# On $\{1,2\}$-distance-balancedness of generalized Petersen graphs 

Gang Ma ${ }^{a, *}$, Jianfeng Wang ${ }^{a}$, Sandi Klavžar ${ }^{b, c, d}$<br>${ }^{a}$ School of Mathematics and Statistics, Shandong University of Technology, Zibo, China<br>${ }^{b}$ Faculty of Mathematics and Physics, University of Ljubljana, Slovenia<br>${ }^{c}$ Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia<br>${ }^{d}$ Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia


#### Abstract

A connected graph $G$ of diameter $\operatorname{diam}(G) \geq \ell$ is $\ell$-distance-balanced if $\left|W_{x y}\right|=\left|W_{y x}\right|$ for every $x, y \in V(G)$ with $d_{G}(x, y)=\ell$, where $W_{x y}$ is the set of vertices of $G$ that are closer to $x$ than to $y$. It is proved that if $k \geq 3$ and $n>k(k+2)$, then the generalized Petersen graph $G P(n, k)$ is not distancebalanced and that $G P(k(k+2), k)$ is distance-balanced. This significantly improves the main result of Yang et al. [Electron. J. Combin. 16 (2009) \#N33]. It is also proved that if $k \geq 6$, where $k$ is even, and $n>\frac{5}{4} k^{2}+2 k$, or if $k \geq 5$, where $k$ is odd, and $n>\frac{7}{4} k^{2}+\frac{3}{4} k$, then $G P(n, k)$ is not 2-distance-balanced. These results partially resolve a conjecture of Miklavič and Šparl [Discrete Appl. Math. 244 (2018) 143-154].


Keywords: Distance-balanced graph; $\ell$-distance-balanced graph; Generalized Petersen graph

AMS Subj. Class. (2020): 05C12

## 1 Introduction

If $G=(V(G), E(G))$ is a connected graph and $x, y \in V(G)$, then the distance, $d_{G}(x, y)$, between $x$ and $y$ is the number of edges on a shortest $x, y$-path. The
*Corresponding author.
Email addresses: math_magang@163.com (G. Ma), jfwang@sdut.edu.cn (J.F. Wang), sandi.klavzar@fmf.uni-lj.si (S. Klavžar).
diameter, $\operatorname{diam}(G)$, of $G$ is the maximum distance between its vertices. The set $W_{x y}$ contains the vertices that are closer to $x$ than to $y$, that is,

$$
W_{x y}=\left\{w \in V(G): d_{G}(w, x)<d_{G}(w, y)\right\} .
$$

Vertices $x$ and $y$ are balanced if $\left|W_{x y}\right|=\left|W_{y x}\right|$. For an integer $\ell \in[\operatorname{diam}(G)]=$ $\{1,2, \ldots, \operatorname{diam}(G)\}$, the graph $G$ is $\ell$-distance-balanced if each pair $x, y$ of its vertices with $d_{G}(x, y)=\ell$ is balanced.

1-distance-balanced were first considered by Handa [12] in 1999. The term "distance-balanced" for these graphs was proposed a decade later in [14]. This has prompted a widespread research into these graphs, see [1-8, 11, 13, 16-19, 22, 24, 26]. It was Frelih who in [9] extended distance-balanced graphs to $\ell$-distance balanced graphs. Also these graphs have already been investigated a lot, see [10, 15, 20, 21, 23].

If $n \geq 3$ and $1 \leq k<n / 2$, then the generalized Petersen $\operatorname{graph} \operatorname{GP}(n, k)$ is the graph with

$$
\begin{aligned}
& V(G P(n, k))=\left\{u_{i}: i \in \mathbb{Z}_{n}\right\} \cup\left\{v_{i}: i \in \mathbb{Z}_{n}\right\}, \\
& E(G P(n, k))=\left\{u_{i} u_{i+1}: i \in \mathbb{Z}_{n}\right\} \cup\left\{v_{i} v_{i+k}: i \in \mathbb{Z}_{n}\right\} \cup\left\{u_{i} v_{i}: i \in \mathbb{Z}_{n}\right\} .
\end{aligned}
$$

As it turned out, in general it is difficult to determine whether a generalized Petersen graphs is $\ell$-distance-balanced for some $\ell$. Back in the seminal paper [14, the following conjecture was proposed for the case $\ell=1$.

Conjecture 1. [14] For any $k \geq 2$, there exists a positive integer $n_{0}$ such that $G P(n, k)$ is not distance-balanced for every $n \geq n_{0}$.

The conjecture has been positively resolved by Yang et al. as follows.
Theorem 2. [26] If $k \geq 2$ and $n>6 k^{2}$, then $G P(n, k)$ is not distance-balanced.
Miklavič and Šparl [23] expanded and specified Conjecture 1 to $\ell$-distancebalancedness as follows.

Conjecture 3. [23] Let $k \geq 2$ be an integer and let

$$
n_{k}= \begin{cases}11 ; & k=2 \\ (k+1)^{2} ; & k \text { odd } \\ k(k+2) ; & k \geq 4 \text { even }\end{cases}
$$

Then $G P(n, k)$ is not $\ell$-distance-balanced for any $n>n_{k}$ and for any $1 \leq \ell<$ $\operatorname{diam}(G P(n, k))$. Moreover, $n_{k}$ is the smallest integer with this property.

Conjecture 3 has by now been confirmed for $k=2$ in [23] and for $k \in\{3,4\}$ in [21]. These results assert that if $k=2$ and $n>11$, or $k=3$ and $n>16$, or $k=4$ and $n>24$, then $G P(n, k)$ is not distance-balanced. These are significant improvements over the bound of Theorem 2 for $k \in\{2,3,4\}$. In the first main result of this paper we improve the bound of Theorem 2 for an arbitrary $k$, where the case $k=2$ is included for completeness.

Theorem 4. Let $n$ and $k$ be integers, where $2 \leq k<n / 2$.
(i) If $k \geq 3$ and $n>k(k+2)$, then $G P(n, k)$ is not distance-balanced. In addition, $G P(k(k+2), k)$ is distance-balanced.
(ii) If $k=2$ and $n>10$, then $G P(n, 2)$ is not distance-balanced. In addition, $G P(10,2)$ is distance-balanced.

In our second main result we deal with 2-distance-balancedness, where the cases $k \in\{2,3,4\}$ are included for completeness.

Theorem 5. Let $n$ and $k$ be integers, where $2 \leq k<n / 2$.
(i) If $k \geq 6$ and $k$ is even, then $G P(n, k)$ is not 2-distance-balanced for any $n>\frac{5}{4} k^{2}+2 k$.
(ii) If $k \geq 5$ and $k$ is odd, then $G P(n, k)$ is not 2-distance-balanced for any $n>$ $\frac{7}{4} k^{2}+\frac{3}{4} k$.
(iii) If $k=2$ and $n>10$, or $k=3$ and $n>10$, or $k=4$ and $n>21$, then $G P(n, k)$ is not 2-distance-balanced. In addition, $\operatorname{GP}(10,2), G P(10,3)$, and $G P(21,4)$ are 2-distance-balanced.

Proofs of Theorems 4 and 5 are respectively given in Sections 2 and 3 ,

## 2 Proof of Theorem 4

Let $x, y$ be vertices of a graph $G$. In addition to the already defined sets $W_{x y}$ and $W_{y x}$, let

$$
{ }_{x} W_{y}=\left\{w \in V(G): d_{G}(w, x)=d_{G}(w, y)\right\} .
$$

Clearly, $\left|W_{x y}\right|+\left|W_{y x}\right|+\left|{ }_{x} W_{y}\right|=|V(G)|$, which in turn implies the following simple, but useful fact.

Lemma 6. Let $x, y$ be vertices of a graph $G$ with $d_{G}(x, y)=\ell$, where $1 \leq \ell \leq$ $\operatorname{diam}(G)$. If $2\left|W_{x y}\right|+\left|{ }_{x} W_{y}\right|>|V(G)|$, then $G$ is not $\ell$-distance-balanced.

As already mentioned, Conjecture 3 holds true for $k=2$. Moreover, $\operatorname{GP}(11,2)$ is not distance-balanced, but $G P(10,2)$ is distance-balanced, see [23, Table 1]). These results cover the case $k=2$ of Theorem (4)

In the rest we assume that $k \geq 3$ and $n \geq k(k+2)$. We consider the vertices $u_{0}$ and $v_{0}$, and the corresponding sets $W_{u_{0} v_{0}}, W_{v_{0} u_{0}}$, and ${ }_{u_{0}} W_{v_{0}}$.
Case 1: $k$ even, $k \geq 4$. In this case we have

- $u_{i}, u_{-i} \in W_{u_{0} v_{0}}$ when $0 \leq i \leq \frac{k}{2}$; there are $2 \frac{k}{2}+1=k+1$ such vertices.
- $u_{i}, u_{-i} \in{ }_{u_{0}} W_{v_{0}}$ when $i=\frac{k+2}{2}$; there are two such vertices.
- $u_{i}, u_{-i} \in W_{v_{0} u_{0}}$ when $\frac{k+2}{2}<i \leq \frac{n}{2}$; there are $n-(k+3)$ such vertices.

Subcase 1.1: $n \bmod k=0$. In this subcase we get

- $v_{i k} \in W_{v_{0} u_{0}}$ when $0 \leq i \leq \frac{n}{k}-1$; there are $\frac{n}{k}$ such vertices.
- $\left\{v_{i}: 0 \leq i \leq n-1\right\} \|$ setminus $\left\{v_{i k}: 0 \leq i \leq \frac{n}{k}-1\right\} \subset W_{u_{0} v_{0}}$; there are $n-\frac{n}{k}$ such vertices.

From the above we obtain

$$
\begin{aligned}
\left|W_{v_{0} u_{0}}\right|-\left|W_{u_{0} v_{0}}\right| & =\left[n-(k+3)+\frac{n}{k}\right]-\left[(k+1)+\left(n-\frac{n}{k}\right)\right] \\
& =\frac{2 n}{k}-2 k-4 .
\end{aligned}
$$

If $n>k(k+2)$, then $\frac{2 n}{k}-2 k-4>0$ and hence $\left|W_{v_{0} u_{0}}\right|>\left|W_{u_{0} v_{0}}\right|$. We can conclude that $G P(n, k)$ is not distance-balanced if $n>k(k+2)$.

Assume now that $n=k(k+2)$. Then $\frac{2 n}{k}-2 k-4=0$ and hence $\left|W_{v_{0} u_{0}}\right|=\left|W_{u_{0} v_{0}}\right|$. Since any two adjacent vertices from the set $\left\{u_{i}: 0 \leq i \leq n-1\right\}$ as well as any two adjacent vertices from $\left\{v_{i}: 0 \leq i \leq n-1\right\}$ are symmetrical, we can conclude that $G P(k(k+2), k)$ is distance-balanced.

Subcase 1.2: $n \bmod k \neq 0$.
In this subcase we have $n \bmod 2 k \neq 0$. If $n>k(k+2)$, then

- $v_{i k}, v_{-i k} \in W_{v_{0} u_{0}}$ when $0 \leq i \leq\left\lfloor\frac{n}{2 k}\right\rfloor$; there are $2\left\lfloor\frac{n}{2 k}\right\rfloor+1$ such vertices.

Hence $\left|W_{v_{0} u_{0}}\right| \geq n-(k+3)+\left(2\left\lfloor\frac{n}{2 k}\right\rfloor+1\right)$ and $\left.\right|_{u_{0}} W_{v_{0}} \mid \geq 2$. From this, we can estimate as follows:

$$
\begin{aligned}
2\left|W_{v_{0} u_{0}}\right|+\left|{ }_{u_{0}} W_{v_{0}}\right| & \geq 2\left[n-(k+3)+\left(2\left\lfloor\frac{n}{2 k}\right\rfloor+1\right)\right]+2 \\
& =2 n+4\left\lfloor\frac{n}{2 k}\right\rfloor-2 k-2 \\
& \geq 2 n+4\left(\frac{k+2}{2}\right)-2 k-2 \\
& =2 n+2>2 n .
\end{aligned}
$$

Applying Lemma 6 we can conclude that $G P(n, k)$ is not distance-balanced.
Case 2: $k$ odd, $k \geq 3$. Now we obtain

- $u_{i}, u_{-i} \in W_{u_{0} v_{0}}$ when $0 \leq i \leq \frac{k+1}{2}$; there are $2\left(\frac{k+1}{2}\right)+1=k+2$ such vertices.
- $u_{i}, u_{-i} \in W_{v_{0} u_{0}}$ when $\frac{k+1}{2}<i \leq \frac{n}{2}$; there are $n-(k+2)$ such vertices.

Case 2.1: $n \bmod k=0$. In this subcase we have

- $v_{i k} \in W_{v_{0} u_{0}}$ when $0 \leq i \leq \frac{n}{k}-1$; there are $\frac{n}{k}$ such vertices.
- $\left\{v_{i}: 0 \leq i \leq n-1\right\} \backslash\left\{v_{i k}: 0 \leq i \leq \frac{n}{k}-1\right\} \subset W_{u_{0} v_{0}}$; there are $n-\frac{n}{k}$ such vertices.

By the above it follows that

$$
\begin{aligned}
\left|W_{v_{0} u_{0}}\right|-\left|W_{u_{0} v_{0}}\right| & =\left[n-(k+2)+\frac{n}{k}\right]-\left[(k+2)+\left(n-\frac{n}{k}\right)\right] \\
& =\frac{2 n}{k}-2 k-4 .
\end{aligned}
$$

If $n>k(k+2)$, then $\left|W_{v_{0} u_{0}}\right|-\left|W_{u_{0} v_{0}}\right|>0$ and $G P(n, k)$ is not distance-balanced. If $n=k(k+2)$, then $\left|W_{v_{0} u_{0}}\right|-\left|W_{u_{0} v_{0}}\right|=0$. Since any two adjacent vertices from $\left\{u_{i}: 0 \leq i \leq n-1\right\}$ as well as any two adjacent vertices from $\left\{v_{i}: 0 \leq i \leq n-1\right\}$ are symmetrical, we can deduce that $G P(k(k+2), k)$ is distance-balanced.

Case 2.2: $n \bmod k \neq 0$.
Now we have $n \bmod 2 k \neq 0$. Assume that $n>k(k+2)$. Then

- $v_{i k}, v_{-i k} \in W_{v_{0} u_{0}}$ when $0 \leq i \leq\left\lfloor\frac{n}{2 k}\right\rfloor+1$; there are $2\left(\left\lfloor\frac{n}{2 k}\right\rfloor+1\right)+1$ such vertices.

Having in mind that $k$ is odd, we have $\left\lfloor\frac{n}{2 k}\right\rfloor \geq \frac{k+1}{2}$. From here we can estimate as follows:

$$
\begin{aligned}
2\left|W_{v_{0} u_{0}}\right|+\left.\right|_{u_{0}} W_{v_{0}} \mid & \geq 2\left[n-(k+2)+\left(2\left\lfloor\frac{n}{2 k}\right\rfloor+3\right)\right]+0 \\
& =2 n+4\left\lfloor\frac{n}{2 k}\right\rfloor-2 k+2 \\
& \geq 2 n+4\left(\frac{k+1}{2}\right)-2 k+2 \\
& =2 n+4>2 n
\end{aligned}
$$

Using Lemma 6 once more we infer that also in this case $G P(n, k)$ is not distancebalanced. This completes the proof of Theorem 4.

## 3 Proof of Theorem 5

For the case $k=2$, Theorem 5 holds because Conjecture 3 is right for $k=2$ [23] and the fact that $G P(11,2)$ is not 2-distance-balanced, but $G P(10,2)$ is 2-distancebalanced (see Table 1 of [23]). For the case $k=3$, Theorem 5 holds because Conjecture 3 is right for $k=3$ [21] and the fact that $G P(n, 3)$ is not 2-distancebalanced when $11 \leq n \leq 16$, but $G P(10,3)$ is 2 -distance-balanced (see Table 1 of [23]). For the case $k=4$, Theorem 5 holds because Conjecture 3 is right for $k=4$ [21] and the fact that $\operatorname{GP}(n, 4)$ is not 2-distance-balanced when $22 \leq n \leq 24$, but $G P(21,4)$ is 2-distance-balanced (see Table 1 of [23]).

In the rest we assume that $k \geq 5$. Note that $d\left(u_{0}, v_{-k}\right)=2$ and $v_{-k}=v_{n-k}$. We will compute $\left|W_{v_{-k} u_{0}}\right|$ and $\left.\right|_{u_{0}} W_{v_{-k}} \mid$. Two cases are discussed according to the parity of $k$.
Case 1: $k$ is even, $k \geq 6$, and $n>\frac{5}{4} k^{2}+2 k$.
We distinguish three subcases which are separated according to which vertices are being addressed.
Subcase 1.1: Vertices $u_{-i}$ and $v_{-i}$, where $1 \leq i \leq k-1$.
Then $u_{-i} \in W_{u_{0} v_{-k}}$ and $v_{-i} \in W_{u_{0} v_{-k}}$ when if $1 \leq i \leq \frac{k}{2}$, and $u_{-i} \in W_{v_{-k} u_{0}}$ and $v_{-i} \in{ }_{u_{0}} W_{v_{-k}}$ when $\frac{k+2}{2} \leq i \leq k-1$. So, there are $\frac{k}{2}-1$ such vertices which are in $W_{v_{-k} u_{0}}$ and $\frac{k}{2}-1$ such vertices which are in ${ }_{u_{0}} W_{v_{-k}}$.
Subcase 1.2: Vertices $u_{i}$, where $0 \leq i \leq n-k$.
For $0 \leq i \leq k$ we have $u_{i} \in W_{u_{0} v_{-k}}$ when $0 \leq i \leq \frac{k}{2}+1$, and $u_{i} \in{ }_{u_{0}} W_{v_{-k}}$ when $\frac{k}{2}+2 \leq i \leq k$. Thus, there are $\frac{k}{2}-1$ such vertices which are in ${ }_{u_{0}} W_{v_{-k}}$.

For $k+1 \leq i \leq n-k$ we have $u_{i} \in{ }_{u_{0}} W_{v_{-k}}$ or $u_{i} \in W_{v_{-k} u_{0}}$. We first consider the vertices $u_{i}$ such that $u_{i} \in W_{v_{-k} u_{0}}$. Note that if $n-2 k<i \leq n-k$, then $u_{i} \in W_{v_{-k} u_{0}}$.

Let $t$ be the largest integer such that the maximum distance of a $v_{n-k}, u_{i}$-path is less than the minimum distance of a $u_{0}, u_{j}$-path, where $n-(t+1) k<i, j \leq n-t k$. That is, $t$ is the maximal integer such that

$$
\begin{gathered}
(t-1)+1+\frac{k}{2}<\left\lfloor\frac{n-t k}{k}\right\rfloor+2 \Longleftrightarrow \\
(t-1)+1+\frac{k}{2}<\left\lfloor\frac{n}{k}\right\rfloor-t+2 \Longleftrightarrow \\
t<\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor-\frac{k}{4}+1 .
\end{gathered}
$$

Because $t$ is the largest integer satisfying the above inequality, we get

$$
t \geq \frac{1}{2}\left(\frac{n}{k}-1\right)-\frac{k}{4}+1=\frac{n}{2 k}-\frac{k}{4}+\frac{1}{2}
$$

By the definition of $t$, if $1 \leq s \leq t$, then $u_{i} \in W_{v_{-k} u_{0}}$, where $n-(s+1) k<i \leq n-s k$. That is, $u_{i} \in W_{v_{-k} u_{0}}$ for any $n-(t+1) k<i \leq n-k$, and there are $k t \geq k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{2}\right)$ such vertices which are in $W_{v_{-k} u_{0}}$.

Note that if $1 \leq j \leq k$, then the difference of the distance of a $v_{n-k}, u_{n-(t+1) k+j^{-}}$ path, and the distance of a $v_{n-k}, u_{n-(t+2) k+j}$-path is -1 . So, among the vertices $u_{i}$, where $n-(t+2) k<i \leq n-(t+1) k$, there are at most two vertices which are not in $W_{v_{-k} u_{0}}$. That is, there are at least $k-2$ vertices among these which are in $W_{v_{-k} u_{0}}$. Using similar discussions we can get that the number of vertices $u_{i}$, where $k<i \leq n-(t+1) k$, which are in $W_{v_{-k} u_{0}}$, is at least

$$
(k-2)+(k-4)+\cdots+2=\frac{k(k-2)}{4} .
$$

Among the vertices $u_{i}$, where $0 \leq i \leq n-k$, there are at least $k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{2}\right)+\frac{k(k-2)}{4}$ vertices which are in $W_{v_{-k} u_{0}}$, and $n-\frac{3}{2} k-1-k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{2}\right)-\frac{k(k-2)}{4}$ vertices which are in ${ }_{u_{0}} W_{v_{-k}} \cup W_{v_{-k} u_{0}}$ and not counted in $W_{v_{-k} u_{0}}$.
Subcase 1.3: Vertices $v_{i}$, where $0 \leq i \leq n-k$.
Firstly, consider vertices $v_{s k}$ such that $v_{s k} \in{ }_{u_{0}} W_{v_{-k}}$. Note that $v_{0} \in{ }_{u_{0}} W_{v_{-k}}$. Let $t$ be the largest integer such that the maximum distance of a $u_{0}, v_{t k}$-path is less than or equal to the minimum distance of a $v_{n-k}, v_{t k}$-path. That is, $t$ is the largest integer such that

$$
t+1 \leq\left\lfloor\frac{n-k-t k}{k}\right\rfloor \Longleftrightarrow t+1 \leq\left\lfloor\frac{n}{k}\right\rfloor-1-t \Longleftrightarrow t \leq \frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor-1
$$

Because $t$ is the largest integer satisfying the above inequality, we get

$$
t>\frac{1}{2}\left(\frac{n}{k}-1\right)-1=\frac{n}{2 k}-\frac{3}{2} .
$$

By the definition of $t$ we have $v_{s k} \in{ }_{u_{0}} W_{v_{-k}}$ if $0 \leq s \leq t$. That is, there are $t+1>\frac{n}{2 k}-\frac{1}{2}$ such vertices which are in ${ }_{u_{0}} W_{v_{-k}}$.

Secondly, consider vertices $v_{n-k-s k}$, such that $v_{n-k-s k} \in W_{v_{-k} u_{0}}$. Note that $v_{n-k} \in W_{v_{-k} u_{0}}$. Let $t$ be the largest integer such that the maximum distance of a $v_{n-k}, v_{n-k-t k}$-path is less than the minimum distance of a $u_{0}, v_{n-k-t k}$-path. So $t$ is the largest integer such that

$$
t<\left\lfloor\frac{n-k-t k}{k}\right\rfloor+1 \Longleftrightarrow t<\left\lfloor\frac{n}{k}\right\rfloor-1-t+1 \Longleftrightarrow t<\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor .
$$

Because $t$ is the largest integer satisfying the above inequality, it can be concluded that

$$
t \geq \frac{1}{2}\left(\frac{n}{k}-1\right)=\frac{n}{2 k}-\frac{1}{2} .
$$

By the definition of $t$ we get that $v_{n-k-s k} \in W_{v_{-k} u_{0}}$ for $0 \leq s \leq t$. That is, there are $t+1 \geq \frac{n}{2 k}+\frac{1}{2}$ such vertices which are in $W_{v_{-k} u_{0}}$.

Thirdly, consider vertices $v_{i}$ with $0<i<n-k, i \neq s k$, and $i \neq n-k-s k$, such that $v_{i} \in{ }_{u_{0}} W_{v_{-k}}$. Note that $v_{i} \in{ }_{u_{0}} W_{v_{-k}}$ if $n-2 k<i<n-k$. Let $t$ be the largest integer such that the maximum distance of a $v_{n-k}, v_{i}$-path is less than or equal to the minimum distance of a $u_{0}, v_{j}$-path, where $n-(t+1) k<i, j \leq n-t k$. IN other words, $t$ is the largest integer such that

$$
\begin{gathered}
(t-1)+\frac{k}{2}+2 \leq\left\lfloor\frac{n-t k}{k}\right\rfloor+1 \Longleftrightarrow \\
(t-1)+\frac{k}{2}+2 \leq\left\lfloor\frac{n}{k}\right\rfloor-t+1 \Longleftrightarrow \\
t \leq \frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor-\frac{k}{4}
\end{gathered}
$$

Because $t$ is the largest integer satisfying the above inequality, we can conclude that

$$
t>\frac{1}{2}\left(\frac{n}{k}-1\right)-\frac{k}{4}=\frac{n}{2 k}-\frac{k}{4}-\frac{1}{2} .
$$

By the definition of $t$, if $1 \leq s \leq t$, then $v_{i} \in{ }_{u_{0}} W_{v_{-k}}$, where $n-(s+1) k<i<n-s k$. That is, there are $t(k-1)>\left(\frac{n}{2 k}-\frac{k}{4}-\frac{1}{2}\right)(k-1)$ such vertices which are in ${ }_{u_{0}} W_{v_{-k}}$.

If $1 \leq j<k$, then the difference between the distance of a $v_{n-k}, v_{n-(t+1) k+j}$-path and the distance of a $v_{n-k}, v_{n-(t+2) k+j}$-path is -1 . So among the vertices $v_{i}$ with
$n-(t+2) k<i<n-(t+1) k$, there are at most two vertices which are not in ${ }_{u_{0}} W_{v_{-k}}$. That is, there are at least $k-3$ vertices among the vertices $v_{i}$, where $n-(t+2) k<i<n-(t+1) k$, which are in ${ }_{u_{0}} W_{v_{-k}}$. Similarly we can get that the number of vertices $v_{i}(0<i<n-(t+1) k$, where $i \neq s k$ and $i \neq n-k-s k$, which are in ${ }_{u_{0}} W_{v_{-k}}$, is at least

$$
(k-3)+(k-5)+\cdots+1=\frac{(k-2)^{2}}{4} .
$$

Among the vertices $v_{i}$, where $0 \leq i \leq n-k$, there are at least $\frac{n}{2 k}+\frac{1}{2}$ vertices which are in $W_{v_{-k} u_{0}}$ and more than

$$
\left(\frac{n}{2 k}-\frac{1}{2}\right)+\left[\left(\frac{n}{2 k}-\frac{k}{4}-\frac{1}{2}\right)(k-1)+\frac{(k-2)^{2}}{4}\right]
$$

vertices which are in ${ }_{u} W_{v_{-k}}$.
Combining the above three subcases, we obtain that

$$
\begin{aligned}
\left|W_{v_{-k} u_{0}}\right| & \geq\left(\frac{k}{2}-1\right)+\left[k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{2}\right)+\frac{k(k-2)}{4}\right]+\left(\frac{n}{2 k}+\frac{1}{2}\right) \\
& =\frac{n}{2}+\frac{n}{2 k}+\frac{k}{2}-\frac{1}{2}
\end{aligned}
$$

which in turn implies that the number of vertices in ${ }_{u_{0}} W_{v_{-k}} \cup W_{v_{-k} u_{0}}$ which are not counted in $\left|W_{v_{-k} u_{0}}\right|$ is at least

$$
\begin{aligned}
\left(\frac{k}{2}-1\right)+ & {\left[n-\frac{3}{2} k-1-k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{2}\right)-\frac{k(k-2)}{4}\right] } \\
& +\left(\frac{n}{2 k}-\frac{1}{2}\right)+\left[\left(\frac{n}{2 k}-\frac{k}{4}-\frac{1}{2}\right)(k-1)+\frac{(k-2)^{2}}{4}\right] \\
= & n-\frac{9}{4} k-1 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2\left|W_{v_{-k} u_{0}}\right|+\left|{ }_{u_{0}} W_{v_{-k}}\right| & \geq 2\left(\frac{n}{2}+\frac{n}{2 k}+\frac{k}{2}-\frac{1}{2}\right)+\left(n-\frac{9}{4} k-1\right) \\
& =2 n+\frac{n}{k}-\frac{5}{4} k-2
\end{aligned}
$$

Since $n>\frac{5}{4} k^{2}+2 k$, we get $2\left|W_{v_{-k} u_{0}}\right|+\left|{ }_{u_{0}} W_{v_{-k}}\right|>2 n$. Lemma 6 yields that $G P(n, k)$ is not 2 -distance-balanced.

Case 2: $k$ is odd, $k \geq 5$, and $n>\frac{7}{4} k^{2}+\frac{3}{4} k$.
Just as in Case 1, we are going to distinguish three subcases separated according to which vertices are being addressed.
Subcase 2.1: Vertices $u_{-i}$ and $v_{-i}$, where $1 \leq i \leq k-1$.
If $1 \leq i<\frac{k+1}{2}$, then $u_{-i} \in W_{u_{0} v_{-k}}$ and $v_{-i} \in W_{u_{0} v_{-k}}$. If $i=\frac{k+1}{2}$, then $u_{-i} \in{ }_{u_{0}} W_{v_{-k}}$ and $v_{-i} \in{ }_{u_{0}} W_{v_{-k}}$, and thus there are two such vertices in ${ }_{u_{0}} W_{v_{-k}}$. If $\frac{k+1}{2}<i \leq k-1$, then $u_{-i} \in W_{v_{-k} u_{0}}$ and $v_{-i} \in{ }_{u_{0}} W_{v_{-k}}$. So, there are $\frac{k-3}{2}$ such vertices in $W_{v_{-k} u_{0}}$ and $\frac{k-3}{2}$ such vertices in ${ }_{u_{0}} W_{v_{-k}}$.
Subcase 2.2: Vertices $u_{i}$, where $0 \leq i \leq n-k$.
If $0 \leq i \leq k$, then $u_{i} \in W_{u_{0} v_{-k}}$ when $0 \leq i \leq \frac{k+1}{2}$, and $u_{i} \in{ }_{u_{0}} W_{v_{-k}}$ when $\frac{k+3}{2} \leq i \leq k$. Thus, there are $\frac{k-1}{2}$ such vertices which are in $u_{0} W_{v_{-k}}$.

If $k+1 \leq i \leq n-k$, then $u_{i} \in{ }_{u_{0}} W_{v_{-k}}$ or $u_{i} \in W_{v_{-k} u_{0}}$. We first consider the vertices $u_{i}$ such that $u_{i} \in W_{v_{-k} u_{0}}$. Note that if $n-2 k<i \leq n-k$, then $u_{i} \in W_{v_{-k} u_{0}}$. Let $t$ be the largest integer such that the maximum distance of a $v_{n-k}, u_{i}$-path is less than the minimum distance of a $u_{0}, u_{i}$-path, where $n-(t+1) k<i \leq n-t k$. In other words, $t$ is the largest integer such that

$$
\begin{gathered}
(t-1)+1+\frac{k+1}{2}<\left\lfloor\frac{n-t k}{k}\right\rfloor+2 \Longleftrightarrow \\
(t-1)+1+\frac{k+1}{2}<\left\lfloor\frac{n}{k}\right\rfloor-t+2 \Longleftrightarrow \\
t<\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor-\frac{k}{4}+\frac{3}{4} .
\end{gathered}
$$

Because $t$ is the largest integer satisfying the above inequality, we get

$$
t \geq \frac{1}{2}\left(\frac{n}{k}-1\right)-\frac{k}{4}+\frac{3}{4}=\frac{n}{2 k}-\frac{k}{4}+\frac{1}{4} .
$$

By the definition of $t$, if $1 \leq s \leq t$, then $u_{i} \in W_{v_{-k} u_{0}}$, where $n-(s+1) k<$ $i \leq n-s k$. That is, $u_{i} \in W_{v_{-k} u_{0}}$ for any $n-(t+1) k<i \leq n-k$, and there are $k t \geq k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{4}\right)$ such vertices which are in $W_{v_{-k} u_{0}}$.

If $1 \leq j \leq k$, then the difference between the distance of a $v_{n-k}, u_{n-(t+1) k+j}$-path and the distance of a $v_{n-k}, u_{n-(t+2) k+j}$-path is -1 . Hence, among the vertices $u_{i}$, where $n-(t+2) k<i \leq n-(t+1) k$, there are at most two vertices which are not in $W_{v_{-k} u_{0}}$. That is, there are at least $k-2$ vertices among these vertices which are in $W_{v_{-k} u_{0}}$. Similarly, the number of vertices $u_{i}$, where $k<i \leq n-(t+1) k$, which are in $W_{v_{-k} u_{0}}$, is at least

$$
(k-2)+(k-4)+\cdots+1=\frac{(k-1)^{2}}{4}
$$

Among the vertices $u_{i}$, where $0 \leq i \leq n-k$, there are at least $k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{4}\right)+\frac{(k-1)^{2}}{4}$ vertices which are in $W_{v_{-k} u_{0}}$, and

$$
n-\frac{3}{2} k-\frac{1}{2}-k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{4}\right)-\frac{(k-1)^{2}}{4}
$$

vertices which are in ${ }_{u_{0}} W_{v_{-k}} \cup W_{v_{-k} u_{0}}$ and not counted in $W_{v_{-k} u_{0}}$.
Subcase 2.3: Vertices $v_{i}$, where $0 \leq i \leq n-k$.
By a similar discussion as in Case 1.3 we obtain that $v_{s k} \in{ }_{u_{0}} W_{v_{-k}}$ if $0 \leq s \leq t$ $\left(t>\frac{n}{2 k}-\frac{3}{2}\right)$, and $v_{n-k-s k} \in W_{v_{-k} u_{0}}$ if $0 \leq s \leq t\left(t \geq \frac{n}{2 k}-\frac{1}{2}\right)$. That is, there are $t+1>\frac{n}{2 k}-\frac{1}{2}$ such vertices which are in ${ }_{u_{0}} W_{v_{-k}}$ and $t+1 \geq \frac{n}{2 k}+\frac{1}{2}$ such vertices which are in $W_{v_{-k} u_{0}}$.

We next consider vertices $v_{i}$, where $0<i<n-k, i \neq s k$, and $i \neq n-k-s k$, such that $v_{i} \in{ }_{u_{0}} W_{v_{-k}}$. If $n-2 k<i<n-k$, then $v_{i} \in{ }_{u_{0}} W_{v_{-k}}$. Let $t$ be the largest integer such that the maximum distance of a $v_{n-k}, v_{i}$-path is less than or equal to the minimum distance of a $u_{0}, v_{j}$-path, where $n-(t+1) k<i, j \leq n-t k$. That is, $t$ is the largest integer such that

$$
\begin{gathered}
(t-1)+\frac{k+1}{2}+2 \leq\left\lfloor\frac{n-t k}{k}\right\rfloor+1 \Longleftrightarrow \\
(t-1)+\frac{k+1}{2}+2 \leq\left\lfloor\frac{n}{k}\right\rfloor-t+1 \Longleftrightarrow \\
t \leq \frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor-\frac{k}{4}-\frac{1}{4}
\end{gathered}
$$

As $t$ is the largest integer satisfying the above inequality, we get

$$
t>\frac{1}{2}\left(\frac{n}{k}-1\right)-\frac{k}{4}-\frac{1}{4}=\frac{n}{2 k}-\frac{k}{4}-\frac{3}{4} .
$$

By the definition of $t$, if $1 \leq s \leq t$, then $v_{i} \in{ }_{u_{0}} W_{v_{-k}}$ where $n-(s+1) k<i<n-s k$. That is, there are $t(k-1)>\left(\frac{n}{2 k}-\frac{k}{4}-\frac{3}{4}\right)(k-1)$ such vertices which are in ${ }_{u_{0}} W_{v_{-k}}$.

If $1 \leq j<k$, then the difference between the distance of a $v_{n-k}, v_{n-(t+1) k+j}$-path and the distance of a $v_{n-k}, v_{n-(t+2) k}+j$-path is -1 . So among the vertices $v_{i}$, where $n-(t+2) k<i<n-(t+1) k$, there are at most two vertices which are not in $u_{0} W_{v_{-k}}$. Consequently, there are at least $k-3$ vertices $v_{i}$, where $n-(t+2) k<i<n-(t+1) k$, which are in $u_{0} W_{v_{-k}}$. Similarly, the number of vertices $v_{i}$, where $0<i<n-(t+1) k$, $i \neq s k$, and $i \neq n-k-s k$, which are in ${ }_{u_{0}} W_{v_{-k}}$, is at least

$$
(k-3)+(k-5)+\cdots+2=\frac{(k-3)(k-1)}{4} .
$$

Among the vertices $v_{i}$, where $0 \leq i \leq n-k$, there are at least $\frac{n}{2 k}+\frac{1}{2}$ vertices which are in $W_{v_{-k} u_{0}}$ and more than

$$
\left(\frac{n}{2 k}-\frac{1}{2}\right)+\left[\left(\frac{n}{2 k}-\frac{k}{4}-\frac{3}{4}\right)(k-1)+\frac{(k-3)(k-1)}{4}\right]
$$

vertices which are in ${ }_{u_{0}} W_{v_{-k}}$.
Combining the above three subcases, we obtain that

$$
\begin{aligned}
\left|W_{v_{-k} u_{0}}\right| & \geq \frac{k-3}{2}+\left[k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{4}\right)+\frac{(k-1)^{2}}{4}\right]+\left(\frac{n}{2 k}+\frac{1}{2}\right) \\
& =\frac{n}{2}+\frac{n}{2 k}+\frac{k}{4}-\frac{3}{4} .
\end{aligned}
$$

Consequently, the number of vertices in ${ }_{u_{0}} W_{v_{-k}} \cup W_{v_{-k} u_{0}}$ which are not counted in $\left|W_{v_{-k} u_{0}}\right|$ is at least

$$
\begin{aligned}
& \frac{k+1}{2}+\left[n-\frac{3}{2} k-\frac{1}{2}-k\left(\frac{n}{2 k}-\frac{k}{4}+\frac{1}{4}\right)-\frac{(k-1)^{2}}{4}\right]+ \\
& \left(\frac{n}{2 k}-\frac{1}{2}\right)+\left[\left(\frac{n}{2 k}-\frac{k}{4}-\frac{3}{4}\right)(k-1)+\frac{(k-3)(k-1)}{4}\right] \\
= & n-\frac{9}{4} k+\frac{3}{4} .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
2\left|W_{v_{-k} u_{0}}\right|+\left|u_{u_{0}} W_{v_{-k}}\right| & \geq 2\left(\frac{n}{2}+\frac{n}{2 k}+\frac{k}{4}-\frac{3}{4}\right)+\left(n-\frac{9}{4} k+\frac{3}{4}\right) \\
& =2 n+\frac{n}{k}-\frac{7}{4} k-\frac{3}{4} .
\end{aligned}
$$

Under the assumption $n>\frac{7}{4} k^{2}+\frac{3}{4} k$ we get $2\left|W_{v_{-k} u_{0}}\right|+\left|{ }_{u_{0}} W_{v_{-k}}\right|>2 n$, hence Lemma 6 yields that $\operatorname{GP}(n, k)$ is not 2-distance-balanced.

## Acknowledgments

This work was supported by Shandong Provincial Natural Science Foundation of China (ZR2022MA077), the research grant NSFC (12371353) of China and IC Program of Shandong Institutions of Higher Learning For Youth Innovative Talents. Sand Klavžar was supported was supported by the Slovenian Research Agency ARIS (research core funding P1-0297 and projects N1-0285, N1-0218).

## References

[1] A. Abiad, B. Brimkov, A. Erey, L Leshock, X. Martínez-Rivera, S. O, S.-Y. Song, J. Williford, On the Wiener index, distance cospectrality and transmission-regular graphs, Discrete Appl. Math. 230 (2017) 1-10.
[2] A. Ali, T. Došlić, Mostar index: Results and perspectives, Appl. Math. Comput. 404 (2021) 126245.
[3] K. Balakrishnan, B. Brešar, M. Changat, S. Klavžar, A. Vesel, P. Žigert Pleteršek, Equal opportunity networks, distance-balanced graphs, and Wiener game, Discrete Opt. 12 (2014) 150-154.
[4] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl, A.R. Subhamathi, Strongly distance-balanced graphs and graph products, European J. Combin. 30 (2009) 1048-1053.
[5] S. Cabello, P. Lukšič, The complexity of obtaining a distance-balanced graph, Electron. J. Combin. 18 (2011) Paper 49.
[6] M. Cavaleri, A. Donno, Distance-balanced graphs and travelling salesman problems, Ars Math. Contemp. 19 (2020) 311-324.
[7] T. Došlić, I. Martinjak, R. Škrekovski, S.Tipurić Spužević, I. Zubac, Mostar index, J. Math. Chem. 56 (2018) 2995-3013.
[8] B. Fernández, A. Hujdurović, On some problems regarding distance-balanced graphs, European J. Combin. 106 (2022) 103593.
[9] B. Frelih, Različni vidiki povezavne regularnosti v grafih, Ph.D. (in Slovene), University of Primorska, 2014.
[10] B. Frelih, Š. Miklavič, On 2-distance-balanced graphs, Ars Math. Contemp. 15 (2018) 81-95.
[11] M. Ghorbani, Z. Vaziri, Graphs with small distance-based complexities, Appl. Math. Comput. 457 (2023), Paper 128188.
[12] K. Handa, Bipartite graphs with balanced ( $a, b$ )-partitions, Ars Combin. 51 (1999) 113-119.
[13] A. Ilić, S. Klavžar, M. Milanović, On distance-balanced graphs. European J. Combin. 31(2010) 733-737.
[14] J. Jerebic, S. Klavžar, D.F. Rall, Distance-balanced graphs, Ann. Combin. 12 (2008) 71-79.
[15] J. Jerebic, S. Klavžar, G. Rus, On $\ell$-distance-balanced product graphs, Graphs Combin. 37 (2021) 369-379.
[16] M. Kramer, D. Rautenbach, Minimum distance-unbalancedness of trees, J. Math. Chem. 59 (2021) 942-950.
[17] K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, Distance-balanced graphs: symmetry conditions, Discrete Math. 306 (2006) 1881-1894.
[18] K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, The strongly distance-balanced property of the generalized Petersen graphs, Ars Math. Contemp. 2 (2009) 4147.
[19] K. Kutnar, Š. Miklavič, Nicely distance-balanced graphs, European J. Combin. 39 (2014) 57-67.
[20] G. Ma, J. Wang, S. Klavžar, On distance-balanced generalized Petersen graphs. Ann. Comb. 28 (2024) 329-349.
[21] G. Ma, J. Wang, S. Klavžar, Non- $\ell$-distance-balanced generalized Petersen graphs $G P(n, 3)$ and $G P(n, 4)$, arXiv:2309.01900 [math.CO] (5 Sep 2023) 32 pp.
[22] Š. Miklavič, P. Šparl, On the connectivity of bipartite distance-balanced graphs, European J. Combin. 33 (2012) 237-247.
[23] Š. Miklavič, P. Šparl, $\ell$-distance-balanced graphs, Discrete Appl. Math. 244 (2018) 143-154.
[24] Š. Miklavič, P. Šparl, Distance-unbalancedness of graphs, Appl. Math. Comput. 405 (2021) 126233.
[25] K. Xu, P. Yao, Minimum distance-unbalancedness of graphs with diameter 2 and given number of edges, Discrete Math. Lett. 9 (2022) 26-30.
[26] R. Yang, X. Hou, N. Li, W. Zhong, A note on the distance-balanced property of generalized Petersen graphs, Electron. J. Combin. 16 (2009) \#N33.

