copies of K_s. The main theorems are also reformulated as Turán-type results on hypergraphs.

Keywords: mutual-visibility set; total mutual-visibility set; Hamming graph; Turán-type problem

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1 Introduction

Let $G = (V(G), E(G))$ be a graph and $X \subseteq V(G)$. Then vertices x and y of G are X-*visible*, if there exists a shortest x,y-path P such that no internal vertex of P belongs to X. The set X is a *mutual-visibility set* if any two vertices from X are X-visible, while X is a *total mutual-visibility set* if any two vertices from $V(G)$ are X-visible. The cardinality of a largest mutual-visibility set (resp. total mutual-visibility set) is the *mutual-visibility number* (resp. *total mutual-visibility number*) $\mu(G)$ (resp. $\mu_t(G)$) of G.

The mutual-visibility sets were introduced by Di Stefano in [7] motivated by mutual visibility in distributed computing and social networks. Although the motivation came from theoretical computer science, it is a graph theory concept. It needs to be said that the term mutual-visibility is also used in other contexts, for instance in robotics, where the mutual visibility problem asks for a distributed algorithm that reposition robots to a configuration where they all can see each other, cf. [1]. Some related research can be found in [3, 6, 12]. The graph theoretic mutual-visibility was further investigated in [4, 5], where the latter paper naturally raised the need to introduce the total mutual-visibility which was in turn investigated in [11, 13].

A graph G is a *Hamming graph* if G is the Cartesian product of complete graphs. In particular, complete graphs are Hamming graphs. In [4, Corollary 3.7] it was shown that $\mu(K_n \,\square\, K_m) = z(n, m; 2, 2)$, where $z(n, m; 2, 2)$ is the Zarankiewitz's number. To determine the latter number is a notorious open problem [14, 15]. On the other hand, it was proved in [13, Proposition 15] that $\mu_t(K_n \,\square\, K_m) = \max\{n, m\}$. In [11] the authors proposed a challenging problem to determine the total mutual-visibility number of Hamming graphs with at least three factors. They provided a total mutual-visibility set of $K_3 \,\square\, K_3 \,\square\, K_2$ of cardinality 4, and in Fig. 1 we give a total mutual-visibility set of $K_2 \,\square\, K_3 \,\square\, K_4$ of cardinality 5.

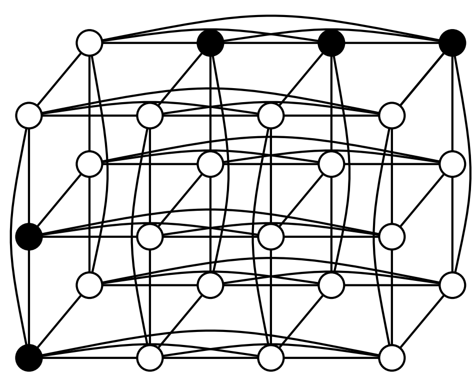

Figure 1: $K_2 \,\square\, K_3 \,\square\, K_4$ with a total mutual-visibility set of cardinality 5 in bold.

In the light of what has just been said, in this paper we focus on the total mutual-visibility in Hamming graphs. In the next section we give equivalent formulations of the problem which later serve as a tool for proof. Our first main result which we prove in Section 3 reads as follows.

Theorem 1 *If $n_1 \geq n_2 \geq n_3 \geq 2$ and $N = n_1 + n_2 + n_3$, then*

$$\mu_{\rm t}(K_{n_1} \,\square\, K_{n_2} \,\square\, K_{n_3}) = \begin{cases} N-4; & n_3 = 2, \\ N-5; & n_3 = 3, \\ N-6; & n_3 \geq 4\,. \end{cases}$$

Note that if $n_1 \geq n_2 \geq n_3 = 1$, then by the above-mentioned result from [13] we have $\mu_{\rm t}(K_{n_1} \,\square\, K_{n_2} \,\square\, K_1) = \mu_{\rm t}(K_{n_1} \,\square\, K_{n_2}) = N - n_2 - 1 = n_1$.

In Section 4, we prove the following:

Theorem 2 *If $r \geq 3$, and $s, n_1, \ldots, n_r$ are positive integers, $N = n_1 + \cdots + n_r$, then the following statements hold:*

(i) $\mu_{\rm t}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) \leq \frac{6}{r!}\, N^{r-2}$;

(ii) $\mu_{\rm t}(K_s^{\square,r}) \leq c'_r\, s^{r-2}$ *with* $c'_r = 3\prod_{i=3}^{r}(i-1)^{i-3}$.

In the subsequent section, we strengthen the result for balanced Hamming graphs by establishing the exact magnitude for their total mutual-visibility number. The result [13, Proposition 15] implies $\mu_{\rm t}(K_s^{\square,2}) = \Theta(s)$. However, the situation is different for higher values of r as our third main result asserts. It will be proved in Section 5 using a probabilistic approach.

Theorem 3 *If $r \geq 3$, then*

$$\mu_{\rm t}(K_s^{\square,r}) = \Theta(s^{r-2}).$$

In the last section we reformulate our problem as a Turán-type question and accordingly restate Theorems 1-3.

In the remainder of the introduction, we recall some definitions and terminology, mainly about the Cartesian product of graphs. The standard shortest-path distance between vertices u and v of a (connected) graph G will be denoted by $d_G(u,v)$. We will use the term clique for a complete graph as well as for its vertex set. If $u, v \in V(G)$, then the *interval* $I_G[u,v]$ between u and v in G is the set of all vertices of G that lie on shortest u,v-paths. The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$, vertices (g,h) and (g',h') are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. Given a vertex $h \in V(H)$, the subgraph of $G \,\square\, H$ induced by the set $\{(g,h) : g \in V(G)\}$, is a *G-layer* and is denoted by G^h. H-layers ${}^gH$ are defined analogously. Each G-layer and each H-layer is isomorphic to G and H, respectively. Moreover, each layer of a Cartesian product is its convex subgraph. We use this fact later on many times, sometimes implicitly. The Cartesian product of r copies of G is denoted by $G^{\square,r}$. We say that $K_s^{\square,r}$ is a *balanced Hamming graph*. For more information on the Cartesian product see the book [9].

2 Equivalent formulations of the problem

In this section we prove two equivalent formulations of the total mutual-visibility problem in Hamming graphs. First we prove that total mutual-visibility sets in Hamming graphs are precisely the vertex sets such that no pair of vertices is at distance 2. Then we reformulate this fact in terms of clique systems in complete multipartite graphs.

We say that a 4-cycle of a Cartesian product graph G is a *Cartesian square* if it is not contained in a single layer of G. This definition also applies to Cartesian product of more than two factors. More precisely, let $G = G_1 \square \cdots \square G_r$, $r \geq 2$. Then the vertices $u, u', u'', u''' \in V(G)$ form a Cartesian square if there exist $i, j \in [r]$, $i < j$, such that

$$\begin{aligned} u &= (u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_r)\,, \\ u' &= (u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{j-1}, u_j', u_{j+1}, \ldots, u_r)\,, \\ u'' &= (u_1, \ldots, u_{i-1}, u_i', u_{i+1}, \ldots, u_{j-1}, u_j', u_{j+1}, \ldots, u_r)\,, \\ u''' &= (u_1, \ldots, u_{i-1}, u_i', u_{i+1}, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_r)\,. \end{aligned}$$

The following results were stated in [11, Lemma 5.8] for two factors, the proof for more factors is analogous. That is, we just need to infer that a Cartesian square is a convex subgraph of a Cartesian product graph.

Lemma 4 *Let $G = H_1 \square \cdots \square H_r$, $r \geq 2$. If X is a total mutual-visibility set of G and C is a Cartesian square of G, then X contains no diametral pair of vertices of C.*

If a Cartesian square of a Cartesian product graph G fulfils the condition of Lemma 4 for a set $X \subseteq V(G)$, then we say that the cycle is *X-suitable*. For the proof of our first characterization, we need the following result.

Proposition 5 *If G is a Hamming graph, $u, v \in V(G)$, and $d_G(u,v) = t$, then the subgraph induced by $I[u,v]$ is isomorphic to the t-cube Q_t.*

Our first equivalent formulation of the total mutual-visibility problem in Hamming graphs now reads as follows.

Theorem 6 *If G is a Hamming graph and $X \subseteq V(G)$, then the following statements are equivalent.*

(i) X is a total mutual-visibility set of G.

(ii) $2 \notin \{d_G(u,v) : \ u,v \in X\}$.

(iii) Each Cartesian square of G is X-suitable.

Proof. Let $G = K_{n_1} \,\square \cdots \square\, K_{n_r}$, where $n_i \geq 2$ for $i \in [r]$, and $r \geq 1$. If $r = 1$, then G is a complete graph which contains no Cartesian square, every subset of $V(G)$ forms a total mutual-visibility set, and $\mathrm{diam}(G) = 1$. Hence the three statements are equivalent for G and we may assume in the rest that $r \geq 2$.

(i) $\Rightarrow$ *(ii)*: Suppose on the contrary that there exist vertices $u, v \in X$ with $d_G(u,v) = 2$. Then u and v lie in a convex C_4, but then the other two vertices of this convex C_4 are not X-visible, a contradiction.

(ii) $\Rightarrow$ *(iii)*: If $2 \notin \{d_G(u,v) : \ u,v \in X\}$, then clearly each Cartesian square contains at most two vertices of X, and if so, these two vertices are adjacent, hence each Cartesian square is X-suitable.

(iii) $\Rightarrow$ *(i)*: Let $X \subseteq V(G)$ and assume that each Cartesian square of G is X-suitable. We need to show that each two vertices $u, v \in V(G)$ are X-visible and proceed by induction on $d_G(u,v) = t$. If $t = 1$, the assertion is clear. Suppose now that $t \geq 2$ and that each pair of vertices at distance at most $t-1$ is X-visible. By Proposition 5, $I_G[u,v]$ induces a t-cube Q_t. Let $v^{(1)}, \ldots, v^{(t)}$ be the neighbors of v in this Q_t. Then each pair of vertices $v^{(i)}$ and $v^{(j)}$ lies in a Cartesian square of G together with the vertex v. Since each Cartesian square of G is X-suitable, we infer that at most one of the vertices $v^{(1)}$ and $v^{(2)}$ lies in X. Assume without loss of generality that $v^{(1)} \notin X$. By the induction hypothesis, $v^{(1)}$ and u are X-visible which in turn implies that u and v are also X-visible by concatenating the corresponding shortest $u, v^{(1)}$-path and the edge $v^{(1)}v$. $\square$

To get another reformulation of the total mutual-visibility problem on Hamming graphs, we consider the complete multipartite graph $K_{n_1,\ldots,n_r}$, where $r \geq 3$ and $n_1 \geq \cdots \geq n_r \geq 2$. Note that each maximal clique in it is a maximum clique that is of order r.

Proposition 7 *If $r \geq 3$ and $n_1 \geq \cdots \geq n_r \geq 2$, then $\mu_{\rm t}(K_{n_1} \,\square \cdots \square\, K_{n_r})$ is equal to the cardinality of a largest family of maximal cliques of $K_{n_1,\ldots,n_r}$ such that no two cliques from the family intersect in an $(r-2)$-clique.*

Proof. Let $r \geq 3$ and $n_1 \geq \cdots \geq n_r \geq 2$ and set $G = K_{n_1} \,\square \cdots \square\, K_{n_r}$. Setting $V(K_{n_j}) = [n_j]$ we have $V(G) = \{(i_1, \ldots, i_r) :\ i_j \in [n_j], j \in [r]\}$. We are going to prove that each total mutual-visibility set of G gives rise to a family of maximal cliques in $K_{n_1,\ldots,n_r}$, such that no two cliques from the family intersect in an $(r-2)$-clique, as well as the other way around, that is, each family of maximal cliques in $K_{n_1,\ldots,n_r}$ gives rise to a total mutual-visibility set of G.

Set $H = K_{n_1,\ldots,n_r}$. Let I_i, $i \in [r]$, be the partite classes of H, where $|I_i| = n_i$, so that $V(H) = \bigcup_{i=1}^{r} I_i$. Let further $I_i = \{u_{i,j} :\ j \in [n_i]\}$.

Let $X = \{x_1, \ldots, x_t\}$ be a total mutual-visibility set of G. For $i \in [t]$, set

$$x_i = (z_1^{(i)}, z_2^{(i)}, \ldots, z_r^{(i)})\,. \tag{1}$$

To each vertex x_i assign an r-clique of H induced by the set of vertices

$$X_i = \{u_{1,z_1^{(i)}}, u_{2,z_2^{(i)}}, \ldots, u_{r,z_r^{(i)}}\}\,. \tag{2}$$

We claim that $\mathcal{X} = \{X_i :\ i \in [t]\}$ is a set of r-cliques of H such that no two cliques from $\mathcal{X}$ intersect in an $(r-2)$-clique. Since the vertices from X_i belong to pairwise different partite classes of H, the graph induced by X_i is isomorphic to K_r. Moreover, if $i' \neq i$, then $d_G(x_i, x_{i'}) \neq 2$, hence $|X_i \cap X_{i'}| \neq r-2$. We have thus seen that the total mutual-visibility set X of G gives rise to a family of maximal cliques of H, such that no two cliques from the family intersect in an $(r-2)$-clique.

To prove the reverse assignment, we proceed in the reverse order as above. That is, we start with a family of k-cliques $\mathcal{X}$ such that no two cliques from the set intersect in an $(r-2)$-clique. Then we use their enumeration as in (2) to produce a total mutual-visibility set of the same cardinality as in (1). □

3 Proof of Theorem 1

The proof of Theorem 1 is divided into two cases. We first deal with the case when $n_3 \in \{2, 3\}$, and then with the case $n_3 \geq 4$. For the proof of the first part, we recall the following result.

Proposition 8 [13, Proposition 4.4] *If $s \geq 3$ and $n \geq 3$, then*

$$\mu_{\rm t}(C_s \,\square\, K_n) = \begin{cases} 0; & s \geq 5, \\ n; & otherwise. \end{cases}$$

Theorem 9 *If $n_1 \geq n_2 \geq n_3 \in \{2, 3\}$, then $\mu_{\rm t}(K_{n_1} \,\square\, K_{n_2} \,\square\, K_{n_3}) = n_1 + n_2 - 2$.*

Proof. Let $n_1 \geq n_2 \geq n_3$, let $n_3 \in \{2,3\}$, and set $G = K_{n_1} \,\square\, K_{n_2} \,\square\, K_{n_3}$. By a straightforward case analysis we infer that $\mu_{\rm t}(K_2 \,\square\, K_2 \,\square\, K_2) = 2$, hence the theorem holds in this case. We may thus assume in the rest that $n_1 \geq 3$.

The vertices from the set $Y = \{(i,1,1) :\ 2 \leq i \leq n_1\} \cup \{(1,j,2) :\ 2 \leq j \leq n_2\}$ are pairwise at distance 1 or 3. Theorem 6 implies that Y is a total mutual-visibility set of G. Hence $\mu_{\rm t}(G) \geq n_1 + n_2 - 2 = |Y|$.

Let $V(K_{n_j}) = [n_j]$, so that $V(G) = \{(i_1,i_2,i_3) :\ i_j \in [n_j], j \in [3]\}$ and let X be a total mutual-visibility set with $|X| = \mu_{\rm t}(G)$. For a vertex $x \in V(G)$, let $X_i(x) = \{v \in X :\ d_G(x,v) = i\}$, $i \in [3]$.

To prove that $\mu_{\rm t}(G) \leq n_1+n_2-2$, note first that if no two vertices of X are adjacent, then $|X| \leq n_3$. Indeed, if $|X| > n_3$, then there exist two vertices $w, w' \in X$ with the same third coordinate. As w and w' are not adjacent, this means then $d_G(w,w') = 2$, a contradiction with Theorem 6. Hence, if no two vertices of X are adjacent, then $|X| \leq n_3 \leq n_2 \leq n_1 + n_2 - 2$.

Let u and u' be two adjacent vertices of X. Assume that u and u' differ in the first coordinate. By the symmetry of Hamming graphs we may assume without loss of generality that $u = (1,1,1)$ and $u' = (2,1,1)$. By Theorem 6, we have $X_2(u) = \emptyset$, so that

$$X = \{u\} \cup X_1(u) \cup X_3(u)\,. \tag{3}$$

We claim that all the vertices from $X_1(u)$ differ from u in the first coordinate. Indeed, if this would not be the case, then there would exist a vertex $u'' = (1,j'',1)$ (or $u'' = (1,1,k'')$) in X, but then u' and u'' are diametral vertices of a Cartesian square which is not possible by Theorem 6. Since $n_1 \geq n_2 \geq n_3$, the claim implies that $|X_1| \leq n_1 - 1$.

Case 1: $|X_1(u)| = n_1 - 1$.
By the above claim, in this case we may assume without loss of generality that $X_1(u) = \{(i,1,1) :\ i \in \{2,\ldots,n_1\}\}$. Consider an arbitrary vertex $w = (w_1,w_2,w_3)$ from $X_3(u)$. Then $w_1 \neq 1$, $w_2 \neq 1$, and $w_3 \neq 1$. Since $(w_1,1,1) \in X$ and $d(w,(w_1,1,1)) = 2$ we get that $X_3(u) = \emptyset$. By (3) we conclude that if $|X_1(u)| = n_1 - 1$, then $|X| = n_1 \leq n_1+n_2-2$ because $n_2 \geq 2$.

Case 2: $|X_1(u)| \leq n_1 - 2$.
If $n_2 = 2$, then we also have $n_3 = 2$, and hence $G = K_{n_1} \,\square\, K_2 \,\square\, K_2 = K_{n_1} \,\square\, C_4$. The assertion of the theorem then follows by Proposition 8. In the rest we may thus assume that $n_2 \geq 3$.

Suppose on the contrary that $\mu_{\rm t}(G) \geq n_1 + n_2 - 1 \geq n_1 + 2$. Then $|X_3(u)| \geq 3$. We distinguish two subcases.

Case 2.1: $n_3 = 2$.
Then the third coordinate of the vertices from $X_3(u)$ is 2. Let $z = (i,j,2) \in X_3(u)$. If no vertex from X is adjacent to z, then $|X| \leq |X_1(u)| + 2 \leq n_1$ and we are done.

Assume hence that $z' \in X$ is adjacent to z, so that $z' = (i', j, 2)$ or $z' = (i, j', 2)$. If $z' = (i', j, 2)$, then the first coordinates of the vertices from X are pairwise different, hence $|X| \leq n_1$ in this subcase. If $z' = (i, j', 2)$, then the vertices from $X_3(u)$ must have pairwise different second coordinates and also not equal to 1, hence in this subcase $|X_3(u)| \leq n_2 - 1$ so that $|X| \leq 1 + (n_1 - 2) + (n_2 - 1) = n_1 + n_2 - 2$.

Case 2.2: $n_3 = 3$.
If the third coordinate of all the vertices of $X_3(u)$ is 2, then by similar arguments to the case $n_3 = 2$ we get $\mu_t(G) \leq n_1 + n_2 - 2$. Analogously, if the third coordinate of all the vertices of $X_3(u)$ is 3, then we also conclude that $\mu_t(G) \leq n_1 + n_3 - 2 \leq n_1 + n_2 - 2$. Hence in the rest we may assume that in $X_3(u)$ there exist vertices $w = (i, j, 2)$ and $w' = (i', j', 3)$. Set $W_1 = X_3(u) \cap X_1(w)$ and $W_3 = X_3(u) \cap X_3(w)$, so that $X_3(u) = \{w\} \cup W_1 \cup W_3$. Further, let $N_i(u) = \{v \in V(G) : d_G(u, v) = i\}$, $i \in [3]$. See Fig. 2 for a schematic presentation of these sets.

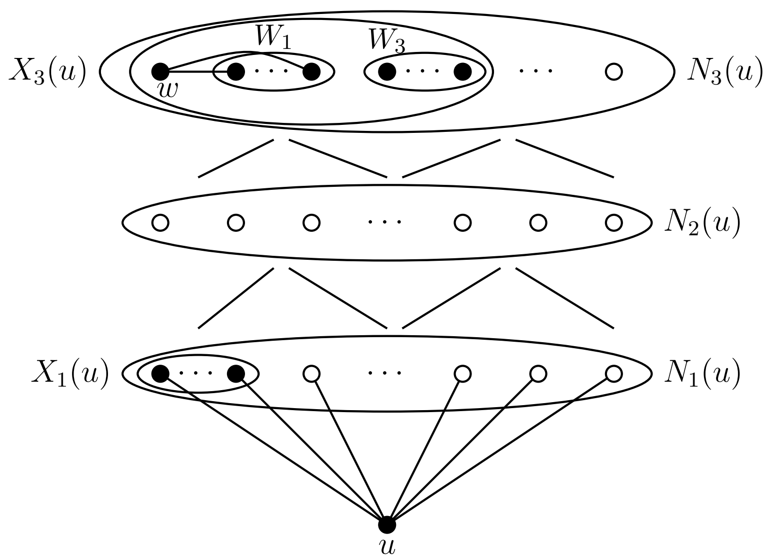

Figure 2: Graph G with a total mutual-visibility set $X = \{u\} \cup X_1(u) \cup X_3(u)$.

Assume first that $W_1 = \emptyset$, that is, $X_3(u) = \{w\} \cup W_3$. Since $n_3 = 3$ and $w = (i, j, 2)$, the third coordinate of the vertices of W_3 is 3. Furthermore, if the second coordinates of all the vertices from W_3 are equal to j', the first coordinates of the vertices from X

are pairwise different. Then we have $|W_3| \leq n_1 - |X_1(u)| - 2$ and hence

$$\begin{aligned} |X| &= |X_1(u)| + |X_3(u)| + 1 \\ &= |X_1(u)| + (|W_3| + 1) + 1 \\ &\leq |X_1(u)| + (n_1 - |X_1(u)| - 2 + 1) + 1 = n_1. \end{aligned}$$

If the first coordinates of all the vertices from W_3 are equal to i', then since $j' \neq j \neq 1$, we have $|W_3| \leq n_2 - 2$. Hence in this case we get

$$\begin{aligned} |X| &= |X_1(u)| + |X_3(u)| + 1 \\ &= |X_1(u)| + (|W_3| + 1) + 1 \\ &\leq n_1 - 2 + (n_2 - 2 + 1) + 1 = n_1 + n_2 - 2. \end{aligned}$$

Assume second that $|W_1| > 0$. If the second coordinates of all the vertices of W_1 are j, then it follows that the third coordinates of the vertices from W_1 equal to 2 and the third coordinates of the vertices from W_3 equal to 3. Hence the first coordinates of the vertices from $X_1(u) \cup W_1$ are pairwise different, since $|W_3| \geq 1$, which then implies that $|W_1| \leq n_1 - |X_1(u)| - 2$. Assume further that the second coordinates of all the vertices from W_3 equal to j'. That means that the first coordinates of the vertices from X are pairwise different, then we have $|W_3| \leq n_1 - |X_1(u)| - |W_1| - 2$. Hence

$$\begin{aligned} |X| &= |X_1(u)| + |X_3(u)| + 1 \\ &= |X_1(u)| + (|W_1| + |W_3| + 1) + 1 \\ &\leq |X_1(u)| + (|W_1| + n_1 - |X_1(u)| - |W_1| - 2 + 1) + 1 = n_1. \end{aligned}$$

Assume that the first coordinates of all the vertices from W_3 are equal to i'. Since $j' \neq j \neq 1$, then we have $|W_3| \leq n_2 - 2$. Hence

$$\begin{aligned} |X| &= |X_1(u)| + |X_3(u)| + 1 \\ &= |X_1(u)| + (|W_1| + |W_3| + 1) + 1 \\ &\leq |X_1(u)| + (n_1 - |X_1(u)| - 2 + n_2 - 2 + 1) + 1 = n_1 + n_2 - 2. \end{aligned}$$

Similarly, we also get $|X| \leq n_1 + n_2 - 2$ when the first coordinates of all the vertices of W_1 are i.

By similar arguments to the above, $\mu_{\rm t}(G) \leq n_1 + n_2 - 2$ holds when u and u' differ in the second or in the third coordinate, in which case all the vertices from $X_1(u)$ differ in this coordinate. Hence in any case we have $\mu_{\rm t}(G) \leq n_1 + n_2 - 2$ and we can conclude that $\mu_{\rm t}(G) = n_1 + n_2 - 2$. $\square$

Theorem 10 *If* $n_1 \geq n_2 \geq n_3 \geq 4$, *then* $\mu_{\rm t}(K_{n_1} \,\square\, K_{n_2} \,\square\, K_{n_3}) = n_1 + n_2 + n_3 - 6$.

Proof. To prove this result we will apply Proposition 7. More precisely, setting $H = K_{n_1,n_2,n_3}$ we are going to prove that the largest set of triangles in H such that no two triangles from the set intersect in a single vertex has cardinality $n_1 + n_2 + n_3 - 6$.

We use the same notation as in the proof of Proposition 7. That is, let I_1, I_2, I_3 be the partition classes of H with $I_i = \{u_{i,j} : \ j \in [n_i]\}$ for each $i \in [3]$. A 3-clique (triangle) induced by the vertices u, v, z is denoted by uvz. Consider first the following sets of triangles in H

$$\mathcal{K}_1 = \{u_{1,j}u_{2,1}u_{3,1} : 3 \leq j \leq n_1\}, \qquad \mathcal{K}_2 = \{u_{1,1}u_{2,j}u_{3,2} : 3 \leq j \leq n_2\},$$

$$\mathcal{K}_3 = \{u_{1,2}u_{2,2}u_{3,j} : 3 \leq j \leq n_3\},$$

and set $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$. It is clear that $|\mathcal{K}| = n_1 + n_2 + n_3 - 6$. Moreover, for any two triangles $t, t' \in \mathcal{K}$, either t and t' are from the same set $\mathcal{K}_i$ and $|t \cap t'| = 2$, or they are from different triangle sets and do not share a vertex. In either case, $|t \cap t'| \neq 1$ as required. By Proposition 7 we thus have $\mu_{\rm t}(K_{n_1} \,\square\, K_{n_2} \,\square\, K_{n_3}) \geq n_1 + n_2 + n_3 - 6$.

To prove the reverse inequality, let $\mathcal{K}$ be a triangle set in H of maximum cardinality such that $|t \cap t'| \neq 1$ holds for every pair of triangles t, t' from $\mathcal{K}$. Given $\mathcal{K}$, we say that uv is a *base edge* of the triangle $uvz \in \mathcal{K}$, if uv is incident to at least two triangles from $\mathcal{K}$.

Claim A. Every triangle $t \in \mathcal{K}$ has at most one base edge.
Proof. Suppose that uv and vz are two base edges of $uvz \in \mathcal{K}$. Then, there exists a vertex $z_1 \neq z$ with $uvz_1 \in \mathcal{K}$, and also a vertex $u_1 \neq u$ with $u_1vz \in \mathcal{K}$. The triangles $t_1 = uvz_1$ and $t_2 = u_1vz$ share the vertex v. As $|t_1 \cap t_2| = 1$ is not possible, there must be another common vertex in $t_1 \cap t_2$. As $u \neq u_1$, $z \neq z_1$, and $u \neq z$ are supposed, the only remaining possibility is $u_1 = z_1$. However, under the assumption $u_1 = z_1$, vertices u, v, z, and u_1 form a 4-clique that is impossible in the 3-partite graph H. ($\square$)

By Claim A, the triangles in $\mathcal{K}$ can be partitioned into classes $\mathcal{K}_1, \ldots, \mathcal{K}_s$ such that, for each $i \in [s]$, the set $\mathcal{K}_i$ either contains all triangles from $\mathcal{K}$ which are incident to a fixed base edge, or $\mathcal{K}_i$ contains just one triangle without a base edge. Note that, by Claim A, the partition is unique. Let V_i denote the set of vertices covered by the triangles in $\mathcal{K}_i$, for every $i \in [s]$.

Claim B. The sets $V_1, \ldots, V_s$ are pairwise vertex-disjoint, and $|\mathcal{K}| = |\bigcup_{i=1}^{s} V_i| - 2s \leq n_1 + n_2 + n_3 - 2s$ holds.
Proof. We first prove that $V_1, \ldots, V_s$ are pairwise vertex-disjoint. If $\mathcal{K}_i$ contains a

triangle t without a base edge, then $|t \cap t'| < 2$ holds for every $t' \in \mathcal{K} \setminus \{t\}$. By the assumption $|t \cap t'| \neq 1$, we may infer that t is vertex-disjoint from every other triangle in $\mathcal{K}$. Now, consider a triangle $t = uvz$ with a base edge uv and the class $\mathcal{K}_i$ that contains t. Suppose that z is also incident to another triangle $t' = zxy$. As $|t \cap t'| \neq 1$, the two triangles share a vertex different from z. It follows then that t contains a base edge different from uv that contradicts Claim A. We may conclude that z belongs to only one class V_i. A similar argument shows that the same is true for the vertices u and v.

Since the triangles in $\mathcal{K}_i$ share a base edge or $\mathcal{K}_i$ contains only one triangle, $|\mathcal{K}_i| = |V_i| - 2$ holds for every $i \in [s]$. As $V_1, \dots, V_s$ are pairwise vertex-disjoint, we may conclude $|\mathcal{K}| = |\bigcup_{i=1}^{s} V_i| - 2s \leq n_1 + n_2 + n_3 - 2s$ as stated. (□)

Claim C. If $s = 1$, then $|\mathcal{K}| \leq n_1$, and if $s = 2$, then $|\mathcal{K}| \leq n_1 + n_2 - 2$.
Proof. If $s = 1$, then all triangles in $\mathcal{K}$ has the same base edge uv and a further vertex which belongs to the third partite class of H. Hence, the number of triangles in $\mathcal{K}$ is at most n_1 and $|\mathcal{K}| \leq n_1 + n_2 - 2$ is true.

If $s = 2$, we have two sets $\mathcal{K}_1$ and $\mathcal{K}_2$ with base edges uv and $u'v'$. If $u, u' \in I_i$ and $v, v' \in I_j$, then the remaining vertices in $V_1 \cup V_2 \setminus \{u, u', v, v'\}$ all belong to the third partite class I_ℓ. By Claim B, V_1 and V_2 are disjoint sets and therefore, $|V_1 \cup V_2| \leq |I_\ell| + 4$ and $|\mathcal{K}| \leq |I_\ell| = n_\ell$. In the other case, the base edges uv and $u'v'$ contain at least one vertex from each partition class of H. Suppose that $u, u' \in I_i$, $v \in I_j$, $v' \in I_\ell$, and that uv and $u'v'$ are the base edges in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. It follows from Claim B that $V_1 \setminus \{u, v\} \subseteq I_\ell \setminus \{v'\}$ and $|\mathcal{K}_1| \leq n_\ell - 1$. Analogously, $|\mathcal{K}_2| \leq n_j - 1$. We may conclude again that

$$|\mathcal{K}| \leq n_\ell + n_j - 2 \leq n_1 + n_2 - 2$$

which finishes the proof of the claim. (□)

By Claim B, we have $|\mathcal{K}| \leq n_1 + n_2 + n_3 - 6$ whenever $s \geq 3$. By Claim C, we get $|\mathcal{K}| \leq n_1 + n_2 - 2$ if $s = 2$. As $n_3 \geq 4$ is supposed, it also implies $|\mathcal{K}| \leq n_1 + n_2 + n_3 - 6$. Similarly, the case of $s = 1$ gives $|\mathcal{K}| \leq n_1 < n_1 + n_2 + n_3 - 6$. We may then infer $|\mathcal{K}| \leq n_1 + n_2 + n_3 - 6$. In view of Proposition 7, we conclude that $\mu_{\rm t}(K_{n_1} \,\square\, K_{n_2} \,\square\, K_{n_3}) \leq n_1 + n_2 + n_3 - 6$. □

Theorem 1 follows by combining Theorems 9 and 10.

Remark 11 *Using the method from the proof of Theorem 10, it is possible to give a shorter proof of Theorem 9. However, we decided to include the present proof because it could provide a different technique to handle higher dimensional Hamming graphs.*

4 Proof of Theorem 2

Recall that for Theorem 2(i) we assume $r \geq 3$ and $N = n_1 + \cdots + n_r$. The main goal is to prove that $\mu_{\rm t}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) = \mathcal{O}(N^{r-2})$, but along the way we will also determine the constant from the statement of the theorem. We are going to show that, for each $r \geq 3$ and each r-dimensional Hamming graph $K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}$, the constant $c_r = \frac{6}{r!}$ satisfies

$$\mu_{\rm t}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) \leq c_r N^{r-2}\,. \tag{4}$$

By Theorem 1, the statement (4) is true for $r = 3$ with the constant $c_3 = 1$. We then proceed by induction on r.

We may suppose that $n_1 \geq \cdots \geq n_r$ and therefore, $n_r \leq \frac{N}{r}$ holds. Let H denote the product $K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}$ and let X be a maximum total mutual-visibility set in H. For an integer $j \in [n_r]$, we define the following set of vertices in H:

$$L(j) = \{(x_1, \ldots, x_{r-1}, j) : x_i \in [n_i] \text{ for } i \in [r-1]\}.$$

The subgraph of H induced by $L(j)$ is isomorphic to the $(r-1)$-dimensional Hamming graph $H' = K_{n_1} \,\square\, \cdots \,\square\, K_{n_{r-1}}$. As it is a convex subgraph of H, the set $X \cap L(j)$ contains no two vertices at distance 2 apart. Hence, the removal of the fixed last coordinate j transforms $L(j) \cap X$ into a total mutual-visibility set X' in H'. Clearly, $|X'| \leq \mu_{\rm t}(H')$ and $|X'| = |X \cap L(j)|$ hold. By the induction hypothesis,

$$|X'| \leq \mu_{\rm t}(H') \leq c_{r-1}(N - n_r)^{r-3} < c_{r-1}\, N^{r-3}. \tag{5}$$

Equivalently, every set $L(j)$ contains less than $c_{r-1}\, N^{r-3}$ vertices from X. On the other hand, every vertex $v \in X$ belongs to exactly one set $L(j)$ and it follows that

$$|X| < n_r\, c_{r-1}\, N^{r-3} \leq \frac{N}{r}\, c_{r-1}\, N^{r-3} = \frac{c_{r-1}}{r}\, N^{r-2}. \tag{6}$$

This proves (4) for the r-dimensional Hamming graphs with the constant $c_r = c_{r-1}/r$. Starting with the constant $c_3 = 1$, we infer that the upper bound (4) holds for every $r \geq 3$ with the constant $c_r = \frac{6}{r!}$.

The assertion (ii) of Theorem 2 is true for $r = 3$ by Theorem 1. We then proceed by induction on r. The formula can be proved along the same lines as the inequality in (i). We set $H = K_s^{\square, r}$, $n_1 = \cdots = n_r = s$, and $N = rs$. Rewriting the inequalities (5) and (6) according to the hypothesis and using $N - n_r = (r-1)s$, we get

$$|X'| \leq \mu_{\rm t}(H') \leq \left(3 \prod_{i=3}^{r-1} (i-1)^{i-3}\right) ((r-1)s)^{r-3} = \left(3 \prod_{i=3}^{r} (i-1)^{i-3}\right) s^{r-3},$$

and we can conclude

$$|X| = \mu_{\rm t}(H) \le \left(3 \prod_{i=3}^{r} (i-1)^{i-3} \right) s^{r-2}$$

which proves Theorem 2.

5 Proof of Theorem 3

Theorem 3 asserts that for every integer $r \ge 3$ it holds that $\mu_{\rm t}(K_s^{\Box,r}) = \Theta(s^{r-2})$. For $r \ge 3$, Theorem 2(*ii*) directly implies $\mu_{\rm t}(K_s^{\Box,r}) = \mathcal{O}(s^{r-2})$. For the lower bound, we give a probabilistic proof based on a similar idea as the proof in [2, Section 4] for a famous hypergraph Turán-problem of Brown, Erdős, and Sós.

Let $r \ge 3$ and $H = K_s^{\Box,r}$. From H, we choose each vertex with probability $p = \frac{2}{r(r-1)s^2}$, independently of the decisions made for other vertices. This way, we obtain a set $S \subseteq V(H)$. The expected value of the size of S is

$$E(|S|) = s^r p = \frac{2}{r(r-1)} s^{r-2}\,.$$

We say that a set $\{u, v\}$ of two vertices from S is a *bad pair* in S, if $d_H(u, v) = 2$. Let B denote the set of all bad pairs that are present in S. To get the pairs of vertices at distance 2 apart in H, we may first choose the two entries where they differ, fix the coordinates for these two entries appropriately, and fix the remaining coordinates arbitrarily. As the vertices of S were selected randomly with probability p, we may estimate the size of B in the following way:

$$\begin{aligned} E(|B|) &= \binom{r}{2} \frac{s^2(s-1)^2}{2} s^{r-2}\, p^2 \\ &\le \frac{r(r-1)s^4}{4} \, \frac{2}{r(r-1)s^2}\, s^{r-2}\, p \\ &= \frac{1}{2}\, s^r\, p = \frac{1}{2}\, E(|S|). \end{aligned}$$

For a set $S \subseteq V(H)$, we remove one vertex from each bad pair. The obtained set S^* contains no bad pairs, that is, S^* is a total mutual-visibility set in H. Moreover, we have

$$E(|S^*|) \ge E(|S|) - E(|B|) \ge \frac{1}{2}\, E(|S|) = \frac{s^{r-2}}{r(r-1)}.$$

There exists at least one set S that results in a total mutual visibility set X with $|X| \geq E(|S^*|)$ after removing one vertex from each bad pair. We may therefore infer

$$\mu_{\rm t}(K_s^{\square,r}) \geq \frac{1}{r(r-1)}\, s^{r-2}.$$

Together with the upper bound, this implies $\mu_{\rm t}(K_s^{\square,r}) = \Theta(s^{r-2})$.

Remark 12 *If we consider r-dimensional Hamming graphs in general, the statement analogous to that in Theorem 3 is not valid. Indeed, assume that, for a fixed $r \geq 3$, there exists an absolute constant $c = c(r)$ such that*

$$\mu_{\rm t}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) \geq c\, N^{r-2}$$

holds for every Hamming graph with $2 \leq n_r \leq \cdots \leq n_1$ and $N = \sum_{i=1}^{r} n_i$. By setting $n_2 = \cdots = n_r = 2$ and choosing an integer $n_1 > 2\, \sqrt[r-3]{4/c}$, the obtained r-dimensional Hamming graph H gives the contradiction

$$\mu_{\rm t}(H) \leq |V(H)| = 2^{r-1}\, n_1 = \frac{2^{r-1}}{n_1^{r-3}}\, n_1^{r-2} < c\, n_1^{r-2} < c\, N^{r-2} \leq \mu_{\rm t}(H).$$

6 A Turán-type problem

In this section, we show that our main results can be reformulated in the language of Turán-type problems on hypergraphs.

A *hypergraph* $H = (V, E)$ is a set system over the vertex set V. More precisely, every (hyper)edge $e \in E$ is a nonempty subset of V. A hypergraph $H' = (V', E')$ is a *subhypergraph* of $H = (V, E)$ if both $V' \subseteq V$ and $E' \subseteq E$ hold. A hypergraph H is r-*uniform* if every $e \in E$ contains exactly r vertices. Note that r-uniform hypergraphs are often called r-*graphs*. Then, 2-uniform hypergraphs correspond to simple graphs. The *complete r-partite r-graph* $\mathcal{K}^{(r)}_{n_1,\dots,n_r}$ is the r-uniform hypergraph on the vertex set $V = V_1 \cup \cdots \cup V_r$, where the partite classes $V_1, \dots, V_r$ are pairwise disjoint and $|V_i| = n_i$ holds for every $i \in [r]$, and moreover, the edge set is defined as $E = \{e \subseteq V : |e \cap V_i| = 1 \text{ for every } i \in [r]\}$. Thus, $\mathcal{K}^{(r)}_{n_1,\dots,n_r}$ contains $\Pi_{i=1}^{r} n_i$ edges.

The basic example for a hypergraph Turán problem takes an n-element vertex set V and asks for the maximum number of edges in a r-uniform hypergraph $H = (V, E)$ that contains no subhypergraph isomorphic to a given (r-uniform) hypergraph F. This maximum number is denoted by $\mathrm{ex}_r(n, F)$. Remark that most of the Turán-type hypergraph problems considered are notoriously hard. Even the tight asymptotics or the exact order of magnitude for $\mathrm{ex}_r(n, F)$ may be hard to identify as $n \to \infty$. For more details on the subject we refer the reader to the book [8] and the survey [10].

We may also consider the version of the problem, where the edges must be selected from the complete r-uniform r-graph $\mathcal{K}^{(r)}_{n_1,\ldots,n_r}$ such that the obtained hypergraph does not contain a subhypergraph isomorphic to a given F. Under this condition, the maximum number of edges will be denoted by $\mathrm{ex}(\mathcal{K}^{(r)}_{n_1,\ldots,n_r}, F)$.

For $r \geq 2$, let F_r denote the r-uniform hypergraph on $r+2$ vertices that contains two edges f_1 and f_2 with $|f_1 \cap f_2| = r-2$, see Fig. 3.

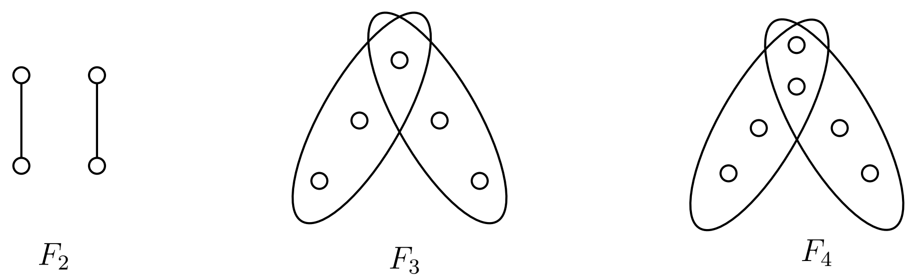

Figure 3: The 2-uniform, 3-uniform and 4-uniform forbidden subhypergraphs in our problem.

By Proposition 7, the maximum size of a total mutual visibility set in the Hamming graph $K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}$ equals the maximum number of r-cliques in the graph $K_{n_1,\ldots,n_r}$ such that no two of them intersect in $r-2$ vertices. The latter problem can be expressed by taking the vertex sets of the maximum cliques in $K_{n_1,\ldots,n_r}$ as edges in an r-uniform r-graph and forbidding the subhypergraph F_r. Therefore, we may conclude

$$\mu_{\mathrm{t}}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) = \mathrm{ex}(\mathcal{K}^{(r)}_{n_1,\ldots,n_r}, F_r)$$

and then, Theorems 1, 2, and 3 can be reformulated as follows.

Proposition 13 *If $n_1 \geq n_2 \geq n_3 \geq 2$ and $n = n_1 + n_2 + n_3$, then*

$$\mathrm{ex}(\mathcal{K}^{(3)}_{n_1,n_2,n_3}, F_3) = \begin{cases} n-4; & n_3 = 2, \\ n-5; & n_3 = 3, \\ n-6; & n_3 \geq 4\,. \end{cases}$$

Proposition 14 *(i) If $r \geq 3$ and n denotes $\sum_{i=1}^{r} n_i$, then*

$$\mathrm{ex}(\mathcal{K}^{(r)}_{n_1,\ldots,n_r}, F_r) = \mathcal{O}(n^{r-2}).$$

(ii) For every integer $r \geq 3$, it holds that

$$\mathrm{ex}(\mathcal{K}^{(r)}_{s,\ldots,s}, F_r) = \Theta(s^{r-2}).$$

The famous problem of Brown, Erdős, and Sós from [2] asks for the maximum number of edges in an r-uniform hypergraph of order n when all subhypergraphs with v vertices and e edges are forbidden. This maximum is denoted by $f^{(r)}(n, v, e) - 1$. Our problem differs from this famous one in two main aspects and consequently, neither lower nor upper bounds on $f^{(r)}(n, r+2, 2) - 1$ can be applied directly to $\mathrm{ex}(\mathcal{K}^{(r)}_{n_1,\dots,n_r}, F_r)$. First, when $f^{(r)}(n, r+2, 2) - 1$ is counted, r-edges intersecting in $r-1$ vertices are also forbidden unlike to our problem setting. Second, in our problem, the edges of the extremal hypergraph must be selected from $\mathcal{K}_{n_1,\dots,n_r}$, while the problem discussed in [2] has no such a restriction.

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