On the Δ-edge stability number of graphs

Saieed Akbari[a*] Reza Hosseini Dolatabadi[b †] Mohsen Jamaali[a‡]
Sandi Klavžar[e,f,g §] Nazanin Movarraei[c¶]

[a] Department of Mathematical Sciences, Sharif University of Technology

[b] Department of Computer Engineering, Sharif University of Technology

[c] School of Mathematics, Institute for Research in Fundamental Sciences

[e] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[f] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[g] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Abstract

The Δ-edge stability number $\mathrm{es}_\Delta(G)$ of a graph G is the minimum number of edges of G whose removal results in a subgraph H with $\Delta(H) = \Delta(G) - 1$. Sets whose removal results in a subgraph with smaller maximum degree are called mitigating sets. It is proved that there always exists a mitigating set which induces a disjoint union of paths of order 2 or 3. Minimum mitigating sets which induce matchings are characterized. It is proved that to obtain an upper bound of the form $\mathrm{es}_\Delta(G) \le c|V(G)|$ for an arbitrary graph G of given maximum degree Δ, where c is a given constant, it suffices to prove the bound for Δ-regular graphs. Sharp upper bounds of this form are derived for regular graphs. It is proved that if $\Delta(G) \ge \frac{|V(G)|-2}{3}$ or the induced subgraph on maximum degree vertices has a $\Delta(G)$-edge coloring, then $\mathrm{es}_\Delta(G) \le |V(G)|/2$.

Keywords: vertex degree; Δ-edge stability number; matching; edge coloring

AMS Subj. Class. (2020): 05C75, 05C07, 05C70, 05C15

*Email: s_akbari@sharif.edu
†Email: nimadolat80@gmail.com
‡Email: mohsen_djamali@yahoo.com
§Email: sandi.klavzar@fmf.uni-lj.si
¶Email: nazanin.movarraei@gmail.com

1 Introduction

Let G be a graph and let $\Delta(G)$ denote its maximum degree. The Δ-*edge stability number*, $\mathrm{es}_\Delta(G)$, of G, is the minimum number of edges of G in which their deletions result in a subgraph H with $\Delta(H) = \Delta(G)-1$. This graph invariant has been for the first time investigated by Borg and Fenech in [7]. The vertex version of the problem, that is, the Δ-vertex stability number, has been studied in [5, 6]. Furthermore, over the last few years, the corresponding problems for the chromatic number and the chromatic index were respectively investigated in [3, 8] and [1, 2, 15, 16, 17].

The above discussion naturally fall within the following broader framework recently proposed in [13] and further elaborated in [4, 14]. For an arbitrary graph invariant τ, the τ-vertex stability number (the τ-edge stability number) is the minimum number of vertices (edges) whose removal results in a subgraph H with $\tau(H) \neq \tau(G)$. The corresponding minimum number of vertices and edges are respectively denoted by $\mathrm{vs}_\tau(G)$ and $\mathrm{es}_\tau(G)$. Following this general framework we use in this paper the notation $\mathrm{es}_\Delta(G)$, although we should add that the Δ-edge stability number and the Δ-vertex stability number of a graph G were in [7, 6] respectively denoted by $\lambda_\mathrm{e}(G)$ and $\lambda(G)$.

In the seminal paper [7] the focus was on the upper bounds of the Δ-edge stability number in terms of the size of the graph, the maximum degree, and the number of vertices of maximum degree. In this paper, we continue with the exploration of the Δ-edge stability number. Our main goal is to find tight bounds for $\mathrm{es}_\Delta(G)$ based on the order of the graph.

The paper is organized as follows. In the rest of the introduction, we briefly define notations used in the paper and recall a result to be used later on. In the subsequent section, we prove several general properties for sets of edges whose removal decreases the maximum degree. In particular, we prove that there always exists such a set which induces a disjoint union of paths of order 2 or 3, and characterize smallest such sets which are matchings. In Section 3 we prove that to obtain an upper bound of the form $\mathrm{es}_\Delta(G) \le c|V(G)|$ for an arbitrary graph G of given maximum degree Δ, where c is a given constant, it sufficed to prove the bound for Δ-regular graphs. In Section 4 sharp upper bound for regular graphs are derived. In the final section, we prove that the Δ-edge stability number is bounded from the above by one-half of the order for each graph in which vertices of maximum degree induce a Class 1 graph, as well as for graphs G with $\Delta(G) \ge \frac{|V(G)|-2}{3}$.

Throughout this paper all graphs are finite and simple, that is, with no loops and multiple edges, and moreover, with at least one edge. Let $G = (V(G), E(G))$ be a graph. The degree of a vertex u of G is denoted by $d_G(u)$. Further, $\delta(G)$ is the minimum degree of G. The subgraph of G induced by a set A of vertices and/or edges will be denoted by $G[A]$. The number of edges between two disjoint sets of vertices $S, T \subseteq V(G)$ is denoted by $e(S,T)$. The open neighborhood $N_G(v)$ of a vertex $v \in V(G)$ is the set of neighbors of v, the closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. If $S \subseteq V(G)$, then the open and the closed neighborhood of S are the respective sets $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = \cup_{v\in S} N_G[v]$. If the graph G is clear from the context, we may omit the subscript G in the above notation. The *independence number* of G is denoted by $\alpha(G)$ and its matching number by $\alpha'(G)$. The *odd girth* of G is the length of a shortest odd cycle in G and is denoted by $\mathrm{og}(G)$.

The *core* of G is the set of vertices of G of maximum degree and is denoted by $\mathrm{Core}(G)$. Clearly, if G is regular, then $\mathrm{Core}(G) = V(G)$. If $S \subseteq E(G)$ is such that $\Delta(G-S) = \Delta(G)-1$, then we say that S is a *mitigating set* of G. In that case, we also say that $G[S]$ is a *mitigating subgraph* of G.

Given a graph G, a function $c : E(G) \to \{c_1, \dots, c_k\}$ with $c(e) \neq c(f)$ for any two adjacent edges e and f is a *proper k-edge coloring* of G. The minimum k for which G admits a proper k-edge coloring is the *chromatic index* of G, and denoted by $\chi'(G)$. We let $[k] = \{1, \dots, k\}$. For any

$i \in [\chi'(G)]$, let C_i denote the set of all edges of G that are colored by c_i in the proper edge coloring c. For any $v \in V(G)$, let $c(v)$ denote the set of colors appearing in v. Vizing's Theorem [19] states that the chromatic index of an arbitrary simple graph lies between the maximum degree $\Delta(G)$ and $\Delta(G)+1$. Graphs with $\chi'(G) = \Delta(G)$ are said to be *Class* 1, while graphs with $\chi'(G) = \Delta(G)+1$ are said to be *Class* 2.

Throughout the following we will use Tutte's Theorem, which states that a graph has a perfect matching if and only if $o(G-S) \leq |S|$ for every $S \subseteq V(G)$, where $o(H)$ denotes the number of odd components of a graph H. We conclude the preliminaries by recalling the following result to be used later on.

Theorem 1.1. [7, Theorem 2.8] *If G is a graph, then* $\mathrm{es}_\Delta(G) = |\operatorname{Core}(G)| - \alpha'(G[\operatorname{Core}(G)])$.

2 Properties of mitigating sets

In this section, we first prove that one can always find a mitigating subgraph of G whose each component is P_2 or P_3, and give an upper bound on the Δ-edge stability number involving the independence number. To derive both results, Hall's Theorem will be applied. Afterwards, we characterize in two ways minimum mitigating sets which are matchings.

Theorem 2.1. *If G is a graph of order n, then the following properties hold.*

(i) There exists a mitigating subgraph of G whose each component is P_2 or P_3.

(ii) $\mathrm{es}_\Delta(G) \leq n - \alpha(G)$*, and the bound is sharp.*

Proof. (i) Let M be a maximum matching of $G[\operatorname{Core}(G)]$, and let $A \subseteq \operatorname{Core}(G)$ be the set of vertices of $\operatorname{Core}(G)$ which are not saturated by M. As M is a maximum matching of $G[\operatorname{Core}(G)]$, the set A is independent. Furthermore, since the vertices in A are of maximum degree, $|N(A')| \geq |A'|$ holds for each $A' \subseteq A$. Therefore, by Hall's Theorem, there exists a matching M' in G that saturates A. By applying Theorem 1.1 we have

$$\begin{aligned}\mathrm{es}_\Delta(G) &\leq |M \cup M'| \leq |M| + |M'| \\ &= \alpha'(G[\operatorname{Core}(G)]) + (|\operatorname{Core}(G)| - 2\alpha'(][\operatorname{Core}(G)])) \\ &= |\operatorname{Core}(G)| - \alpha'(G[\operatorname{Core}(G)]) \\ &= \mathrm{es}_\Delta(G)\,,\end{aligned}$$

hence equality must hold in the first line. Therefore, $M \cup M'$ is a mitigating set of G.

Since M is a maximum matching of $G[\operatorname{Core}(G)]$, there is no pair of edges of M' which meet an edge of M. Thus, any component of $M \cup M'$ is P_2 or P_3.

(ii) Let $I \subseteq V(G)$ be an independent set of size $\alpha(G)$, and $A = \operatorname{Core}(G) \cap I$. By Hall's Theorem, there exists a matching M in G that saturates A. Assume $B \subseteq V(G) \setminus I$ is the set of vertices of $V(G) \setminus A$ which are saturated by M. Let S be the set of edges containing M as well as one edge adjacent to each vertex of $V(G) \setminus (I \cup B)$. Clearly, $G \setminus S$ has no vertex of degree Δ. Since

$$|S| = |M| + (n - |I \cup B|) = |M| + n - (\alpha(G) + |M|) = n - \alpha(G)\,,$$

the bound is proved. To see that it is sharp, consider an arbitrary regular, bipartite graph G with a perfect matching. Then $\mathrm{es}_\Delta(G) = \alpha(G) = n/2$. $\square$

To characterize minimum mitigating sets, we need a lemma which can be deduced from Tutte's Theorem.

Lemma 2.2. *Let G be a graph and $A \subseteq V(G)$. If for all $S \subseteq V(G)$ we have*

$$o_{G[A]}(G \setminus S) \leq |S|,$$

then there exists a matching in G that saturates A, where $o_{G[A]}(G \setminus S)$ is the number of odd components of $G \setminus S$ which are contained in A.

Proof. Let $n = |V(G)|$ and $B = G \setminus A$. Let H be the graph obtained by the disjoint union of K_n and G and joining each vertex of K_n to any vertex of B. Clearly, H is of order $2n$. Moreover, as the order of H is even, we see that H has a perfect matching if and only if G has a matching that saturates A. Note that maybe some edges in the matching have one endpoint in A and another in B. Assume for the contrary that no matching in G covers A. Therefore, by Tutte's Theorem there exists $S \subseteq V(H)$ such that

$$o(H \setminus S) \geq |S| + 2\,. \tag{1}$$

If $V(K_n) \subseteq S$, then $|S| \geq \frac{|V(H)|}{2}$ and thus, (1) does not hold. Hence there exists $v \in V(K_n) \setminus S$. Obviously, there are at least $|S| + 1$ odd components of $H \setminus S$ that are contained in A, which is a contradiction, and the lemma is proved. □

We now characterize minimum mitigating sets which are matchings as follows.

Theorem 2.3. *If G is a graph, and $S \subseteq V(G)$, then the following statements are equivalent:*

(1) G has a matching that saturates Core(G).

(2) G has a minimum mitigating set which is a matching.

(3) For every $S \subseteq$ Core(G), $\mathrm{es}_\Delta(G[N[S]]) \leq \frac{|V(G[N[S]])|}{2}$.

Proof. We first show that (1) and (2) are equivalent. It is clear that if there is a minimum mitigating set which is a matching, this matching saturates Core(G). For the reverse, suppose M is a matching that saturates Core(G) and let L be a minimum mitigating set for G. We claim that L could be chosen as a matching using induction on $\mathrm{es}_\Delta(G)$. If $\mathrm{es}_\Delta(G) = 1$, the claim is obvious. Hence, assume that $\mathrm{es}_\Delta(G) \geq 2$. Clearly, $|M| \geq |L|$. If $L \setminus M = \varnothing$, L must be a matching and the claim is proved. Thus, assume $e \in L \setminus M$. Since $|L| \geq 2$, we have $\Delta(G \setminus e) = \Delta(G)$. Therefore, M is a matching that saturates Core$(G \setminus e)$. By the induction hypothesis, there exists a minimum mitigating set for $G \setminus e$, say L', which is a matching and $|L'| = \mathrm{es}_\Delta(G) - 1$. Obviously, $L'' = L' \cup \{e\}$ is a minimum mitigating set for G. If e is not adjacent to any edge in L', the claim is proved. Since L' is matching and L'' is a minimum mitigating set, e is adjacent to at most one edge in L', say e'. Let $e = uv$ and $e' = vw$ for some $u, v, w \in V(G)$. It is easy to see that $d(u) = \Delta$. Since $uv \notin M$, there exists $e_1 = uy_1 \in M$. Replace e with e_1 in L'' and the resulting mitigating set is still minimum and is a matching, unless there exists $e'_1 = y_1x_1 \in L''$ and $d(x_1) = \Delta$. Otherwise, if we remove e'_1 from L'', the resulting set of edges is still a mitigating set. Replace e'_1 with some $e_2 = x_1y_2 \in M$ to obtain a new minimum mitigating set and continue this operation and in each step replace e'_i with e_{i+1}. Note that if $d(x_i) < \Delta$ or there is no edge in L' saturating y_{i+1}, there is no need to continue the operation. Moreover, since M is a matching, all x_i and y_i are distinct. Therefore, this operation will be stopped after finitely many steps. So, when the replacement is done, the resulting minimum mitigating set is a matching and the claim is proved.

We next prove that (1) and (3) are equivalent. Let $S \subseteq V(G)$, and set $H(S) = G[N[S]]$. It is clear that if G has a matching that saturates $\mathrm{Core}(G)$, then (3) holds. For the converse, set $n = |V(G)|$. Let $A = G[\mathrm{Core}(G)]$ and $B = G \setminus A$. By contradiction assume that G has no matching that covers A. Therefore, by Lemma 2.2, there exists $S \subseteq V(G)$ such that

$$o_{G[A]}(G \setminus S) \geq |S| + 1\,,$$

Suppose $C_1, C_2, \ldots, C_{|S|+1} \subseteq A$ are some odd components contained in A. Let $C = \bigcup_{i=1}^{|S|+1} C_i$. Clearly, $|V(H(C))| \leq |S| + |V(C)|$. Since $V(C_i)$ has odd cardinality and consists of vertices of degree Δ, we have

$$\mathrm{es}_\Delta\,(H(C_i)) \geq \frac{|V(C_i)| + 1}{2}\,.$$

Therefore,

$$\mathrm{es}_\Delta\,(H(C)) \geq \sum_{i=1}^{|S|+1} \frac{|V(C_i)| + 1}{2} \geq \frac{|V(C)|}{2} + \frac{|S| + 1}{2} > \frac{|V(C)| + |S|}{2} \geq \frac{|V(H(C))|}{2}\,,$$

a contradiction and we are done. □

3 Graph's regularization and its applications

In this section we prove that to derive an upper bound of the form $\mathrm{es}_\Delta(G) \leq c|V(G)|$ for an arbitrary graph, where c is a given constant, it suffices to prove the bound for regular graphs.

For a graph G we construct its *regularization* $R(G)$ as follows. If G is regular, then set $R(G) = G$. Assume now that G is not regular and set $A_G = V(G) \setminus \mathrm{Core}(G)$. Then the graph $G^{(1)}$ is obtained from two disjoint copies of G, say G' and G'', by adding a matching between the corresponding vertices in $A_{G'}$ and $A_{G''}$. Note that $\Delta(G^{(1)}) = \Delta(G)$ and $\delta(G^{(1)}) = \delta(G) + 1$. If $G^{(1)}$ is not yet regular, we repeat the same construction on $G^{(1)}$ to construct $G^{(2)}$. Repeating the construction $\Delta(G) - \delta(G)$ times we arrive at the regularization $R(G)$ of G:

$$R(G) = G^{(\Delta(G) - \delta(G))}\,,$$

which is a $\Delta(G)$-regular graph.

A key property of $R(G)$ is the following.

Lemma 3.1. *If G is a graph, then*

$$\frac{|V(R(G))|}{\mathrm{es}_\Delta(R(G))} \leq \frac{|V(G)|}{\mathrm{es}_\Delta(G)}\,.$$

Proof. There is nothing to prove if G is regular, hence assume in the rest that $\Delta(G) - \delta(G) \geq 1$. We first claim that

$$\frac{|V(G^{(1)})|}{\mathrm{es}_\Delta(G^{(1)})} \leq \frac{|V(G)|}{\mathrm{es}_\Delta(G)}\,.$$

Let M be a mitigating set of $G^{(1)}$ and let G' and G'' be the two copies of G in $G^{(1)}$. As every edge between G' and G'' connects vertices of not-maximum degree in G' and G'', the sets $M \cap E(G')$

and $M \cap E(G'')$ are mitigating sets of G' and G'' respectively. Thus, $\mathrm{es}_\Delta(G^{(1)}) \geq 2\,\mathrm{es}_\Delta(G)$. Since $|V(G^{(1)})| = 2|V(G)|$, the claim is proved. Proceeding by induction we analogously infer that

$$\frac{|V(G^{(i)})|}{\mathrm{es}_\Delta(G^{(i)})} \leq \frac{|V(G)|}{\mathrm{es}_\Delta(G)}$$

holds for each $i \in \{2, \ldots, \Delta(G) - \delta(G)\}$. Thus the assertion. □

The announced reduction to regular graphs now reads as follows.

Theorem 3.2. *If there exists a constant $0 < c_\Delta < 1$, such that $\mathrm{es}_\Delta(H) \leq c_\Delta |V(H)|$ holds for every Δ-regular graph H, then $\mathrm{es}_\Delta(G) \leq c_\Delta |V(G)|$ holds for every graph G with $\Delta(G) = \Delta$.*

Proof. Let G be an arbitrary graph with $\Delta(G) = \Delta$. As there is nothing to prove if G is regular, assume this is not the case. Then by Lemma 3.1 and the theorem's assumption we get

$$\mathrm{es}_\Delta(G) \leq \mathrm{es}_\Delta(R(G)) \frac{|V(G)|}{|V(R(G))|} \leq c_\Delta |V(R(G))| \frac{|V(G)|}{|V(R(G))|} = c_\Delta |V(G)|$$

and we are done. □

Another applications of Lemma 3.1 is the following bound on the Δ-edge stability number of a graph in terms of its odd girth.

Theorem 3.3. *If G be a graph of order n and $\mathrm{og}(G) = 2k+1$, $k \geq 1$, then $\mathrm{es}_\Delta(G) \leq \frac{k+1}{2k+1} n$.*

Proof. We claim that $\mathrm{og}(G^{(i)}) \geq 2k+1$ holds for each $i \in [\Delta - \delta]$. Consider first $G^{(1)}$ and let W_1 be an arbitrary odd closed walk in it. Assume that W_1 passes through G' as well as through G''. Let $e = u'u''$ and $f = v'v''$ be two edges of W_1 between G' and G'' (where $u', v' \in V(G')$ and $u'', v'' \in V(G''))$ occurring consecutively on W_1. Let $W''_{u''v''}$ be the u'', v''-subwalk of W_1. Then $W''_{u''v''}$ is contained in G''. Now replace the $u' - u'' - W''_{u''v''} - v'' - v'$ subwalk of W_1 by the walk $u' - W'_{u'v'} - v'$, where $W'_{u'v'}$ is the isomorphic copy of $W''_{u''v''}$ in G'. Repeating this process if necessary, we arrive at a closed walk of $G^{(1)}$ which lies completely in G' and is shorter (or of equal length if W_1 already lies completely in G') than W_1. As G' is isomorphic to G, this proved the claim for $G^{(1)}$. The argument for $G^{(i)}$, $i \in \{2, \ldots, \Delta - \delta\}$ is then analogous.

$R(G)$ is a regular graph, and it was proved by Hajnal [9] and Tutte [18] (see also [12]) that such a graph has a $\{1,2\}$-factor F. Clearly for any component C of F, one can remove $\lceil \frac{|V(C)|}{2} \rceil$ edges of C which saturate $V(C)$. Since $\lceil \frac{|V(C)|}{2} \rceil \leq \frac{k+1}{2k+1} |V(C)|$, we obtain

$$\mathrm{es}_\Delta(R(G)) \leq \frac{k+1}{2k+1} |V(R(G))|\,.$$

By Lemma 3.1 we then get

$$\mathrm{es}_\Delta(G) \leq \mathrm{es}_\Delta(R(G)) \frac{|V(G)|}{|V(R(G))|} \leq \frac{k+1}{2k+1} |V(R(G))| \frac{|V(G)|}{|V(R(G))|} = \frac{k+1}{2k+1} |V(G)|$$

and we are done. □

4 The regular case

In view of Theorem 3.2, in this section we take a closer look to regular graphs. For this sake we first recall the following fundamental result due to Henning and Yeo.

Theorem 4.1 (Henning and Yeo, [10]). *Let G be a connected, k-regular graph of order n. If $k \geq 2$ is even, then*

$$\alpha'(G) \geq \min\left\{\frac{(k^2+4)n}{2(k^2+k+2)}, \frac{n-1}{2}\right\},$$

and if $k \geq 3$ is odd, then

$$\alpha'(G) \geq \frac{(k^3-k^2-2)n-2k+2}{2(k^3-3k)}.$$

Moreover, both bounds are tight.

From Theorem 4.1 we can deduce the following consequence.

Corollary 4.2. *Let G be a connected, k-regular graph of order n. If $k \geq 2$ is even, then*

$$\mathrm{es}_\Delta(G) \leq \max\left\{\left(1-\frac{k^2+4}{2(k^2+k+2)}\right)n, \frac{n+1}{2}\right\},$$

and if $k \geq 3$ is odd, then

$$\mathrm{es}_\Delta(G) \leq \frac{(k^3+k^2-6k+2)n+2k-2}{2(k^3-3k)}.$$

Moreover, both bounds are tight.

Proof. By Theorem 1.1 and the assumption that G is regular, we have $\mathrm{es}_\Delta(G) = n - \alpha'(G)$. Therefore, if k is even, then using Theorem 4.1 we can estimate as follows:

$$\begin{aligned}\mathrm{es}_\Delta(G) &\leq n - \min\left\{\frac{(k^2+4)n}{2(k^2+k+2)}, \frac{n-1}{2}\right\} \\ &= \max\left\{n - \frac{(k^2+4)n}{2(k^2+k+2)}, n - \frac{n-1}{2}\right\} \\ &= \max\left\{\left(1-\frac{k^2+4}{2(k^2+k+2)}\right)n, \frac{n+1}{2}\right\}.\end{aligned}$$

The estimate for odd k is derived analogously.

The tightness follows by the tightness of the estimates from Theorem 4.1. □

As discussed in [10], the bound $(n-1)/2$ from Theorem 4.1 is only necessary to cover some cases when n is very small or $k = 2$. In particular, it is not needed for $k \geq 4$, cf. [10, Corollary 1]. Therefore, we can also state the following the following easier-to-read corollary for all "non-trivial" even k.

Corollary 4.3. *If G is a connected graph of order n with maximum degree k, where $k \geq 4$ is even, then*

$$\mathrm{es}_\Delta(G) \leq \left(1-\frac{k^2+4}{2(k^2+k+2)}\right)n.$$

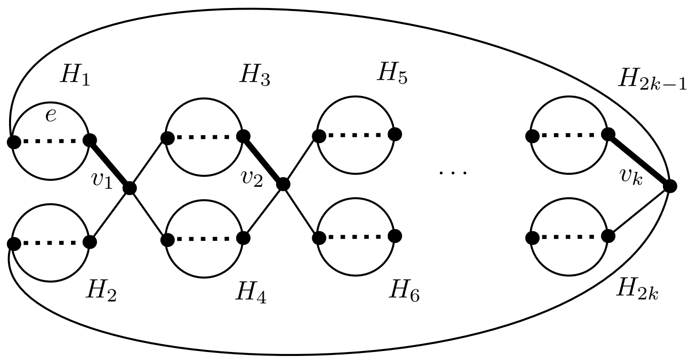

Figure 1: The graph G_k

If $k = 4$, then the bound of Corollary 4.3 reads as $\mathrm{es}_\Delta(G) \leq \frac{6}{11}n$. We next construct 2-connected, 4-regular graphs for which the equality holds in this bound. Let $H_i = K_5 \setminus e$, $i \in [2k]$, and let G_k be the graph of order $n = 11k$ as shown in Fig. 1.

We claim that $\alpha'(G) = 5k$ and $\mathrm{es}_\Delta(G) = 6k$. Since $\alpha'(K_5 \setminus e) = 2$, and every matching of G has at most one edge incident with v_j, $j \in [k]$, we infer that $\alpha'(G) \leq 2(2k) + k = 5k$. Note that a 2-matching of each H_i together with the bold edges from Fig. 1 represent a maximum matching of size $5k$ in G. So, $\alpha'(G) = 5k$. Since G is 4-regular, by Theorem 1.1, $\mathrm{es}_\Delta(G) = 6k$ and hence the conclusion is reached after a direct computation.

From the second estimate of Corollary 4.2, we can deduce also the following consequence.

Corollary 4.4. *For each $\epsilon > 0$ and sufficiently large n, for every graph G of order n with odd maximum degree k, we have*

$$\mathrm{es}_\Delta(G) \leq \left(\frac{k^3 + k^2 - 6k + 2}{2(k^3 - 3k)} + \epsilon \right) n.$$

We know the following function is monotone

$$f(k) = \frac{k^3 + k^2 - 6k + 2}{2(k^3 - 3k)}.$$

Moreover,

$$\lim_{k \to \infty} f(k) = \frac{1}{2},$$

By Corollary 4.3 for $k > 2$ even, we have

$$\mathrm{es}_\Delta(G) \leq \frac{n+1}{2}.$$

Combining Corollaries 4.3 and 4.4 with the above discussion, and having Theorem 3.2 in mind, we get the following result by setting $k = 3$.

Corollary 4.5. *For each $\epsilon > 0$ and sufficiently large n, for every connected graph G of order n, we have*

$$\mathrm{es}_\Delta(G) \leq \left(\frac{5}{9} + \epsilon\right) n.$$

5 Graphs with the Δ-edge stability number at most half the order

In this section we are interested in families of graphs for which the Δ-edge stability number is bounded from the above by one half of the order. First, as a consequence of Theorem 2.1(ii), this holds true for bipartite graphs.

Corollary 5.1. *If G is a bipartite graph of order n, then $\mathrm{es}_\Delta(G) \leq n/2$.*

Our next goal is to show that this bound also holds for each graph G whose $G[\mathrm{Core}(G)]$ is Class 1. To this end, let us first prove the following.

Theorem 5.2. *Let G be a graph with maximum degree Δ. If $G[\mathrm{Core}(G)]$ has a proper Δ edge-coloring, then G has a matching that saturates $\mathrm{Core}(G)$.*

Proof. Let $n = |V(G)|$. Let $A = G[\mathrm{Core}(G)]$ and $B = G \setminus V(A)$. Suppose that G has no matching that saturates A. By Lemma 2.2, there exists $S \subseteq V(G)$ such that

$$o_{G[A]}(G \setminus S) \geq |S| + 1$$

Now, we claim that if $C \subseteq A$ is an odd component of $G \setminus S$, then

$$e_G(C, S) \geq \Delta. \tag{2}$$

Consider a proper Δ-edge coloring of A and extend it to a coloring of $G \setminus E(B)$ such that for every $u \in A$, all edges adjacent to u have Δ distinct colors.

If there are at most $\Delta - 1$ edges with one endpoint in $V(C)$ and another in S, then there exists a color t which has not appeared on these edges. Since every vertex in C has degree Δ, the color t appears at each vertex of C. Therefore, the edges with color t in C form a perfect matching in C, a contradiction, and the claim is proved.

By the claim, we have

$$e_G(A \setminus S, S) \geq (|S| + 1)\,\Delta.$$

Now, by the Pigeonhole Principle, there exists $u \in S$ such that

$$e_G(A, \{u\}) \geq \Delta + 1$$

which is a contradiction. $\square$

A matching that saturates every vertex of $\mathrm{Core}(G)$ is clearly a mitigating set, hence from Theorem 5.2 we immediately get:

Corollary 5.3. *If G is a graph of order n such that $G[\mathrm{Core}(G)]$ is Class 1, then $\mathrm{es}_\Delta(G) \leq \frac{n}{2}$. In particular, if G is Class 1 graph, then $\mathrm{es}_\Delta(G) \leq \frac{n}{2}$.*

In our final result, we prove that also in graphs with large maximum degree the Δ-edge stability number is bounded from the above by one-half of the order. To prove the result we state the following lemma for which we recall that a graph is *factor critical* if every vertex deleted subgraph has a perfect matching.

Lemma 5.4. *If G does not have a perfect matching and $S \subseteq V(G)$ is the biggest set such that*

$$o(G \setminus S) > |S|, \tag{3}$$

then each component of $G \backslash S$ is factor critical.

Proof. Suppose that C is a component of $G \setminus S$ of even order and $v \in C$. Note that, $C \setminus \{v\}$ has at least one odd component. Therefore, $S \cup \{v\}$ satisfies (3), a contradiction. So, $G \setminus S$ has no even component. Suppose C is an odd component of $G \backslash S$ which is not factor critical. Thus if we remove $v \in C$ from C, the remaining vertices do not have a perfect matching. Hence, there is $S' \subseteq C \setminus \{v\}$ such that $o\left(G[C \setminus \{v\}] \setminus S'\right) > |S'| + 1$. Thus, $S \cup S' \cup \{v\}$ satisfies (3), a contradiction. □

Our final result now reads as follows.

Theorem 5.5. *Let G be a connected graph of order n with maximum degree Δ. If $\Delta \geq \frac{n-2}{3}$, then $es_{\Delta}(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Let $A = V(Core(G))$ and $B = V(G) \backslash A$. If there exists a matching in G which saturates A, then we get the result. Now, assume G does not contain such matchings. So, by Lemma 2.2, there exists $S \subseteq V(G)$ such that $o_{G[A]}(G \setminus S) \geq |S| + 1$. Thus, there exist at least $|S| + 1$ odd components in $G \setminus S$ that are contained in A. We name them as $C_1, C_2, \ldots, C_{|S|+1}$.

We claim that if C is a component in $G \backslash S$ whose vertices are contained in A and $e\left(V(C), S\right) < \Delta$, then the order of C is at least $\Delta + 1$, moreover, there exists $v \in V(C)$ such that $N[v] \subseteq V(C)$. First, suppose that C is of order at least Δ. If every vertex of C has a neighbor in S, then it is clear that $e(V(C), S) \geq \Delta$, a contradiction. So, there exists a vertex $v \in C$ such that $N[v] \subseteq V(C)$. Thus, $|V(C)| \geq \Delta + 1$, as desired. If $|V(C)| < \Delta$, since every vertex of C have degree Δ, so for every $u \in V(C)$ we have $e(\{u\}, S) \geq \Delta - |V(C)| + 1$. Hence $e(V(C), S) \geq |V(C)|(\Delta - |V(C)| + 1) \geq \Delta$, contradiction. So the claim is proved.

Note that if $e(V(C), S) \geq \Delta$ for every $C \in \{C_1, \ldots, C_{|S|+1}\}$, then $e(\cup V(C_i), S) \geq \Delta(|S| + 1)$ and by the pigeonhole principal, there exists $w \in S$ such that $d(w) \geq \Delta + 1$, a contradiction.

Now, since $\Delta \geq \frac{n-2}{3}$, if C is a component of $G \backslash S$ and $e(V(C), S) < \Delta$, then by the claim we have $|V(C)| \geq \frac{n+1}{3}$. Hence, we have at most two C_i with this property that $e(V(C_i), S) < \Delta$, say C_1 and C_2. Let $L = \bigcup_{i=1}^{|S|+1} V(C_i)$ and consider two cases.

Case 1. $|V(C_1)| > \Delta$ and for every $i > 1$, $|V(C_i)| \leq \Delta$.
So, by the claim, $e(L \setminus V(C_1), S) \geq \Delta|S|$. On the other hand since for every $s \in S$, $d(s) \leq \Delta$, we have $e(L \setminus V(C_1), S) \leq \Delta|S|$. Hence, $e(L \setminus V(C_1), S) = \Delta|S|$ and it means that all neighbors of each vertex of S is in $V(C_2), \ldots, V(C_{|S|+1})$. So, C_1 is a component of G which contradicts the connectivity of G.

Case 2. $|V(C_1)| > \Delta$, $|V(C_2)| > \Delta$, and for every $i > 2$, $|V(C_i)| \leq \Delta$.
Since, $|V(C_1)| \geq \frac{n+1}{3}$ and $|V(C_2)| \geq \frac{n+1}{3}$, so $|V(G) \setminus (V(C_1) \cup V(C_2))| < \frac{n-1}{3}$. If there exists a vertex $u \in V(G) \setminus (V(C_1) \cup V(C_2))$ of degree Δ, since $N[u] \subseteq V(G) \setminus (V(C_1) \cup V(C_2))$ and $|N[u]| > \frac{n}{3}$, we get a contradiction. So, $L \setminus (V(C_1) \cup V(C_2)) = \varnothing$ and thus, $|S| = 1$.

Note that if we choose the set S with the maximum size, by Lemma 5.4 all components of $G \backslash S$ that are in A and have odd orders are factor critical. Let $S = \{s\}$. We choose two arbitrary edges su and sv such that $u \in V(C_1)$ and $v \in V(C_2)$. As C_1 and C_2 are factor critical, there exist matchings M and N that saturate vertices of $C_1 \setminus \{u\}$ and $C_2 \setminus \{v\}$, respectively. Since all vertices of degree Δ are in $L \cup S$, it is clear that $M \cup N \cup \{su, sv\}$ is a mitigating set for G. Note that $|M| + |N| + |\{su, sv\}| = \frac{|L|+2}{2} \le \frac{n+1}{2}$, thus $\mathrm{es}_\Delta(G) \le \frac{n+1}{2}$ and we are done. □

Remark 5.6. To see that the bound $\Delta \ge \frac{n-2}{3}$ of Theorem 5.5 cannot be lowered in general, consider the following example. For each odd $t \ge 7$, let G_t be the graph constructed as follows. Take three disjoint copies of K_t, remove one edge from each of them, add a vertex u, and join u to the endpoints of the deleted edges, see Fig. 2.

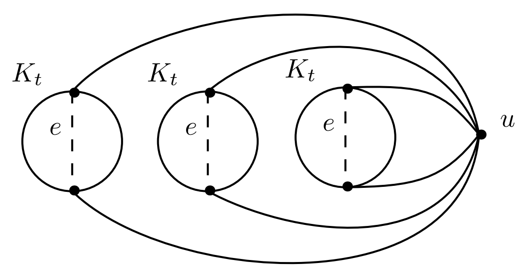

Figure 2: The graph G_t

In G_t we need at least $\frac{t+1}{2}$ edges to saturate the vertices of each K_t. Hence, $\mathrm{es}_\Delta(G) \ge \frac{3(t+1)}{2}$. Setting $n = |V(G_t)|$, we have $n = 3t + 1$. As $t \ge 7$ we have $\Delta(G_t) = \frac{n-4}{3}$. In summary,

$$\Delta(G_t) = \frac{n-4}{3} < \frac{n-2}{3} \quad \text{and} \quad \mathrm{es}_\Delta(G_t) \ge \frac{n}{2} + 1\,,$$

hence the assumption on the maximum degree in Theorem 5.5 is tight.

As in a graph G we have $\Delta(G) \ge \frac{2|E(G)|}{|V(G)|}$, Theorem 5.5 yields:

Corollary 5.7. *If G is a graph of order n and size at least $\frac{n(n-2)}{6}$, then $es_\Delta(G) \le \lceil \frac{n}{2} \rceil$.*

Acknowledgements

This research was in part supported by a grant from IPM (No. 1402050012). Sandi Klavžar was supported by the Slovenian Research Agency ARIS (research core funding P1-0297 and projects J1-2452, N1-0285).

References

[1] S. Akbari, A. Beikmohammadi, S. Klavžar, N. Movarraei, On the chromatic vertex stability number of graphs, European J. Combin. 102 (2022) Paper 103504.

[2] S. Akbari, A. Beikmohammadi, B. Brešar, T. Dravec, M.M. Habibollahi, N. Movarraei, On the chromatic edge stability index of graphs, European J. Combin. 111 (2023) Paper 103690.

[3] S. Akbari, S. Klavžar, N. Movarraei, M. Nahvi, Nordhaus-Gaddum and other bounds for the chromatic edge-stability number, European J. Combin. 84 (2020) Paper 103042.

[4] S. Alikhani, M.R. Piri, On the edge chromatic vertex stability number of graphs, AKCE Int. J. Graphs Comb. 20 (2023) 29–34.

[5] P. Borg, Decreasing the maximum degree of a graph, Discrete Math. 345 (2022) Paper 112994.

[6] P. Borg, K. Fenech, Reducing the maximum degree of a graph by deleting vertices, Australas. J. Combin. 69 (2017) 29–40.

[7] P. Borg, K. Fenech, Reducing the maximum degree of a graph by deleting edges, Australas. J. Combin. 73 (2019) 247–260.

[8] B. Brešar, S. Klavžar, N. Movarraei, Critical graphs for the chromatic edge-stability number, Discrete Math. 343 (2020) Paper 111845.

[9] A. Hajnal, A theorem on k-saturated graphs, Canadian J. Math. 17 (1965) 720–724.

[10] M.A. Henning, A. Yeo, Tight lower bounds on the size of a maximum matching in a regular graph, Graphs Combin. 23 (2007) 647–657.

[11] B. Jackson, Cycles through vertices of large maximum degree, J. Graph Theory 19 (1995) 157–168.

[12] M. Kano, Factors of regular graphs, J. Combin. Theory Ser. B 41 (1986) 27–36.

[13] A. Kemnitz, M. Marangio, On the ρ-edge stability number of graphs, Discuss. Math. Graph Theory 42 (2022) 249–262.

[14] A. Kemnitz, M. Marangio, On the vertex stability numbers of graphs, Discrete Appl. Math. 344 (2024) 1–9.

[15] A. Kemnitz, M. Marangio, N. Movarraei, On the chromatic edge stability number of graphs, Graphs Combin. 34 (2018) 1539–1551.

[16] M. Knor, M. Petruševski, R. Škrekovski, On chromatic vertex stability of 3-chromatic graphs with maximum degree 4, Discrete Math. Lett. 10 (2022) 75–80.

[17] H. Lei, X. Lian, X. Meng, Y. Shi, Y. Wang, On critical graphs for the chromatic edge-stability number, Discrete Math. 346 (2023) Paper 113307.

[18] W.T. Tutte, The factors of graphs, Canad. J. Math. 4 (1952) 314–328.

[19] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz 3 (1964) 25–30.