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Maker-Breaker domination game critical graphs

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Abstract

The Maker-Breaker domination game (MBD game) is a two-player game played on a graph G by Dominator and Staller. They alternately select unplayed vertices of G. The goal of Dominator is to form a dominating set with the set of vertices selected by him while that of Staller is to prevent this from happening. In this paper MBD game critical graphs are introduced. Their existence is established and critical graphs are characterized for most of the cases in which the first player can win the game in one or two moves.

Keywords: Maker-Breaker game; Maker-Breaker domination game; Maker-Breaker domination number; Maker-Breaker domination game critical graph

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1 Introduction

Erdős and Selfridge introduced the Maker-Breaker game in [10]. This is a two-person game played on an arbitrary hypergraph $\mathcal{H}$. The players named Maker and Breaker alternately select an unplayed vertex of $\mathcal{H}$ during the game. Maker aims to occupy all the vertices of some hyperedge, on the other hand, Breaker's goal is to prevent Maker from doing it. The game has been extensively researched, both in general and in specific cases, cf. the book [16], the recent paper [18], and references therein.

In this paper, we are interested in the domination version of the Maker-Breaker game which was introduced in 2020 by Duchêne, Gledel, Parreau, and Renault [8]. The *Maker-Breaker domination game* (*MBD game* for short) is a game played on a graph $G = (V(G), E(G))$ by two players named Dominator and Staller. These names were chosen so that the players are named in line with the previously intensively researched domination game [1, 2]. Just as in the general case, the two players alternately select unplayed vertices of G. The aim of Dominator is to select all the vertices of some dominating set of G, while Staller aims to select at least one vertex from every dominating set of G. There are two variants of this game depending on which player has the first move. A *D-game* is the MBD game in which Dominator has the first move and an *S-game* is the MBD game in which Staller has the first move.

The following graph invariants are naturally associated with the MBD game [3, 15]. The *Maker-Breaker domination number*, $\gamma_{\rm MB}(G)$, is the minimum number of moves of Dominator to win the D-game on G when both players play optimally. That is, $\gamma_{\rm MB}(G)$ is the minimum number of moves of Dominator such that he wins in this number of moves no matter how Staller is playing. If Dominator has no winning strategy in the D-game, then set $\gamma_{\rm MB}(G) = \infty$. The *Staller-Maker-Breaker domination number*, $\gamma_{\rm SMB}(G)$, is the minimum number of moves of Staller to win the D-game on G when both players play optimally, where $\gamma_{\rm SMB}(G) = \infty$ if Staller has no winning strategy. In a similar manner, $\gamma'_{\rm MB}(G)$ and $\gamma'_{\rm SMB}(G)$ are the two parameters associated with the S-game. Briefly, we will call $\gamma_{\rm MB}(G)$, $\gamma'_{\rm MB}(G)$, $\gamma_{\rm SMB}(G)$, and $\gamma'_{\rm SMB}(G)$ the *MBD numbers of* G.

In the seminal paper [8] it was proved, among other things, that deciding the winner of the MBD game can be solved efficiently on trees, but it is PSPACE-complete even for bipartite graphs and split graphs. In [9] the authors give a complex linear algorithm for the Maker-Maker version of the game played on forests. The paper [3] focuses on $\gamma_{\rm SMB}(G)$ and $\gamma'_{\rm SMB}(G)$ and among other results establishes an appealing exact formula for $\gamma'_{\rm SMB}(G)$ where G is a path. In [4], for every positive integer k, trees T with $\gamma_{\rm SMB}(T) = k$ are characterized and exact formulas for $\gamma_{\rm SMB}(G)$ and $\gamma'_{\rm SMB}(G)$ derived for caterpillars. In the main result of [12], $\gamma_{\rm MB}$ and $\gamma'_{\rm MB}$ are determined for

Cartesian products of K_2 by a path. In [7], the MBD game is further studied on Cartesian products of paths, stars, and complete bipartite graphs. The total version of the MBD game was introduced in [14] and further investigated in [11].

It is known from [7, Lemma 2.3] that all the four graph invariants associated with the MBD game are monotonic for adding and deleting an edge. This motivates us to introduce MBD game critical graphs. For the classical domination game, this aspect has already been studied in [5, 17, 19, 20]. In the preliminaries, additional definitions are listed and known results about the MBD game needed later on are stated. MBD game critical graphs are formally defined in Section 3. In the same section the existence of the MBD game critical graphs is established for all four related invariants, in three of the four cases explicit constructions are provided. In Section 4 critical graphs are characterized for most of the cases in which Dominator wins the game in one or two moves. Parallel results for graphs in which Staller wins the game in one or two moves are derived in Section 5.

2 Preliminaries

Let $G = (V(G), E(G))$ be a graph. The order of G is denoted by $n(G)$. For a vertex $v \in V(G)$, its open neighbourhood is denoted by $N(v)$ and its closed neighbourhood by $N[v]$. The degree of v is $\deg(v) = |N(v)|$. The minimum and the maximum degree of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. An isolated vertex is a vertex of degree 0, a leaf is a vertex of degree 1. A *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex that is adjacent to at least two leaves. A set $S \subseteq V(G)$ is x-free for $x \in V(G)$ if $x \notin S$.

A *dominating set* of G is a set $D \subseteq V(G)$ such that each vertex from $V(G)\setminus D$ has a neighbour in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. If X is a dominating set of G with $|X| = \gamma(G)$, then X is a *γ-set* of G. A vertex of degree $n(G) - 1$ is a *dominating vertex*. If G is a connected bipartite graph, then its vertex x is a *bipartite dominating vertex* if x is adjacent to all the vertices of the bipartition set of G which does not contain x. An edge of a graph G is a *dominating edge* if it is adjacent to all the other edges of G.

The *outcome* $o(G)$ of the MBD game played on G can be one of $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{N}$, where $o(G) = \mathcal{D}$, if Dominator has a winning strategy no matter who starts the game; $o(G) = \mathcal{S}$, if Staller has a winning strategy no matter who starts the game; and $o(G) = \mathcal{N}$, if the first player has a winning strategy. See [8] that the fourth possible option for the outcome never happens.

In the rest of the preliminaries, we recall known results needed later. For $X \subseteq V(G)$, let $G|X$ denote the graph G in which vertices from X are considered as being already dominated. Then we have:

Theorem 2.1 (Continuation Principle [15]) *Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$ then $\gamma_{\rm MB}(G|A) \le \gamma_{\rm MB}(G|B)$ and $\gamma'_{\rm MB}(G|A) \le \gamma'_{\rm MB}(G|B)$.*

Proposition 2.2 [3] *If G is a graph, then the following properties hold.*

1. *If $o(G) = \mathcal{D}$ then $o(G + e) = \mathcal{D}$ for every $e \notin E(G)$.*
2. *If $o(G) = \mathcal{S}$ then $o(G - e) = \mathcal{S}$ for every $e \in E(G)$.*
3. *If $o(G) = \mathcal{N}$ then $o(G + e) \in \{\mathcal{N}, \mathcal{D}\}$ for every $e \notin E(G)$.*
4. *If $o(G) = \mathcal{N}$ then $o(G - e) \in \{\mathcal{N}, \mathcal{S}\}$ for every $e \in E(G)$.*

Lemma 2.3 [7] *If G is a graph, then the following properties hold.*

(i) $\gamma_{\rm MB}(G) \le \gamma_{\rm MB}(G - e)$ *for every* $e \in E(G)$.

(ii) $\gamma'_{\rm MB}(G) \le \gamma'_{\rm MB}(G - e)$ *for every* $e \in E(G)$.

(iii) $\gamma_{\rm SMB}(G) \le \gamma_{\rm SMB}(G + e)$ *for every* $e \notin E(G)$.

(iv) $\gamma'_{\rm SMB}(G) \le \gamma'_{\rm SMB}(G + e)$ *for every* $e \notin E(G)$.

3 MBD game critical graphs

In this section, we introduce MBD game critical graphs. It is known from [3] that the outcome of the MBD game of a graph G may change when an edge is removed or added. It was observed in [7] that the MBD number of a graph G never decreases by removing an edge and that its SMBD number never decreases by adding an edge. Hence the MBD numbers and the SMBD numbers of G are monotonic with respect to the deletion and addition of an edge. This motivates us to define the MBD game critical graphs as follows.

Definition 3.1 *A graph G is*

- $\gamma_{\rm MB}$*-critical, if* $\gamma_{\rm MB}(G) < \infty$ *and* $\gamma_{\rm MB}(G) \ne \gamma_{\rm MB}(G - e)$, *for any* $e \in E(G)$;
- $\gamma'_{\rm MB}$*-critical, if* $\gamma'_{\rm MB}(G) < \infty$ *and* $\gamma'_{\rm MB}(G) \ne \gamma'_{\rm MB}(G - e)$, *for any* $e \in E(G)$;
- $\gamma_{\rm SMB}$*-critical, if* $\gamma_{\rm SMB}(G) < \infty$ *and* $\gamma_{\rm SMB}(G) \ne \gamma_{\rm SMB}(G - e)$, *for any* $e \in E(G)$;

- $\gamma'_{\rm SMB}$*-critical, if* $\gamma'_{\rm SMB}(G) < \infty$ *and* $\gamma'_{\rm SMB}(G) \neq \gamma'_{\rm SMB}(G-e)$*, for any* $e \in E(G)$*.*

Note that if G is a graph with $\gamma_{\rm MB}(G) = \infty$, then by Lemma 2.3(i) we have $\gamma_{\rm MB}(G-e) \geq \gamma_{\rm MB}(G) = \infty$, thus it is not possible that $\gamma_{\rm MB}(G) \neq \gamma_{\rm MB}(G-e)$ holds for any edge $e \in E(G)$. Therefore we could omit the condition $\gamma_{\rm MB}(G) < \infty$ from the definition of $\gamma_{\rm MB}$-critical graphs. By the same reasoning, we could also omit the condition $\gamma'_{\rm MB}(G) < \infty$ from the definition of $\gamma'_{\rm MB}$-critical graphs.

Lemma 3.2 *If G is a graph and $e \in E(G)$, then the following properties hold.*

(i) If G is $\gamma_{\rm MB}$-critical, then $\gamma_{\rm MB}(G) < \gamma_{\rm MB}(G-e)$.

(ii) If G is $\gamma'_{\rm MB}$-critical, then $\gamma'_{\rm MB}(G) < \gamma'_{\rm MB}(G-e)$.

(iii) If G is $\gamma_{\rm SMB}$-critical, then $\gamma_{\rm SMB}(G) > \gamma_{\rm SMB}(G-e)$.

(iv) If G is $\gamma'_{\rm SMB}$-critical, then $\gamma'_{\rm SMB}(G) > \gamma'_{\rm SMB}(G-e)$.

Proof. Assume that G is $\gamma_{\rm MB}$-critical. Then $\gamma_{\rm MB}(G) \neq \gamma_{\rm MB}(G-f)$ holds for any $f \in E(G)$, hence (i) follows by Lemma 2.3(i). Statements (ii), (iii), and (iv) follow by the same argument by respectively applying Lemma 2.3(ii), (iii), and (iv). □

If G is $\gamma_{\rm MB}$-critical and $\gamma_{\rm MB}(G) = k$, then we say that G is a *k-$\gamma_{\rm MB}$-critical*. We analogously define *k-$\gamma'_{\rm MB}$-critical*, *k-$\gamma_{\rm SMB}$-critical*, and *k-$\gamma'_{\rm SMB}$-critical* graphs.

It is clear that the disjoint union of k copies of K_2 is a k-$\gamma_{\rm MB}$-critical graph. To show that there exist connected such graphs, consider graphs G_k, $k \geq 1$, constructed as follows. First, take the disjoint union of k copies of $K_{2,2}$ with respective bipartitions $\{x_i, x'_i\}$, $\{y_i, y'_i\}$, where $i \in [k]$. Then add vertex w and make it adjacent to x_i and x'_i for $i \in [k]$. Finally, add a vertex w' and the edge ww'. See Fig. 1.

Proposition 3.3 *If $k \geq 1$, then G_k is a $(k+1)$-$\gamma_{\rm MB}$-critical graph.*

Proof. We first note that $\gamma(G_k) = k+1$, so that $\gamma_{\rm MB}(G_k) \geq k+1$. Assume now that Dominator starts the game on G_k by selecting the vertex w. Then in the rest of the game, he can select one of the vertices x_i and x'_i for each $i \in [k]$. It follows that $\gamma_{\rm MB}(G_k) \leq k+1$.

To show that $\gamma_{\rm MB}(G_k - e) > k+1$ for any edge $e \in E(G_k)$, by the symmetry of G_k it suffices to consider three typical edges. Let first $e = ww'$. Then Dominator must start the game by playing w'. Since the domination number of the large component of $G_k - ww'$ is $k+1$, it follows that $\gamma_{\rm MB}(G_k - ww') \geq k+2$. Consider next the edge $e = wx_1$. Clearly, w is an optimal first move of Dominator. Now Staller replies with the move x_1. Since x_1, y_1, and y'_1 are not yet dominated, Dominator will need

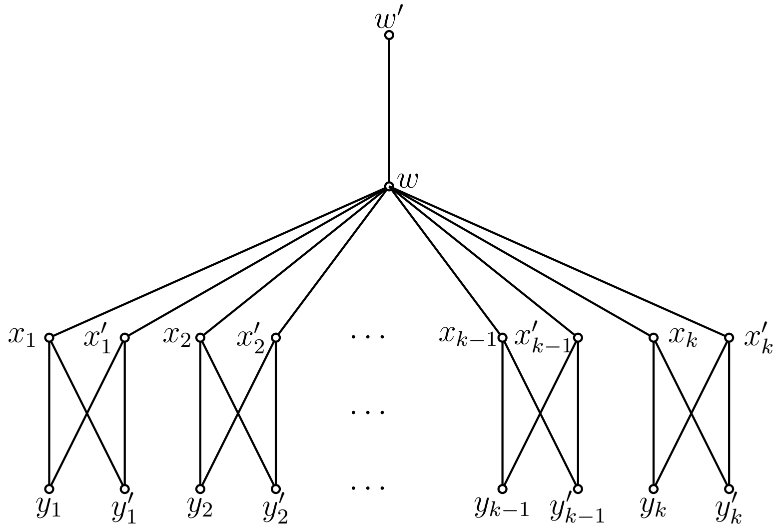

Figure 1: The graph G_k which is a connected $(k+1)$-$\gamma_{\rm MB}$ critical graph

two moves to dominate them. This in turn implies that $\gamma_{\rm MB}(G_k - wx_1) \geq k+2$. Consider finally the edge $e = x_1y_1$. In this case, we again see that w is an optimal first move of Dominator and if now Staller replies by playing x'_1 we can see as in the previous case that $\gamma_{\rm MB}(G_k - x_1y_1) \geq k+2$. $\square$

To show that there exist connected k-$\gamma'_{\rm MB}$-critical graphs, consider graphs H_k, $k \geq 1$, obtained as follows. First, take the disjoint union of $k+1$ copies of $K_{2,3}$ whose respective bipartitions are $\{x_i, x'_i\}$, $\{y_i, y'_i, y''_i\}$, where $i \in [k+1]$. Then add all possible edges between x_{k+1} and $x_1, x'_1, \ldots, x_k, x'_k$, and between x'_{k+1} and $x_1, x'_1, \ldots, x_k, x'_k$. See Fig. 2.

Proposition 3.4 *If $k \geq 1$, then H_k is a $(k+1)$-$\gamma'_{\rm MB}$-critical graph.*

Proof. Since $\gamma(H_k) = k+1$ and because during the S-game Dominator is able to select one vertex from each of the sets $\{x_i, x'_i\}$, $i \in [k+1]$, we infer that $\gamma'_{\rm MB}(H_k) = k+1$.

To show that H_k is $(k+1)$-$\gamma'_{\rm MB}$-critical, by the symmetry of H_k it suffices to consider three typical edges of H_k. Assume first that $e = x_{k+1}y_{k+1}$. Then in the S-game played on $H_k - x_{k+1}y_{k+1}$, Staller's strategy is that she first selects the vertex x'_{k+1}. Then Dominator must reply by choosing the vertex y_{k+1} for otherwise Staller wins in her next move by selecting y_{k+1}. Then Staller selects x_{k+1} as her second move. Because y'_{k+1} and y''_{k+1} are adjacent only to both x_{k+1} and x'_{k+1}, Staller will

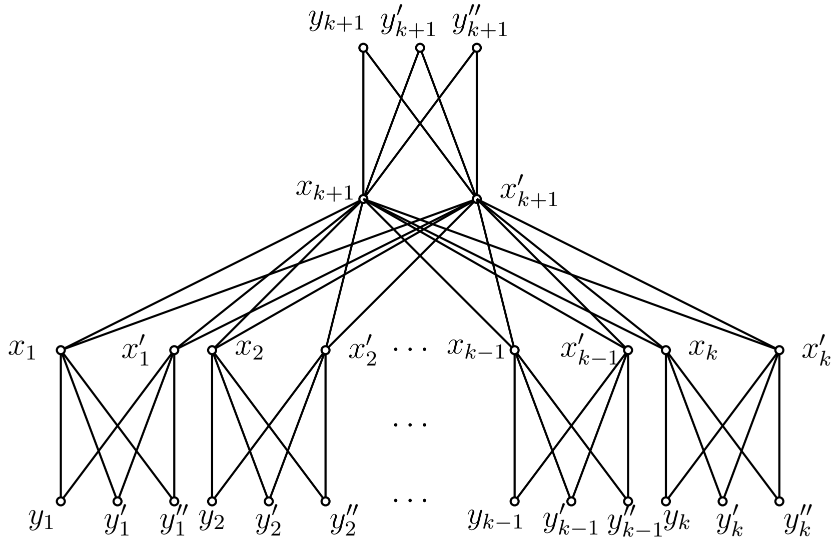

Figure 2: The graph H_k which is a connected $(k+1)$-$\gamma'_{\rm MB}$-critical graph

be able to win the game in her third move by either selecting y'_{k+1} or y''_{k+1}. As a consequence, $\gamma'_{\rm MB}(H_k - x_{k+1}y_{k+1}) = \infty$. The next typical edge to be considered is $e = x_1x_{k+1}$. Then Staller in her strategy first selects the vertex x'_{k+1}. This move forces Dominator to play x_{k+1}. As her second move, Staller then plays x_1 which in turn forces Dominator to play x'_1. But now since x_1 is not yet dominated, we can conclude that in the rest of the game, Dominator must play at least k more moves. Hence also in this case, at least $k+2$ vertices will be selected by him. The last typical edge to be considered is the edge x_1y_1. In this case, Staller first selects x'_1 which forces Dominator to play y_1 (otherwise Staller will win in her next move). Now Staller plays x_1 and then she wins in her next move. Hence $\gamma'_{\rm MB}(H_k - x_1y_1) = \infty$. □

We next construct (connected) examples of k-$\gamma'_{\rm SMB}$-critical graphs. To this end, we recall from [3, Theorem 5.2] that for an odd n,

$$\gamma'_{\rm SMB}(P_n) = \lfloor \log_2(n) \rfloor + 1\,. \tag{1}$$

Proposition 3.5 *If $k \geq 1$, then P_{2^k+1} is a $(k+1)$-$\gamma'_{\rm SMB}$-critical graph.*

Proof. Let $n = 2^k + 1$ for some positive integer k. Then by (1), $\gamma'_{\rm SMB}(P_n) = k+1$. Let e be an arbitrary edge of the path. Then at least one component of $P_n - e$ is a path P of odd order strictly smaller than 2^k. Then the strategy of Staller is to select vertices from P. Since the order of P is strictly less than 2^k it follows from [3] that $\gamma'_{\rm SMB}(P_n - e) = \gamma'_{\rm SMB}(P) < k + 1$. □

From Proposition 3.5 we can deduce that the disjoint union of P_{2^k+1} and an isolated vertex is a $(k+1)$-$\gamma_{\rm SMB}$-critical graph. In the following proposition, we give an existential proof of the existence of k-$\gamma_{\rm SMB}$-critical graphs.

Proposition 3.6 *For any integer $k > 1$, there exists a k-$\gamma_{\rm SMB}$-critical graph.*

Proof. Let G be an arbitrary graph with $\gamma_{\rm SMB}(G) = k$. (It is known from [3, Theorem 3.2] that such graphs exist.) If G is k-$\gamma_{\rm SMB}$-critical we are done. Otherwise G contains an edge e such that $\gamma_{\rm SMB}(G-e) = k$. Continuing the process of removing such edges from G we arrive at a graph G' with $\gamma_{\rm SMB}(G') = k$ and such that $\gamma_{\rm SMB}(G' - e) \neq k$ for every $e \in E(G')$. Hence G' is k-$\gamma_{\rm SMB}$-critical. □

Note that the proof of Proposition 3.6 reveals that any graph G contains a spanning subgraph G' which is is $\gamma_{\rm SMB}$-critical.

4 MBD game critical graphs with small MBD numbers

In this section, we describe critical graphs for the cases in which Dominator wins the game in one or two moves.

Theorem 4.1 *A connected graph G is 1-$\gamma_{\rm MB}$-critical if and only if $G = K_{1,n}, n \geq 1$.*

Proof. First, assume that G is connected and 1-$\gamma_{\rm MB}$-critical. Since G is 1-$\gamma_{\rm MB}$-critical, we have $\gamma_{\rm MB}(G) = 1$ and $\gamma_{\rm MB}(G - e) > 1$ for every $e \in E(G)$. Since $\gamma_{\rm MB}(G) = 1$, G contains at least one dominating vertex. Suppose first that G contains two dominating vertices x, y. Then for any edge $e \in E(G) \setminus \{xy\}$ it holds that $\gamma_{\rm MB}(G - e) = 1$. Since $\gamma_{\rm MB}(G - f) > 1$ holds for any edge $f \in E(G)$, this implies that $E(G) = \{xy\}$ and thus G is isomorphic to K_2 or equivalently to $K_{1,1}$. Suppose now that G has exactly one dominating vertex, say u. If possible, suppose that there exists an edge e of G that is not incident with u. Then $\gamma_{\rm MB}(G - e) = 1$, which leads to a contradiction. Hence every edge of G is incident with u. Thus G is isomorphic to $K_{1,n}, n \geq 1$.

Conversely, let G be a star $K_{1,n}$ for $n \geq 1$. Clearly, G is connected and $\gamma_{\rm MB}(G) = 1$. Deletion of any edge of G results in a disconnected graph and hence Dominator cannot win this game in one move. Therefore G is 1-$\gamma_{\rm MB}$-critical. □

Let $K'_{2,n}$, $n \geq 1$, be the complete split graph (cf. [13]) consisting of a clique of order 2 and an independent set of order n, where every vertex in the independent set is adjacent to both vertices of the clique. (We use this notation because $K'_{2,n}$ can be obtained from $K_{2,n}$ by adding a single edge.)

Theorem 4.2 *A connected graph G is 1-$\gamma'_{\rm MB}$-critical if and only if $G = K'_{2,n}$, $n \geq 1$.*

Proof. Assume that G is a 1-$\gamma'_{\rm MB}$-critical graph. Therefore, Dominator can win this game by selecting one vertex in his first move as a second player. Thus G has at least two dominating vertices say u and v. Now we show that $e = uv$ is a dominating edge. Suppose on the contrary that there exists an edge $f = xy$ such that $x, y \notin \{u, v\}$. Then $\gamma'_{\rm MB}(G - f) = 1$, which is a contradiction. Therefore, every edge of G is incident with either u or v or both. Hence the edge $e = uv$ is a dominating edge of G. Since u and v are dominating vertices, every vertex other than u and v has degree two in G. It follows that $G = K'_{2,n}$ for some $n \geq 1$.

Conversely, assume that $G = K'_{2,n}$ for some $n \geq 1$. Then G has two dominating vertices and hence $\gamma'_{\rm MB}(G) = 1$. Moreover, $G - e$ has at most one dominating vertex for every $e \in E(G)$. Thus in the S-game played on $G - e$, Dominator needs at least two moves because Staller can choose the dominating vertex of $G - e$ in her first move if there is such a vertex. Therefore G is 1-$\gamma'_{\rm MB}$-critical. □

Theorem 4.3 *A connected graph G is 2-$\gamma_{\rm MB}$-critical if and only if G is obtained from a star $K_{1,n}$, $n \geq 1$, with center u and a $K_{2,m}$, $m \geq 2$, whose bipartition is $\{x_1, x_2\}$, $\{y_1, y_2, \ldots, y_m\}$, by adding the edges ux_1 and ux_2.*

Proof. Assume that G is connected and 2-$\gamma_{\rm MB}$-critical. Since $\gamma_{\rm MB}(G) = 2$, Dominator cannot finish the game in one move and hence G has no dominating vertex.

Let u be an optimal first move of Dominator in the D-game played on G. Suppose that there exists an edge e with both end-vertices in $N(u)$. Then, Dominator can win the D-game on $G-e$ in two moves using the same strategy as that of the D-game on G which is a contradiction with the fact that G is a 2-$\gamma_{\rm MB}$-critical graph. Thus $N(u)$ is an independent set.

Further, we prove that $|V(G) \setminus N[u]| \geq 2$. Since u is not a dominating vertex, $|V(G) \setminus N[u]| \geq 1$. If possible suppose that there exists exactly one vertex $x \in V(G) \setminus N[u]$. Since G is connected, x is adjacent to at least one vertex, say w, in $N(u)$. Let $e = uw$ be an edge of G and consider a D-game on $G - e$. Dominator selects u as his first move in $G - e$. The only undominated vertices of G are x and w. Therefore, Dominator can finish this game in his next move by selecting either x or w depending on the Staller's move and this contradicts that G is 2-$\gamma_{\rm MB}$-critical.

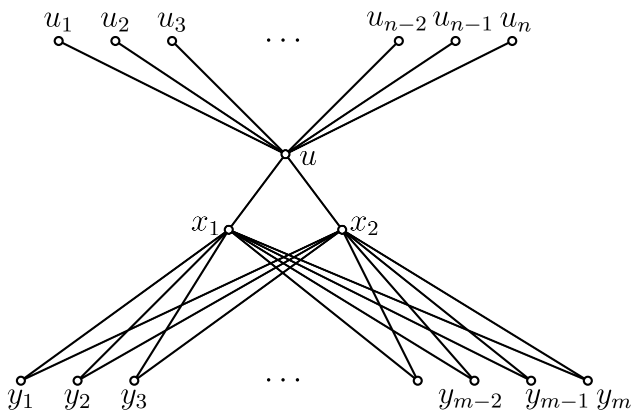

Figure 3: An example of connected 2-$\gamma_{\rm MB}$-critical graph

Since Dominator has a winning strategy with two moves, there exists $a, b \in V(G) \setminus \{u\}$ that are both adjacent to all vertices from $V(G) \setminus N(u)$. If both a, b are from $V(G) \setminus N(u)$, then $\gamma_{\rm MB}(G - e) = 2$ holds for any edge e between $N(u)$ and $V(G) \setminus N[u]$ (at least one such edge must exist, since G is connected), which contradicts the fact that G is 2-$\gamma_{\rm MB}$-critical. Thus assume that $a \in N(u)$. Since $\gamma_{\rm MB}(G - ua) > 2$, Dominator cannot finish the D-game played on $G - ua$ by first selecting u and then in his second move one vertex from $\{a, b\}$ (that was not selected by Staller in her first move). This is possible only if Staller in her first move selects a and $ab \notin E(G)$. Since a dominates all vertices of $V(G) \setminus N[u]$, this implies that $b \in N(u)$. Hence all vertices of G that dominate the whole $G - N(u)$ are from $N(u)$. Hence in the D-game played on G Dominator's optimal second move will be a vertex from $N(u)$.

Let a and b be two vertices of $N(u)$ that dominate all vertices in $V(G) \setminus N[u]$. Suppose that there exists a vertex $x \in V(G) \setminus N[u]$ that is adjacent to $y \in V(G) \setminus \{a, b\}$. Then $\gamma_{\rm MB}(G - xy) = 2$, a contradiction. Hence vertices in $V(G) \setminus N[u]$ have degree 2 in G and are adjacent to a and b.

Finally, we show that $|N(u)| \geq 3$. If possible suppose that a and b are the only neighbours of u in G. Let $e = ua$. Then $\gamma_{\rm MB}(G - e) = 2$. Indeed, Dominator can select b as his first optimal move in a D-game played on $G - e$. The only vertex undominated after this move is a. Therefore Dominator can finish the game by selecting a vertex in $V(G) \setminus N[u]$ as his next move. Hence we again get a contradiction. Thus $|N[u]| \geq 3$.

By the above properties of the graph G we can deduce that vertices in $\{u\} \cup (N(u) \setminus \{a, b\})$ induce $K_{1,n}$ for some $n \geq 1$, vertices in the set $\{a, b\} \cup (V(G) \setminus N[u])$

induce $K_{2,m}$ for $m \geq 2$ and au, bu are the only edges between $K_{1,n}$ and $K_{2,m}$ in G.

Conversely, let G be a graph obtained from $K_{1,n}$, $n \geq 1$, with center u, and from $K_{2,m}$, $m \geq 2$, with bipartition $\{x_1, x_2\}$, $\{y_1, y_2, \ldots, y_m\}$, by adding the edges ux_1 and ux_2. Clearly, G is connected and Dominator can finish a D-game on G by selecting u as his first move and then selecting either x_1 or x_2 with respect to the Staller's first move. So $\gamma_{\rm MB}(G) = 2$.

Let e be a pendant edge incident to u. Clearly, the graph $G - e$ has an isolated vertex. Dominator selects this isolated vertex as his first optimal move in a D-game on $G - e$. And the remaining part of $G - e$ has no dominating vertices. Therefore Dominator needs at least three moves to finish a D-game on $G - e$.

Now let $e = ux_1$. Consider a D-game on $G - e$. Since u is a support vertex Domiantor first selects u. Then Staller selects x_1. If Dominator selects x_2 as his next move then x_1 remains undominated. And if Dominator selects a vertex in $V(G) \setminus N[u]$, then there is an undominated vertex in $V(G) \setminus N[u]$. Therefore Dominator needs at least three moves to finish the game in $G - e$. A similar argument also holds for $e = ux_2$.

Now let e be an edge whose one end vertex is x_1 and the other end vertex lies in $V(G) \setminus N[u]$. Consider a D-game on $G - e$. Clearly, u and x_2 are support vertices. Definitely, Staller can select one of these support vertices in her turn. Therefore Dominator must select the pendant vertex adjacent to the support vertex selected by Staller. Clearly, this restriction does not allow Dominator to finish the game on $G - e$ in two moves. Hence $\gamma_{\rm MB}(G - e) > 2$ in this case and this is the same when e is an edge between x_2 and $V(G) \setminus N[u]$.

From all these cases we conclude that G is 2-$\gamma_{\rm MB}$ critical. □

For 2-$\gamma'_{\rm MB}$-critical graphs we do not have a complete characterization but can give the following necessary conditions.

Proposition 4.4 *If G is connected 2-$\gamma'_{\rm MB}$-critical graph, then the following properties hold.*

(i) $n(G) \geq 5$.

(ii) $\delta(G) \geq 2$.

(iii) $\Delta(G) \leq n(G) - 2$.

Moreover, all the bounds are sharp.

Proof. (i) Let G be a connected 2-$\gamma'_{\rm MB}$-critical graph. Since $\gamma'_{\rm MB}(G) = 2$, G has at least four vertices (note that Staller has two moves and Dominator has two moves). Hence we need to exclude all connected graphs of order 4. Clearly, K_4 and $K_4 - e$ are connected graphs having four vertices. But it is clear that $\gamma'_{\rm MB}(K_4) = \gamma'_{\rm MB}(K_4 - e) = 1$. The remaining connected graphs with four vertices are P_4, C_4 and K_3 with a pendant edge. We can easily verify that $\gamma'_{\rm MB}(P_4) = \gamma'_{\rm MB}(C_4) = 2$. Since P_4 is isomorphic to $C_4 - e$, C_4 is not a critical graph. Let v_1, v_2, v_3, v_4 be the vertices of P_4 and let e be the edge v_2v_3. Clearly, $P_4 - e$ is the disjoint union of two K_2's. Therefore $\gamma'_{\rm MB}(P_4 - e) = 2$. Hence P_4 is not a critical graph. Finally, let H be a graph with vertices x_1, x_2, x_3 that induce a K_3 and y that is adjacent to x_1, and there are no other vertices or edges in H. Then $\gamma'_{\rm MB}(H - x_1x_3) = 2$, as $H - x_1x_3$ is isomorphic to P_4. Hence H is not 2-$\gamma'_{\rm MB}$-critical. Thus there is no connected 2-$\gamma'_{\rm MB}$ critical graph having four vertices.

(ii) Suppose on the contrary that $\delta(G) = 1$. Let $u \in V(G)$ such that $\deg(u) = 1$ and let v be the only neighbor of u. Since G is connected and has at least 5 vertices by (i), we have $\deg(v) \geq 2$.

Now consider an S-game on G. Assume that Staller selects v as her first optimal move. Therefore Dominator must play u as his first reply, otherwise, Staller will win this game by selecting u. Since $\gamma'_{\rm MB}(G) = 2$, Dominator can dominate all the vertices of G other than u and v in his next move. Hence there exist $a, b \in V(G) \setminus \{u, v\}$ that dominate all the vertices of $V(G) \setminus \{u, v\}$ or, equivalently, the subgraph of G induced by $V(G) \setminus \{u, v\}$ has two dominating vertices. Let e be an arbitrary edge incident with v but not with u. Then $\gamma'_{\rm MB}(G - e) = 2$, a contradiction with G being 2-$\gamma'_{\rm MB}$-critical. Thus we conclude that $\delta(G) \geq 2$.

(iii) Let G be connected and 2-$\gamma'_{\rm MB}$-critical. Since $\gamma'_{\rm MB}(G) \neq 1$, G has at most one dominating vertex. For the purpose of contradiction assume that G contains a dominating vertex u, i.e. $\deg(u) = n(G) - 1$. It is known from [6, Proposition 4.2] that $\gamma'_{\rm MB}(G) = 2$ implies that for any vertex u there exists a vertex v such that $\{v, v_1\}$ and $\{v, v_2\}$ are two u-free γ-sets. Since $\deg(v) < n - 1$, there is a vertex w in G which is not adjacent to v. We consider two cases for the remaining part.

Case 1: $w \notin \{v_1, v_2\}$.
Since $\{v, v_1\}$ and $\{v, v_2\}$ are two u-free γ-sets of G, the vertex w must be dominated by both v_1 and v_2. Let $e = uw$. Consider an S-game on $G - e$. If Staller first selects u, then Dominator can select all the vertices from either $\{v, v_1\}$ or $\{v, v_2\}$ and win the game in two moves.

If Staller first selects a vertex other than u, then Dominator must select u. In this case, the only undominated vertex is w. Since w is dominated by all the vertices from the set $A = \{w, v_1, v_2\}$, Dominator can select any vertex of A (depending on Staller's move) and win the game in two moves. This contradicts that G is 2-$\gamma'_{\rm MB}$-critical.

Case 2: $w \in \{v_1, v_2\}$.
In this case, v_1v_2 must be an edge of G. Let $e = uv_1$. Consider an S-game on $G - e$. If Staller selects a vertex from the set $\{v_1, v_2\}$, then Dominator in his first move selects u. Then v_1 is the only still not dominated vertex. Since v_1 is dominated by both v_1 and v_2, Dominator can select one of them (depending on Staller's move) and win in two moves. Therefore $\gamma'_{\rm MB}(G - e)=2$, a contradiction.

Now, if Staller selects u, then Dominator can select all the vertices from either $\{v, v_1\}$ or $\{v, v_2\}$ in his first two moves and finishes the game. Finally, if Staller selects one of the vertices in $\{v_1, v_2\}$, then Dominator selects the other one. Now Dominator selects one vertex from $\{u, v\}$ (depending on the Staller's move) and wins the game in his second move. This leads to a contradiction to the assumption that G is 2-$\gamma'_{\rm MB}$-critical.

To prove that all three bounds are sharp, consider the complete bipartite graphs $K_{2,n}$, $n \geq 3$, which are connected and 2-$\gamma'_{\rm MB}$-critical. □

To conclude the section we identify a large class of connected graphs which are 2-$\gamma'_{\rm MB}$-critical.

Theorem 4.5 *Let G be a bipartite graph with bipartition V_1, V_2, where $|V_i| \geq 3$, $i \in [2]$. If G has exactly two bipartite dominating vertices in each V_i, $i \in [2]$, and every vertex has degree two except bipartite dominating vertices, then G is 2-$\gamma'_{\rm MB}$-critical.*

Proof. Let G be a graph satisfying the above properties. Let $v_{i,1}$ and $v_{i,2}$ be the bipartite dominating vertices of V_i, $i \in [2]$. Clearly, $\{v_{1,1}, v_{2,1}\}$, and $\{v_{1,1}, v_{2,2}\}$ are two $v_{1,2}$-free γ-sets of cardinality 2. Also $\{v_{1,2}, v_{2,1}\}$, and $\{v_{1,2}, v_{2,2}\}$ are two $v_{1,1}$-free γ sets of cardinality 2. Therefore Dominator can select two vertices from one set irrespective of Staller's move and win the game in two moves. Thus $\gamma'_{\rm MB}(G) = 2$. Now we show that G is critical. Any edge of G either has two bipartite dominating vertices as endpoints or has exactly one bipartite dominating vertex as an endpoint. Let first $e = v_{1,1}v_{2,1}$. Consider an S-game on $G - e$. Clearly, the only bipartite dominating vertices in $G - e$ are $v_{1,2}$ and $v_{2,2}$. In her strategy Staller first selects one of the bipartite dominating vertices, say $v_{1,2}$. If the first optimal move of Dominator is in V_2, then by the Theorem 2.1, $v_{2,2}$ is an optimal first move of Dominator. Now all the vertices in V_2 except $v_{2,2}$ are undominated. Thus Staller selects $v_{1,1}$ as her next move. Any vertex in $V_1 \setminus \{v_{1,1}, v_{1,2}\}$ dominates only $v_{2,1}$ and $v_{2,2}$ in V_2. But V_2 has at least three vertices and Dominator cannot finish the game in two moves. Now assume that an optimal first move of Dominator is in V_1. Let the first optimal move of Dominator be $v_{1,1}$ after the same first move of Staller. Now Staller selects

$v_{2,1}$. So the undominated vertices are $v_{2,1}$ and all vertices in V_1 except $v_{1,1}$. Since G is bipartite, Dominator needs at least two more moves to finish the game in this case. Finally, assume that Dominator selects an unplayed vertex other than $v_{1,1}$ in V_1 after the same first move of Staller. This vertex only Dominates itself and both the vertices $v_{2,1}$ and $v_{2,2}$. It is clear that there are still undominated vertices in both V_1 and V_2. Therefore Dominator needs at least three moves to finish the game. Hence we can conclude that $\gamma'_{\rm MB}(G-e) > 2$.

If e is an edge between any two bipartite dominating vertices, then $\gamma'_{\rm MB}(G-e) > 2$ is proved by similar arguments as above.

Finally, let $e = ab$ be an edge, where exactly one of its end vertices, say a, is a bipartite dominating vertex. Clearly, b is the leaf of $G-e$. Therefore Staller first selects the support vertex of $G-e$ and then Dominator must select the leaf b of $G-e$ as his first move. We can see that there are still undominated vertices in both V_1 and V_2. So Dominator needs at least two more moves to finish the game. Thus we can conclude that $\gamma'_{\rm MB}(G-e) > 2$.

Therefore G is a connected 2-$\gamma'_{\rm MB}$-critical graph. □

5 MBD game critical graphs with small SMBD numbers

In this section, we characterise MBD game critical graphs with small SMBD numbers. It is known that $\gamma_{\rm SMB}(G) = 1$ if and only if G has at least two isolated vertices. Let G be a graph with at least two isolated vertices. It is clear that $G-e$ has again at least two isolated vertices for any edge e of G. This indeed proves that G is not 1-$\gamma_{\rm SMB}$-critical. Hence there do not exist a 1-$\gamma_{\rm SMB}$-critical graphs.

Theorem 5.1 *A graph G is 2-$\gamma_{\rm SMB}$-critical if and only if G is the disjoint union of exactly one copy of K_1, at least one copy of $K_{1,n}$, $n \geq 2$ and possibly some copies of K_2.*

Proof. Assume that G is 2-$\gamma_{\rm SMB}$-critical. Hence $\gamma_{\rm SMB}(G) = 2$ and $\gamma_{\rm SMB}(G-e) = 1$ for any edge $e \in E(G)$. Since $\gamma_{\rm SMB}(G) = 2$ it follows from [3] that G is exactly a graph that has either at least two strong support vertices or at least one strong support vertex and exactly one isolated vertex. Suppose first that G has no isolated vertices. Since $\gamma_{\rm SMB}(G-e) = 1$ for every $e \in E(G)$, it holds that $G-e$ has two isolated vertices for any edge $e \in E(G)$. Since G has no isolated vertices, each component of G is K_2. Hence $G = \ell K_2$ for some positive integer ℓ, which leads to the contradiction as $2 \neq \gamma_{\rm SMB}(\ell K_2) = \infty$.

Now let G be a graph with at least one isolated vertex and at least one strong support vertex. Since $\gamma_{\rm SMB}(G) \neq 1$, G has exactly one isolated vertex. Since for any edge $e \in E(G)$ it holds that $G - e$ has at least two isolated vertices, every edge of G is a pendant edge of G. Thus G is the disjoint union of exactly one copy of K_1, at least one copy of $K_{1,n}$, $n \geq 2$ and possibly some copies of K_2.

Conversely, assume that G is the disjoint union of exactly one copy of K_1, at least one copy of $K_{1,n}$, $n \geq 2$, and possibly some copies of K_2. Consider a D-game on G. Clearly, Dominator must first select the isolated vertex. Then Staller selects the centre of a $K_{1,n}$ (there is at least one such component), which allows her to win in her second move by choosing an unplayed leaf adjacent to the centre. Thus $\gamma_{\rm SMB}(G) = 2$. Let e be an edge of G. Clearly, $G - e$ has at least two isolated vertices. Therefore Staller can win the game in one move on $G - e$. This proves that G is 2-$\gamma_{\rm SMB}$-critical. □

A graph G is 1-$\gamma'_{\rm SMB}$-critical if $\gamma'_{\rm SMB}(G) = 1$ and $\gamma'_{\rm SMB}(G-e) < \gamma'_{\rm SMB}(G)$ for any edge $e \in E(G)$. This is possible only when G has at least one vertex and no edges. Thus we conclude that a graph G is 1-$\gamma'_{\rm SMB}$-critical if and only if G is a totally disconnected graph.

Theorem 5.2 *A graph G is 2-$\gamma'_{\rm SMB}$ critical if and only if G is the disjoint union of at least one copy of $K_{1,n}, n \geq 2$, and possibly some copies of K_2.*

Proof. Assume that G is 2-$\gamma'_{\rm SMB}$ critical. That is, $\gamma'_{\rm SMB}(G) = 2$ and $\gamma'_{\rm SMB}(G-e) = 1$ for any edge $e \in E(G)$. It is known from [6] that $\gamma'_{\rm SMB}(G) = 2$ if and only if G has a strong support vertex. Also, $\gamma'_{\rm SMB}(G-e) = 1$ if and only if $G - e$ has an isolated vertex. This is possible only when every edge of G is incident with a vertex of degree 1. Thus we conclude that G has at least one $K_{1,n}, n \geq 2$, and possibly some copies of K_2.

Conversely, assume that G is the disjoint union of at least one copy of $K_{1,n}, n \geq 2$ and possibly some copies of K_2. Since G has a strong support vertex, $\gamma'_{\rm SMB}(G) = 2$. It is clear that the removal of any edge from G results in a graph having an isolated vertex. Thus $\gamma'_{\rm SMB}(G-e) = 1$ for any edge $e \in E(G)$. Thus G is 2-$\gamma'_{\rm SMB}$-critical. □

6 Concluding remarks

The graph G' from the proof of Proposition 3.6 need not be connected. In fact, we believe that for any $k \geq 1$ there are no connected k-$\gamma_{\rm SMB}$-critical graphs. This is clearly true for $k = 1$, as any 1-$\gamma_{\rm SMB}$-critical graph must contain at least two isolated vertices. Moreover, Theorem 5.1 implies that there do not exist connected 2-$\gamma_{\rm SMB}$-critical graphs. For the general case we pose:

Conjecture 6.1 *If G is a connected graph, then G is not $\gamma_{\rm SMB}$-critical.*

In view of Proposition 4.4 and Theorem 4.5 we pose:

Open Problem 6.2 *Find a characterization of 2-$\gamma'_{\rm MB}$-critical graphs.*

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Declaration of interests

The authors declare that they have no conflict of interest.

Data availability

Our manuscript has no associated data.

References

[1] B. Brešar, S. Klavžar, D.F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

[2] B. Brešar, M.A. Henning, S. Klavžar, D.F. Rall, Domination Games Played on Graphs, SpringerBriefs in Mathematics, 2021.

[3] Cs. Bujtás, P. Dokyeesun, Fast winning strategies for Staller in the Maker-Breaker domination game, Discrete Appl. Math. 344 (2024) 10–22.

[4] Cs. Bujtás, P. Dokyeesun, S. Klavžar, Maker-Breaker domination game on trees when Staller wins, Discrete Math. Theor. Comput. Sci. 25(2) (2023) Paper 12.

[5] Cs. Bujtás, S. Klavžar, G. Košmrlj, Domination game critical graphs, Discuss. Math. Graph Theory 35 (2015) 781–796.

[6] A. Divakaran, T. James, S. Klavžar, L. S. Nair, Maker-Breaker domination game played on corona products of graphs, (arXiv).

[7] P. Dokyeesun, Maker-Breaker domination game on Cartesian products of graphs, arXiv:2310.04103v2 [math.CO] (11 Oct 2023).

[8] E. Duchêne, V. Gledel, A. Parreau, G. Renault, Maker-Breaker domination game, Discrete Math. 343 (2020) 111955.

[9] E. Duchêne, A. Dumas, N. Oijid, A. Parreau, E. Rémila, The Maker-Maker domination game in forests, Discrete Appl. Math. 348 (2024) 6–34.

[10] P. Erdős, J.L. Selfridge, On a combinatorial game, J. Combin. Theory Ser. A 14 (1973) 298–301.

[11] J. Forcan, M. Mikalački, Maker-Breaker total domination game on cubic graphs, Discrete Math. Theor. Comput. Sci. 24(1) (2022) Paper 20.

[12] J. Forcan, J. Qi, Maker-Breaker domination number for Cartesian products of path graphs P_2 and P_n, Discrete Math. Theor. Comput. Sci. 25(2) (2023) Paper 21.

[13] J. Ge, Y. Liao, B. Zhang, Resistance distances and the Moon-type formula of a vertex-weighted complete split graph, Discrete Appl. Math. 359 (2024) 10–15.

[14] V. Gledel, M.A. Henning, V. Iršič, S. Klavžar, Maker-Breaker total domination game, Discrete Appl. Math. 282 (2020) 96–107.

[15] V. Gledel, V. Iršič, S. Klavžar, Maker-Breaker domination number, Bull. Malays. Math. Sci. Soc. 42 (2019) 1773–1789.

[16] D. Hefetz, M. Krivelevich, M. Stojaković, T. Szabó, Positional Games, Birkhäuser/Springer, Basel, 2014.

[17] M.A. Henning, S. Klavžar, D.F. Rall, Game total domination critical graphs, Discrete Appl. Math. 250 (2018) 28–37.

[18] M.L. Rahman, T. Watson, 6-uniform Maker-Breaker game is PSPACE-complete, Combinatorica 43 (2023) 595–612.

[19] K. Xu, X. Li, On domination game stable graphs and domination game edge-critical graphs, Discrete Appl. Math. 250 (2018) 47-–56.

[20] C. Worawannotai, K. Charoensitthichai, 4-total domination game critical graphs, Discrete Math. Algorithms Appl. 16 (2024) Paper 2350061.