On $\{1, 2\}$ -distance-balancedness of generalized Petersen graphs

Gang Ma^{*a*,*}, Jianfeng Wang^{*a*}, Sandi Klavžar^{*b*,*c*,*d*}

^aSchool of Mathematics and Statistics, Shandong University of Technology, Zibo, China

^bFaculty of Mathematics and Physics, University of Ljubljana, Slovenia

^cFaculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

^dInstitute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Abstract

A connected graph G of diameter diam $(G) \geq \ell$ is ℓ -distance-balanced if $|W_{xy}| = |W_{yx}|$ for every $x, y \in V(G)$ with $d_G(x, y) = \ell$, where W_{xy} is the set of vertices of G that are closer to x than to y. It is proved that if $k \geq 3$ and n > k(k+2), then the generalized Petersen graph GP(n,k) is not distance-balanced and that GP(k(k+2),k) is distance-balanced. This significantly improves the main result of Yang et al. [Electron. J. Combin. 16 (2009) #N33]. It is also proved that if $k \geq 6$, where k is even, and $n > \frac{5}{4}k^2 + 2k$, or if $k \geq 5$, where k is odd, and $n > \frac{7}{4}k^2 + \frac{3}{4}k$, then GP(n,k) is not 2-distance-balanced. These results partially resolve a conjecture of Miklavič and Šparl [Discrete Appl. Math. 244 (2018) 143–154].

Keywords: Distance-balanced graph; ℓ -distance-balanced graph; Generalized Petersen graph

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1 Introduction

If G = (V(G), E(G)) is a connected graph and $x, y \in V(G)$, then the distance, $d_G(x, y)$, between x and y is the number of edges on a shortest x, y-path. The

^{*}Corresponding author.

Email addresses: math_magang@163.com (G. Ma), jfwang@sdut.edu.cn (J.F. Wang), sandi.klavzar@fmf.uni-lj.si (S. Klavžar).

diameter, diam(G), of G is the maximum distance between its vertices. The set W_{xy} contains the vertices that are closer to x than to y, that is,

$$W_{xy} = \{ w \in V(G) : d_G(w, x) < d_G(w, y) \}.$$

Vertices x and y are balanced if $|W_{xy}| = |W_{yx}|$. For an integer $\ell \in [\operatorname{diam}(G)] = \{1, 2, \dots, \operatorname{diam}(G)\}$, the graph G is ℓ -distance-balanced if each pair x, y of its vertices with $d_G(x, y) = \ell$ is balanced.

1-distance-balanced were first considered by Handa [12] in 1999. The term "distance-balanced" for these graphs was proposed a decade later in [14]. This has prompted a widespread research into these graphs, see [1–8, 11, 13, 16–19, 22, 24–26]. It was Frelih who in [9] extended distance-balanced graphs to ℓ -distance balanced graphs. Also these graphs have already been investigated a lot, see [10, 15, 20, 21, 23].

If $n \geq 3$ and $1 \leq k < n/2$, then the generalized Petersen graph GP(n,k) is the graph with

$$V(GP(n,k)) = \{u_i : i \in \mathbb{Z}_n\} \cup \{v_i : i \in \mathbb{Z}_n\},\ E(GP(n,k)) = \{u_i u_{i+1} : i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} : i \in \mathbb{Z}_n\} \cup \{u_i v_i : i \in \mathbb{Z}_n\}.$$

As it turned out, in general it is difficult to determine whether a generalized Petersen graphs is ℓ -distance-balanced for some ℓ . Back in the seminal paper [14], the following conjecture was proposed for the case $\ell = 1$.

Conjecture 1. [14] For any $k \ge 2$, there exists a positive integer n_0 such that GP(n,k) is not distance-balanced for every $n \ge n_0$.

The conjecture has been positively resolved by Yang et al. as follows.

Theorem 2. [26] If $k \ge 2$ and $n > 6k^2$, then GP(n, k) is not distance-balanced.

Miklavič and Sparl [23] expanded and specified Conjecture 1 to ℓ -distancebalancedness as follows.

Conjecture 3. [23] Let $k \ge 2$ be an integer and let

$$n_k = \begin{cases} 11; & k = 2, \\ (k+1)^2; & k \text{ odd}, \\ k(k+2); & k \ge 4 \text{ even.} \end{cases}$$

Then GP(n,k) is not ℓ -distance-balanced for any $n > n_k$ and for any $1 \leq \ell < \text{diam}(GP(n,k))$. Moreover, n_k is the smallest integer with this property.

Conjecture 3 has by now been confirmed for k = 2 in [23] and for $k \in \{3, 4\}$ in [21]. These results assert that if k = 2 and n > 11, or k = 3 and n > 16, or k = 4 and n > 24, then GP(n, k) is not distance-balanced. These are significant improvements over the bound of Theorem 2 for $k \in \{2, 3, 4\}$. In the first main result of this paper we improve the bound of Theorem 2 for an arbitrary k, where the case k = 2 is included for completeness.

Theorem 4. Let n and k be integers, where $2 \le k < n/2$.

- (i) If $k \ge 3$ and n > k(k+2), then GP(n,k) is not distance-balanced. In addition, GP(k(k+2),k) is distance-balanced.
- (ii) If k = 2 and n > 10, then GP(n, 2) is not distance-balanced. In addition, GP(10, 2) is distance-balanced.

In our second main result we deal with 2-distance-balancedness, where the cases $k \in \{2, 3, 4\}$ are included for completeness.

Theorem 5. Let n and k be integers, where $2 \le k < n/2$.

- (i) If $k \ge 6$ and k is even, then GP(n,k) is not 2-distance-balanced for any $n > \frac{5}{4}k^2 + 2k$.
- (ii) If $k \ge 5$ and k is odd, then GP(n,k) is not 2-distance-balanced for any $n > \frac{7}{4}k^2 + \frac{3}{4}k$.
- (iii) If k = 2 and n > 10, or k = 3 and n > 10, or k = 4 and n > 21, then GP(n,k) is not 2-distance-balanced. In addition, GP(10,2), GP(10,3), and GP(21,4) are 2-distance-balanced.

Proofs of Theorems 4 and 5 are respectively given in Sections 2 and 3.

2 Proof of Theorem 4

Let x, y be vertices of a graph G. In addition to the already defined sets W_{xy} and W_{yx} , let

$$_{x}W_{y} = \{w \in V(G): d_{G}(w, x) = d_{G}(w, y)\}.$$

Clearly, $|W_{xy}| + |W_{yx}| + |_xW_y| = |V(G)|$, which in turn implies the following simple, but useful fact.

Lemma 6. Let x, y be vertices of a graph G with $d_G(x, y) = \ell$, where $1 \leq \ell \leq \text{diam}(G)$. If $2|W_{xy}| + |_xW_y| > |V(G)|$, then G is not ℓ -distance-balanced.

As already mentioned, Conjecture 3 holds true for k = 2. Moreover, GP(11, 2) is not distance-balanced, but GP(10, 2) is distance-balanced, see [23, Table 1]). These results cover the case k = 2 of Theorem 4.

In the rest we assume that $k \geq 3$ and $n \geq k(k+2)$. We consider the vertices u_0 and v_0 , and the corresponding sets $W_{u_0v_0}$, $W_{v_0u_0}$, and $u_0W_{v_0}$.

Case 1: k even, $k \ge 4$. In this case we have

- $u_i, u_{-i} \in W_{u_0v_0}$ when $0 \le i \le \frac{k}{2}$; there are $2\frac{k}{2} + 1 = k + 1$ such vertices.
- $u_i, u_{-i} \in u_0 W_{v_0}$ when $i = \frac{k+2}{2}$; there are two such vertices.
- $u_i, u_{-i} \in W_{v_0 u_0}$ when $\frac{k+2}{2} < i \leq \frac{n}{2}$; there are n (k+3) such vertices.

Subcase 1.1: $n \mod k = 0$. In this subcase we get

- $v_{ik} \in W_{v_0u_0}$ when $0 \le i \le \frac{n}{k} 1$; there are $\frac{n}{k}$ such vertices.
- $\{v_i: 0 \le i \le n-1\}$ || set $v_{ik}: 0 \le i \le \frac{n}{k}-1\} \subset W_{u_0v_0}$; there are $n-\frac{n}{k}$ such vertices.

From the above we obtain

$$|W_{v_0u_0}| - |W_{u_0v_0}| = \left[n - (k+3) + \frac{n}{k}\right] - \left[(k+1) + (n - \frac{n}{k})\right]$$
$$= \frac{2n}{k} - 2k - 4.$$

If n > k(k+2), then $\frac{2n}{k} - 2k - 4 > 0$ and hence $|W_{v_0u_0}| > |W_{u_0v_0}|$. We can conclude that GP(n,k) is not distance-balanced if n > k(k+2).

Assume now that n = k(k+2). Then $\frac{2n}{k} - 2k - 4 = 0$ and hence $|W_{v_0u_0}| = |W_{u_0v_0}|$. Since any two adjacent vertices from the set $\{u_i: 0 \le i \le n-1\}$ as well as any two adjacent vertices from $\{v_i: 0 \le i \le n-1\}$ are symmetrical, we can conclude that GP(k(k+2), k) is distance-balanced.

Subcase 1.2: $n \mod k \neq 0$.

In this subcase we have $n \mod 2k \neq 0$. If n > k(k+2), then

• $v_{ik}, v_{-ik} \in W_{v_0u_0}$ when $0 \le i \le \lfloor \frac{n}{2k} \rfloor$; there are $2\lfloor \frac{n}{2k} \rfloor + 1$ such vertices.

Hence $|W_{v_0u_0}| \ge n - (k+3) + (2\lfloor \frac{n}{2k} \rfloor + 1)$ and $|u_0W_{v_0}| \ge 2$. From this, we can estimate as follows:

$$2|W_{v_0u_0}| + |_{u_0}W_{v_0}| \ge 2\left[n - (k+3) + (2\left\lfloor\frac{n}{2k}\right\rfloor + 1)\right] + 2$$

= $2n + 4\left\lfloor\frac{n}{2k}\right\rfloor - 2k - 2$
 $\ge 2n + 4\left(\frac{k+2}{2}\right) - 2k - 2$
= $2n + 2 > 2n$.

Applying Lemma 6 we can conclude that GP(n, k) is not distance-balanced.

Case 2: k odd, $k \ge 3$. Now we obtain

- $u_i, u_{-i} \in W_{u_0v_0}$ when $0 \le i \le \frac{k+1}{2}$; there are $2(\frac{k+1}{2}) + 1 = k + 2$ such vertices.
- $u_i, u_{-i} \in W_{v_0 u_0}$ when $\frac{k+1}{2} < i \leq \frac{n}{2}$; there are n (k+2) such vertices.

Case 2.1: $n \mod k = 0$. In this subcase we have

- $v_{ik} \in W_{v_0u_0}$ when $0 \le i \le \frac{n}{k} 1$; there are $\frac{n}{k}$ such vertices.
- $\{v_i: 0 \le i \le n-1\} \setminus \{v_{ik}: 0 \le i \le \frac{n}{k}-1\} \subset W_{u_0v_0}$; there are $n-\frac{n}{k}$ such vertices.

By the above it follows that

$$|W_{v_0u_0}| - |W_{u_0v_0}| = \left[n - (k+2) + \frac{n}{k}\right] - \left[(k+2) + (n - \frac{n}{k})\right]$$
$$= \frac{2n}{k} - 2k - 4.$$

If n > k(k+2), then $|W_{v_0u_0}| - |W_{u_0v_0}| > 0$ and GP(n,k) is not distance-balanced. If n = k(k+2), then $|W_{v_0u_0}| - |W_{u_0v_0}| = 0$. Since any two adjacent vertices from $\{u_i: 0 \le i \le n-1\}$ as well as any two adjacent vertices from $\{v_i: 0 \le i \le n-1\}$ are symmetrical, we can deduce that GP(k(k+2), k) is distance-balanced.

Case 2.2: $n \mod k \neq 0$. Now we have $n \mod 2k \neq 0$. Assume that n > k(k+2). Then

• $v_{ik}, v_{-ik} \in W_{v_0 u_0}$ when $0 \le i \le \lfloor \frac{n}{2k} \rfloor + 1$; there are $2(\lfloor \frac{n}{2k} \rfloor + 1) + 1$ such vertices.

Having in mind that k is odd, we have $\lfloor \frac{n}{2k} \rfloor \geq \frac{k+1}{2}$. From here we can estimate as follows:

$$2|W_{v_0u_0}| + |_{u_0}W_{v_0}| \ge 2\left[n - (k+2) + (2\left\lfloor\frac{n}{2k}\right\rfloor + 3)\right] + 0$$

= $2n + 4\left\lfloor\frac{n}{2k}\right\rfloor - 2k + 2$
 $\ge 2n + 4\left(\frac{k+1}{2}\right) - 2k + 2$
= $2n + 4 > 2n.$

Using Lemma 6 once more we infer that also in this case GP(n, k) is not distancebalanced. This completes the proof of Theorem 4.

3 Proof of Theorem 5

For the case k = 2, Theorem 5 holds because Conjecture 3 is right for k = 2 [23] and the fact that GP(11, 2) is not 2-distance-balanced, but GP(10, 2) is 2-distancebalanced (see Table 1 of [23]). For the case k = 3, Theorem 5 holds because Conjecture 3 is right for k = 3 [21] and the fact that GP(n, 3) is not 2-distancebalanced when $11 \le n \le 16$, but GP(10, 3) is 2-distance-balanced (see Table 1 of [23]). For the case k = 4, Theorem 5 holds because Conjecture 3 is right for k = 4 [21] and the fact that GP(n, 4) is not 2-distance-balanced when $22 \le n \le 24$, but GP(21, 4) is 2-distance-balanced (see Table 1 of [23]).

In the rest we assume that $k \geq 5$. Note that $d(u_0, v_{-k}) = 2$ and $v_{-k} = v_{n-k}$. We will compute $|W_{v_{-k}u_0}|$ and $|_{u_0}W_{v_{-k}}|$. Two cases are discussed according to the parity of k.

Case 1: k is even, $k \ge 6$, and $n > \frac{5}{4}k^2 + 2k$.

We distinguish three subcases which are separated according to which vertices are being addressed.

Subcase 1.1: Vertices u_{-i} and v_{-i} , where $1 \le i \le k - 1$.

Then $u_{-i} \in W_{u_0v_{-k}}$ and $v_{-i} \in W_{u_0v_{-k}}$ when if $1 \leq i \leq \frac{k}{2}$, and $u_{-i} \in W_{v_{-k}u_0}$ and $v_{-i} \in u_0 W_{v_{-k}}$ when $\frac{k+2}{2} \leq i \leq k-1$. So, there are $\frac{k}{2}-1$ such vertices which are in $W_{v_{-k}u_0}$ and $\frac{k}{2}-1$ such vertices which are in $u_0 W_{v_{-k}}$.

Subcase 1.2: Vertices u_i , where $0 \le i \le n-k$. For $0 \le i \le k$ we have $u_i \in W_{u_0v_{-k}}$ when $0 \le i \le \frac{k}{2} + 1$, and $u_i \in u_0 W_{v_{-k}}$ when $\frac{k}{2} + 2 \le i \le k$. Thus, there are $\frac{k}{2} - 1$ such vertices which are in $u_0 W_{v_{-k}}$.

For $k+1 \leq i \leq n-k$ we have $u_i \in {}_{u_0}W_{v_{-k}}$ or $u_i \in W_{v_{-k}u_0}$. We first consider the vertices u_i such that $u_i \in W_{v_{-k}u_0}$. Note that if $n-2k < i \leq n-k$, then $u_i \in W_{v_{-k}u_0}$.

Let t be the largest integer such that the maximum distance of a v_{n-k} , u_i -path is less than the minimum distance of a u_0 , u_j -path, where $n - (t+1)k < i, j \le n - tk$. That is, t is the maximal integer such that

$$\begin{aligned} (t-1)+1+\frac{k}{2} < \left\lfloor \frac{n-tk}{k} \right\rfloor + 2 \iff \\ (t-1)+1+\frac{k}{2} < \left\lfloor \frac{n}{k} \right\rfloor - t + 2 \iff \\ t < \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor - \frac{k}{4} + 1. \end{aligned}$$

Because t is the largest integer satisfying the above inequality, we get

$$t \ge \frac{1}{2}\left(\frac{n}{k} - 1\right) - \frac{k}{4} + 1 = \frac{n}{2k} - \frac{k}{4} + \frac{1}{2}$$

By the definition of t, if $1 \leq s \leq t$, then $u_i \in W_{v_{-k}u_0}$, where $n - (s+1)k < i \leq n - sk$. That is, $u_i \in W_{v_{-k}u_0}$ for any $n - (t+1)k < i \leq n-k$, and there are $kt \geq k(\frac{n}{2k} - \frac{k}{4} + \frac{1}{2})$ such vertices which are in $W_{v_{-k}u_0}$.

Note that if $1 \leq j \leq k$, then the difference of the distance of a $v_{n-k}, u_{n-(t+1)k+j}$ path, and the distance of a $v_{n-k}, u_{n-(t+2)k+j}$ -path is -1. So, among the vertices u_i , where $n - (t+2)k < i \leq n - (t+1)k$, there are at most two vertices which are not in $W_{v_{-k}u_0}$. That is, there are at least k-2 vertices among these which are in $W_{v_{-k}u_0}$. Using similar discussions we can get that the number of vertices u_i , where $k < i \leq n - (t+1)k$, which are in $W_{v_{-k}u_0}$, is at least

$$(k-2) + (k-4) + \dots + 2 = \frac{k(k-2)}{4}.$$

Among the vertices u_i , where $0 \le i \le n-k$, there are at least $k(\frac{n}{2k} - \frac{k}{4} + \frac{1}{2}) + \frac{k(k-2)}{4}$ vertices which are in $W_{v_{-k}u_0}$, and $n - \frac{3}{2}k - 1 - k(\frac{n}{2k} - \frac{k}{4} + \frac{1}{2}) - \frac{k(k-2)}{4}$ vertices which are in $u_0 W_{v_{-k}} \cup W_{v_{-k}u_0}$ and not counted in $W_{v_{-k}u_0}$.

Subcase 1.3: Vertices v_i , where $0 \le i \le n - k$.

Firstly, consider vertices v_{sk} such that $v_{sk} \in {}_{u_0}W_{v_{-k}}$. Note that $v_0 \in {}_{u_0}W_{v_{-k}}$. Let t be the largest integer such that the maximum distance of a u_0, v_{tk} -path is less than or equal to the minimum distance of a v_{n-k}, v_{tk} -path. That is, t is the largest integer such that

$$t+1 \le \left\lfloor \frac{n-k-tk}{k} \right\rfloor \iff t+1 \le \left\lfloor \frac{n}{k} \right\rfloor - 1 - t \iff t \le \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor - 1.$$

Because t is the largest integer satisfying the above inequality, we get

$$t > \frac{1}{2}\left(\frac{n}{k} - 1\right) - 1 = \frac{n}{2k} - \frac{3}{2}.$$

By the definition of t we have $v_{sk} \in {}_{u_0}W_{v_{-k}}$ if $0 \leq s \leq t$. That is, there are $t+1 > \frac{n}{2k} - \frac{1}{2}$ such vertices which are in ${}_{u_0}W_{v_{-k}}$.

Secondly, consider vertices v_{n-k-sk} , such that $v_{n-k-sk} \in W_{v_{-k}u_0}$. Note that $v_{n-k} \in W_{v_{-k}u_0}$. Let t be the largest integer such that the maximum distance of a v_{n-k}, v_{n-k-tk} -path is less than the minimum distance of a u_0, v_{n-k-tk} -path. So t is the largest integer such that

$$t < \left\lfloor \frac{n-k-tk}{k} \right\rfloor + 1 \iff t < \left\lfloor \frac{n}{k} \right\rfloor - 1 - t + 1 \iff t < \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor.$$

Because t is the largest integer satisfying the above inequality, it can be concluded that

$$t \ge \frac{1}{2}\left(\frac{n}{k} - 1\right) = \frac{n}{2k} - \frac{1}{2}$$

By the definition of t we get that $v_{n-k-sk} \in W_{v_{-k}u_0}$ for $0 \le s \le t$. That is, there are $t+1 \ge \frac{n}{2k} + \frac{1}{2}$ such vertices which are in $W_{v_{-k}u_0}$.

Thirdly, consider vertices v_i with 0 < i < n-k, $i \neq sk$, and $i \neq n-k-sk$, such that $v_i \in u_0 W_{v_{-k}}$. Note that $v_i \in u_0 W_{v_{-k}}$ if n-2k < i < n-k. Let t be the largest integer such that the maximum distance of a v_{n-k} , v_i -path is less than or equal to the minimum distance of a u_0 , v_j -path, where $n - (t+1)k < i, j \leq n-tk$. IN other words, t is the largest integer such that

$$(t-1) + \frac{k}{2} + 2 \le \left\lfloor \frac{n-tk}{k} \right\rfloor + 1 \iff (t-1) + \frac{k}{2} + 2 \le \left\lfloor \frac{n}{k} \right\rfloor - t + 1 \iff t \le \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor - \frac{k}{4}.$$

Because t is the largest integer satisfying the above inequality, we can conclude that

$$t > \frac{1}{2} \left(\frac{n}{k} - 1 \right) - \frac{k}{4} = \frac{n}{2k} - \frac{k}{4} - \frac{1}{2}$$

By the definition of t, if $1 \le s \le t$, then $v_i \in {}_{u_0}W_{v_{-k}}$, where n - (s+1)k < i < n-sk. That is, there are $t(k-1) > (\frac{n}{2k} - \frac{k}{4} - \frac{1}{2})(k-1)$ such vertices which are in ${}_{u_0}W_{v_{-k}}$.

If $1 \leq j < k$, then the difference between the distance of a $v_{n-k}, v_{n-(t+1)k+j}$ -path and the distance of a $v_{n-k}, v_{n-(t+2)k+j}$ -path is -1. So among the vertices v_i with n - (t+2)k < i < n - (t+1)k, there are at most two vertices which are not in $_{u_0}W_{v_{-k}}$. That is, there are at least k-3 vertices among the vertices v_i , where n - (t+2)k < i < n - (t+1)k, which are in $_{u_0}W_{v_{-k}}$. Similarly we can get that the number of vertices v_i (0 < i < n - (t+1)k, where $i \neq sk$ and $i \neq n - k - sk$, which are in $_{u_0}W_{v_{-k}}$, is at least

$$(k-3) + (k-5) + \dots + 1 = \frac{(k-2)^2}{4}$$

Among the vertices v_i , where $0 \le i \le n-k$, there are at least $\frac{n}{2k} + \frac{1}{2}$ vertices which are in $W_{v_{-k}u_0}$ and more than

$$\left(\frac{n}{2k} - \frac{1}{2}\right) + \left[\left(\frac{n}{2k} - \frac{k}{4} - \frac{1}{2}\right)(k-1) + \frac{(k-2)^2}{4}\right]$$

vertices which are in $_{u_0}W_{v_{-k}}$.

Combining the above three subcases, we obtain that

$$|W_{v_{-k}u_0}| \ge \left(\frac{k}{2} - 1\right) + \left[k\left(\frac{n}{2k} - \frac{k}{4} + \frac{1}{2}\right) + \frac{k(k-2)}{4}\right] + \left(\frac{n}{2k} + \frac{1}{2}\right)$$
$$= \frac{n}{2} + \frac{n}{2k} + \frac{k}{2} - \frac{1}{2},$$

which in turn implies that the number of vertices in $_{u_0}W_{v_{-k}} \cup W_{v_{-k}u_0}$ which are not counted in $|W_{v_{-k}u_0}|$ is at least

$$\begin{pmatrix} \frac{k}{2} - 1 \end{pmatrix} + \left[n - \frac{3}{2}k - 1 - k\left(\frac{n}{2k} - \frac{k}{4} + \frac{1}{2}\right) - \frac{k(k-2)}{4} \right]$$
$$+ \left(\frac{n}{2k} - \frac{1}{2}\right) + \left[\left(\frac{n}{2k} - \frac{k}{4} - \frac{1}{2}\right)(k-1) + \frac{(k-2)^2}{4} \right]$$
$$= n - \frac{9}{4}k - 1.$$

Therefore,

$$2|W_{v_{-k}u_0}| + |_{u_0}W_{v_{-k}}| \ge 2\left(\frac{n}{2} + \frac{n}{2k} + \frac{k}{2} - \frac{1}{2}\right) + \left(n - \frac{9}{4}k - 1\right)$$
$$= 2n + \frac{n}{k} - \frac{5}{4}k - 2.$$

Since $n > \frac{5}{4}k^2 + 2k$, we get $2|W_{v_{-k}u_0}| + |_{u_0}W_{v_{-k}}| > 2n$. Lemma 6 yields that GP(n, k) is not 2-distance-balanced.

Case 2: k is odd, $k \ge 5$, and $n > \frac{7}{4}k^2 + \frac{3}{4}k$. Just as in Case 1, we are going to distinguish

Just as in Case 1, we are going to distinguish three subcases separated according to which vertices are being addressed.

Subcase 2.1: Vertices u_{-i} and v_{-i} , where $1 \leq i \leq k-1$. If $1 \leq i < \frac{k+1}{2}$, then $u_{-i} \in W_{u_0v_{-k}}$ and $v_{-i} \in W_{u_0v_{-k}}$. If $i = \frac{k+1}{2}$, then $u_{-i} \in u_0 W_{v_{-k}}$ and $v_{-i} \in u_0 W_{v_{-k}}$, and thus there are two such vertices in $u_0 W_{v_{-k}}$. If $\frac{k+1}{2} < i \leq k-1$, then $u_{-i} \in W_{v_{-k}u_0}$ and $v_{-i} \in u_0 W_{v_{-k}}$. So, there are $\frac{k-3}{2}$ such vertices in $W_{v_{-k}u_0}$ and $\frac{k-3}{2}$ such vertices in $u_0 W_{v_{-k}}$.

Subcase 2.2: Vertices u_i , where $0 \le i \le n-k$. If $0 \le i \le k$, then $u_i \in W_{u_0v_{-k}}$ when $0 \le i \le \frac{k+1}{2}$, and $u_i \in u_0 W_{v_{-k}}$ when $\frac{k+3}{2} \le i \le k$. Thus, there are $\frac{k-1}{2}$ such vertices which are in $u_0 W_{v_{-k}}$.

If $k + 1 \leq i \leq n - k$, then $u_i \in u_0 W_{v_{-k}}$ or $u_i \in W_{v_{-k}u_0}$. We first consider the vertices u_i such that $u_i \in W_{v_{-k}u_0}$. Note that if $n - 2k < i \leq n - k$, then $u_i \in W_{v_{-k}u_0}$. Let t be the largest integer such that the maximum distance of a v_{n-k}, u_i -path is less than the minimum distance of a u_0, u_i -path, where $n - (t+1)k < i \leq n - tk$. In other words, t is the largest integer such that

$$(t-1)+1+\frac{k+1}{2} < \left\lfloor \frac{n-tk}{k} \right\rfloor + 2 \iff$$
$$(t-1)+1+\frac{k+1}{2} < \left\lfloor \frac{n}{k} \right\rfloor - t + 2 \iff$$
$$t < \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor - \frac{k}{4} + \frac{3}{4}.$$

Because t is the largest integer satisfying the above inequality, we get

$$t \ge \frac{1}{2} \left(\frac{n}{k} - 1 \right) - \frac{k}{4} + \frac{3}{4} = \frac{n}{2k} - \frac{k}{4} + \frac{1}{4}.$$

By the definition of t, if $1 \leq s \leq t$, then $u_i \in W_{v_{-k}u_0}$, where $n - (s+1)k < i \leq n - sk$. That is, $u_i \in W_{v_{-k}u_0}$ for any $n - (t+1)k < i \leq n - k$, and there are $kt \geq k(\frac{n}{2k} - \frac{k}{4} + \frac{1}{4})$ such vertices which are in $W_{v_{-k}u_0}$.

If $1 \leq j \leq k$, then the difference between the distance of a $v_{n-k}, u_{n-(t+1)k+j}$ -path and the distance of a $v_{n-k}, u_{n-(t+2)k+j}$ -path is -1. Hence, among the vertices u_i , where $n - (t+2)k < i \leq n - (t+1)k$, there are at most two vertices which are not in $W_{v_{-k}u_0}$. That is, there are at least k-2 vertices among these vertices which are in $W_{v_{-k}u_0}$. Similarly, the number of vertices u_i , where $k < i \leq n - (t+1)k$, which are in $W_{v_{-k}u_0}$, is at least

$$(k-2) + (k-4) + \dots + 1 = \frac{(k-1)^2}{4}$$

Among the vertices u_i , where $0 \le i \le n-k$, there are at least $k(\frac{n}{2k} - \frac{k}{4} + \frac{1}{4}) + \frac{(k-1)^2}{4}$ vertices which are in $W_{v_{-k}u_0}$, and

$$n - \frac{3}{2}k - \frac{1}{2} - k\left(\frac{n}{2k} - \frac{k}{4} + \frac{1}{4}\right) - \frac{(k-1)^2}{4}$$

vertices which are in $_{u_0}W_{v_{-k}} \cup W_{v_{-k}u_0}$ and not counted in $W_{v_{-k}u_0}$.

Subcase 2.3: Vertices v_i , where $0 \le i \le n - k$.

By a similar discussion as in Case 1.3 we obtain that $v_{sk} \in {}_{u_0}W_{v_{-k}}$ if $0 \le s \le t$ $(t > \frac{n}{2k} - \frac{3}{2})$, and $v_{n-k-sk} \in W_{v_{-k}u_0}$ if $0 \le s \le t$ $(t \ge \frac{n}{2k} - \frac{1}{2})$. That is, there are $t + 1 > \frac{n}{2k} - \frac{1}{2}$ such vertices which are in ${}_{u_0}W_{v_{-k}}$ and $t + 1 \ge \frac{n}{2k} + \frac{1}{2}$ such vertices which are in $W_{v_{-k}u_0}$.

We next consider vertices v_i , where 0 < i < n - k, $i \neq sk$, and $i \neq n - k - sk$, such that $v_i \in {}_{u_0}W_{v_{-k}}$. If n - 2k < i < n - k, then $v_i \in {}_{u_0}W_{v_{-k}}$. Let t be the largest integer such that the maximum distance of a v_{n-k} , v_i -path is less than or equal to the minimum distance of a u_0, v_j -path, where $n - (t+1)k < i, j \leq n - tk$. That is, t is the largest integer such that

$$(t-1) + \frac{k+1}{2} + 2 \le \left\lfloor \frac{n-tk}{k} \right\rfloor + 1 \iff (t-1) + \frac{k+1}{2} + 2 \le \left\lfloor \frac{n}{k} \right\rfloor - t + 1 \iff t \le \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor - \frac{k}{4} - \frac{1}{4}.$$

As t is the largest integer satisfying the above inequality, we get

$$t > \frac{1}{2} \left(\frac{n}{k} - 1 \right) - \frac{k}{4} - \frac{1}{4} = \frac{n}{2k} - \frac{k}{4} - \frac{3}{4}$$

By the definition of t, if $1 \le s \le t$, then $v_i \in {}_{u_0}W_{v_{-k}}$ where n - (s+1)k < i < n - sk. That is, there are $t(k-1) > (\frac{n}{2k} - \frac{k}{4} - \frac{3}{4})(k-1)$ such vertices which are in ${}_{u_0}W_{v_{-k}}$.

If $1 \leq j < k$, then the difference between the distance of a $v_{n-k}, v_{n-(t+1)k+j}$ -path and the distance of a $v_{n-k}, v_{n-(t+2)k} + j$ -path is -1. So among the vertices v_i , where n-(t+2)k < i < n-(t+1)k, there are at most two vertices which are not in $u_0 W_{v_{-k}}$. Consequently, there are at least k-3 vertices v_i , where n-(t+2)k < i < n-(t+1)k, which are in $u_0 W_{v_{-k}}$. Similarly, the number of vertices v_i , where 0 < i < n-(t+1)k, $i \neq sk$, and $i \neq n-k-sk$, which are in $u_0 W_{v_{-k}}$, is at least

$$(k-3) + (k-5) + \dots + 2 = \frac{(k-3)(k-1)}{4}.$$

Among the vertices v_i , where $0 \le i \le n-k$, there are at least $\frac{n}{2k} + \frac{1}{2}$ vertices which are in $W_{v_{-k}u_0}$ and more than

$$\left(\frac{n}{2k} - \frac{1}{2}\right) + \left[\left(\frac{n}{2k} - \frac{k}{4} - \frac{3}{4}\right)(k-1) + \frac{(k-3)(k-1)}{4}\right]$$

vertices which are in $_{u_0}W_{v_{-k}}$.

Combining the above three subcases, we obtain that

$$|W_{v_{-k}u_0}| \ge \frac{k-3}{2} + \left[k\left(\frac{n}{2k} - \frac{k}{4} + \frac{1}{4}\right) + \frac{(k-1)^2}{4}\right] + \left(\frac{n}{2k} + \frac{1}{2}\right)$$
$$= \frac{n}{2} + \frac{n}{2k} + \frac{k}{4} - \frac{3}{4}.$$

Consequently, the number of vertices in $_{u_0}W_{v_{-k}} \cup W_{v_{-k}u_0}$ which are not counted in $|W_{v_{-k}u_0}|$ is at least

$$\frac{k+1}{2} + \left[n - \frac{3}{2}k - \frac{1}{2} - k\left(\frac{n}{2k} - \frac{k}{4} + \frac{1}{4}\right) - \frac{(k-1)^2}{4}\right] + \left(\frac{n}{2k} - \frac{1}{2}\right) + \left[\left(\frac{n}{2k} - \frac{k}{4} - \frac{3}{4}\right)(k-1) + \frac{(k-3)(k-1)}{4}\right]$$
$$= n - \frac{9}{4}k + \frac{3}{4}.$$

Consequently,

$$2|W_{v_{-k}u_0}| + |_{u_0}W_{v_{-k}}| \ge 2\left(\frac{n}{2} + \frac{n}{2k} + \frac{k}{4} - \frac{3}{4}\right) + \left(n - \frac{9}{4}k + \frac{3}{4}\right)$$
$$= 2n + \frac{n}{k} - \frac{7}{4}k - \frac{3}{4}.$$

Under the assumption $n > \frac{7}{4}k^2 + \frac{3}{4}k$ we get $2|W_{v_{-k}u_0}| + |_{u_0}W_{v_{-k}}| > 2n$, hence Lemma 6 yields that GP(n,k) is not 2-distance-balanced.

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