Injective colorings of Sierpiński-like graphs and Kneser graphs

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Abstract

Two relationships between the injective chromatic number and, respectively, chromatic number and chromatic index, are proved. They are applied to determine the injective chromatic number of Sierpiński graphs and to give a short proof that Sierpiński graphs are Class 1. Sierpiński-like graphs are also considered, including generalized Sierpiński graphs over cycles and rooted products. It is proved that the injective chromatic number of a rooted product of two graphs lies in a set of six possible values. Sierpiński graphs and Kneser graphs $K(n,r)$ are considered with respect of being perfect injectively colorable, where a graph is perfect injectively colorable if it has an injective coloring in which every color class forms an open packing of largest cardinality. In particular, all Sierpiński graphs and Kneser graphs $K(n,r)$ with $n \geq 3r-1$ are perfect injectively colorable graph, while $K(7,3)$ is not.

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1 Introduction

Throughout the paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The (*open*) *neighborhood* of a vertex v is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$ is its *closed neighborhood* (we omit the index G if the graph G is clear from the context). The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For terminology and notation not explicitly defined here, we refer to [34].

Recall that a (*vertex*) *coloring* of G is a labeling of the vertices of G so that any two adjacent vertices have distinct labels. The *chromatic number* of G, denoted $\chi(G)$, is the smallest number of labels in a coloring of G. For some additional information on coloring problems, we refer the reader to [19].

A function $f : V(G) \to \{1, \ldots, k\}$ is an *injective k-coloring* if no vertex v is adjacent to two vertices u and w with $f(u) = f(w)$. For an injective k-coloring f, the set of color classes $\big\{\{v \in V(G) \mid f(v) = i\} : 1 \le i \le k\big\}$ is also called an *injective k-coloring* of G (or simply an *injective coloring* if k is clear from the context). The minimum k for which a graph G admits an injective k-coloring is the *injective chromatic number* of G, and is denoted by $\chi_i(G)$. An injective k-coloring for which $k = \chi_i(G)$ is called a *$\chi_i(G)$-coloring*. The study of injective coloring was initiated in [14], and then intensively pursued, see, for example, [5, 8, 25, 29]. In particular, the injective colorings of some products and graphs operations have been studied in [3, 30, 33].

A set $B \subseteq V(G)$ is an *open packing* in G if $N(u) \cap N(v) = \emptyset$ for all distinct vertices $u, v \in B$, and the maximum cardinality of an open packing in G is the *open packing number*, $\rho^{\rm o}(G)$, of G. An open packing of cardinality $\rho^{\rm o}(G)$ is an *$\rho^{\rm o}(G)$-set*. The concept was introduced in [15], and was studied in several papers mainly due to its relation with total domination. It was noticed in [4] that an injective coloring of a graph G is equivalent to a partition of $V(G)$ into open packings, i.e., the vertices colored with a same color in the injective coloring form an open packing in G. In connection with this, the following concept was introduced in [4], and further on, partially investigated for hypercubes in [3]. A graph G is *perfect injectively colorable* if it admits an injective coloring in which every color class forms an open packing of maximum cardinality. Note that such an injective coloring is necessarily a $\chi_i(G)$-

coloring.

Section 2 is devoted to two auxiliary lemmas based on establishing some helpful relationships between injective coloring and, respectively, vertex coloring and edge coloring. They will be efficiently used in Section 3 in order to prove for each Sierpiński graph S_p^n that (i) $\chi_i(S_p^n) = p = \Delta(S_p^n)$ with $p \geq 3$ and $n \geq 1$, and that (ii) S_p^n belongs to Class 1 with $n, p \geq 2$. (Recall that a simple graph G is Class 1 if $\chi'(G) = \Delta(G)$, in which χ' stands for the edge-chromatic number.) Note that the assertion (ii) was already proved by Hinz and Parisse in 2012 ([17]). However, our proof is much shorter based on the present, different approach.

Injective coloring of the rooted product graph $G \circ_v H$, as a Sierpiński-type product graph, is discussed in Section 4. It is readily seen that $\chi_i(G \circ_v H)$ can be bounded from below and above by $\max\{\chi_i(G), \chi_i(H)\}$ and $\chi_i(G) + \chi_i(H)$, respectively. We prove that $\chi_i(G \circ_v H)$ only assumes 6 values from this interval. And, as an immediate result, this leads to a closed formula for this parameter in the case of corona product graphs given in [30].

We also investigate the perfect injectively colorability of Sierpiński graphs and Kneser graphs. It is proved that each Sierpiński graph S_p^n with $p \geq 3$ and $n \geq 1$ is perfect injectively colorable, while this is not the case for generalized Sierpiński graphs by giving a special counterexample. Finally, we prove that all Kneser graphs $K(n, r)$ with $n \geq 3r - 1$ are perfect injectively colorable. Moreover, this is a best possible result as the Kneser graph $K(7, 3)$ does not satisfy this property.

2 Two lemmas on injective colorings versus (edge) colorings

In this section we prove two relationships between the injective chromatic number and, respectively, chromatic number and chromatic index. Their proofs are not difficult, but we will later demonstrate that the results can be very useful.

For the first result, consider the following concept. Let G be a graph. A collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of cliques in G is an *edge clique cover* of G if every edge of G belongs to some $C_i \in \mathcal{C}$. For more information on edge clique covers see the survey [31] and recent papers [7, 28]. We say that an edge clique cover $\mathcal{C}$ is *sparse* if every vertex of G belongs to at most two cliques in $\mathcal{C}$. Note that not every graph has a sparse edge clique cover. For instance, among the triangle-free graphs G only the graphs with $\Delta(G) \leq 2$ admit sparse edge clique covers.

Let $\mathcal{G}$ be the class of graphs that admit a sparse edge clique cover. If $G \in \mathcal{G}$ and $\mathcal{C} = \{C_1, \ldots, C_k\}$ is a sparse edge clique cover of G, then we introduce the graph $G^{\mathcal{C}}$ constructed from G as follows. First, considering the vertex sets of the cliques

C_i to be pairwise disjoint in $G^{\mathcal{C}}$, we set $V(G^{\mathcal{C}}) = \bigcup_{i=1}^{k} V(C_i)$. Note that by this convention, $|V(G^{\mathcal{C}})| = \sum_{i=1}^{k} |V(C_i)|$. Second, two vertices in $G^{\mathcal{C}}$ are adjacent if they are either in the same clique from $\mathcal{C}$ or they correspond to the same vertex from two cliques of $\mathcal{C}$. See Fig. 1 for an example of this construction.

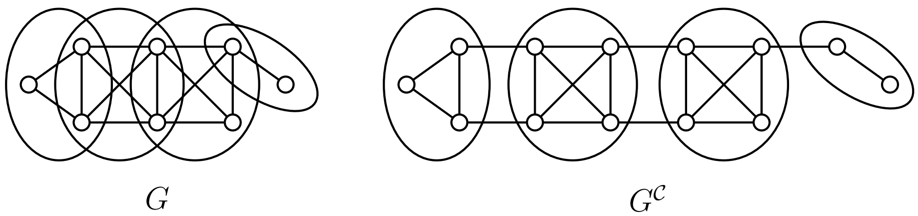

Figure 1: A graph G, its sparse edge clique cover $\mathcal{C}$ (consisting of the circled cliques), and the derived graph $G^{\mathcal{C}}$

Our first result now reads as follows.

Lemma 2.1. *If $G \in \mathcal{G}$ with a sparse edge clique cover $\mathcal{C} = \{C_1, \ldots, C_k\}$, then $\chi_i(G^{\mathcal{C}}) \leq \chi(G)$.*

Proof. Let $c : V(G) \to [k]$ be a proper coloring of G, where $k = \chi(G)$. Let the coloring $c^* : V(G^{\mathcal{C}}) \to [k]$ be defined by $c^*(x) = c(x)$ if $x \in V(G)$ belongs to only one clique of $\mathcal{C}$, and $c^*(x_i) = c(x) = c^*(x_j)$, when x belongs to C_i and C_j. See Fig. 2 for an example of such a derived coloring.

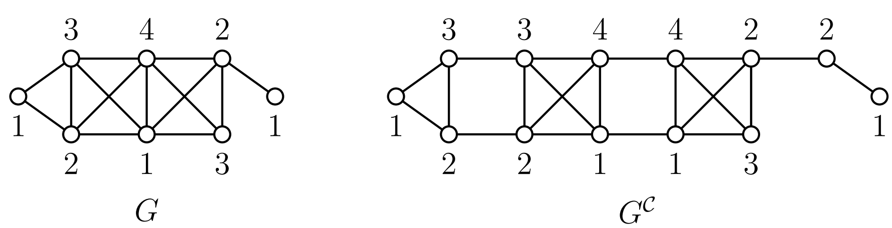

Figure 2: An optimal coloring of G and an optimal injective coloring of $G^{\mathcal{C}}$

We claim that c^* is injective, so we need to show that no two vertices in the neighborhood of any vertex x in $G^{\mathcal{C}}$ receive the same color. Assume that x is a vertex of G that lies in only one clique, say C_i, of $\mathcal{C}$. Then, its neighbors are all in C_i, and since c is a proper coloring, they have pairwise different colors (according to c^* and c). Secondly, assume that $x = x_i$, that is, x belongs to two cliques of G from $\mathcal{C}$, one of which being C_i, and let the other be C_j. In this case, $N_{G^{\mathcal{C}}}(x_i) = V(C_i) \cup \{x_j\}$.

Then, the neighbors of x_i in C_i have pairwise different colors (by c and c^*), which are all different from $c(x) = c^*(x_i) = c^*(x_j)$. In both cases, all neighbors of x get pairwise different colors by c^*. □

Our second result detects a new family of Class 1 graphs based on their injective chromatic number. (Recall that by Vizing's theorem, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$ holds for any graph G, where the graphs achieving the lower bound are said to belong to Class 1.)

Lemma 2.2. *If $\chi_i(G) = \Delta(G)$, then G belongs to Class 1.*

Proof. Let $\Delta = \Delta(G)$, and let $c : V(G) \to \{0, 1, \ldots, \Delta - 1\}$ be an injective coloring of G. Define an edge coloring $c' = E(G) \to \{0, 1, \ldots, \Delta - 1\}$ arising from c as follows. For each edge $uv \in E(G)$, let $c'(uv) = c(u) + c(v) \pmod{\Delta}$. To see that c' is a proper edge coloring of G, let uv and uw be two incident edges in G. Since $c(v) \neq c(w)$, we infer that $c'(uv) = c(u) + c(v) \neq c(u) + c(w) = c'(uw)$, where summations are taken with respect to modulo Δ. Thus, $\chi'(G) \le \Delta$, which by Vizing's theorem implies that G is in Class 1. □

3 Sierpiński graphs

This section is devoted to obtain the injective chromatic number of Sierpiński graphs, and to give some consequences of these computations. In particular, we give a short proof of the fact that all Sierpiński graphs belong to Class 1, a result which was first proved in [17] with a lengthy argument.

When the Switching Tower of Hanoi game was introduced in [21], it was natural to introduce Sierpiński graphs. This family of graphs has subsequently attracted a great deal of interest for various reasons, see a very comprehensive 2017 survey paper [16], where, in addition to an overview of the results on the Sierpiński graphs, a classification of Sierpiński-type graphs is proposed. For some later related papers we refer to [1, 10, 26].

For $p \ge 1$ and $n \ge 1$, the *Sierpiński graph* S_p^n has the vertex set $V(S_p^n) = [p]^n$, and two vertices $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are adjacent if there exists an index $d \in [n]$ such that (i) $u_i = v_i$ for $i \in [d-1]$, (ii) $u_d \neq v_d$, and (iii) $v_i = u_d$ and $u_i = v_d$ for $i \in \{d+1, \ldots, n\}$. The family of *Sierpiński triangle graphs* $\widehat{S}_p^n$ was first introduced by Jakovac in [18]. These graphs can be defined in various ways, but for our purposes we do it for $p \ge 3$ as follows. If $p \ge 3$ and $n \ge 1$, then $\widehat{S}_p^n$ is the graph obtained from S_p^n as follows. For any edge uv which does not lie in a complete graph of order p, remove the edge uv and identify the vertices u and v. See Fig. 3 where the graphs S_3^3 and $\widehat{S}_3^3$ are drawn.

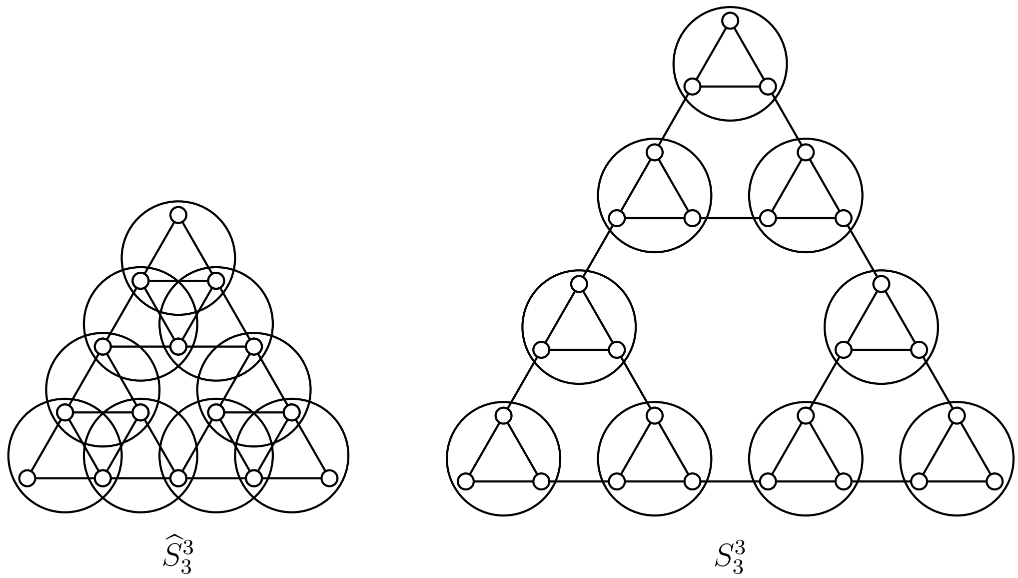

Figure 3: The graph $\widehat{S}_3^3$ and its sparse edge clique cover (left), and the graph S_3^3 (right)

Theorem 3.1. *If $p \geq 3$ and $n \geq 1$, then $\chi_i(S_p^n) = p$. Moreover, S_p^n is perfect injectively colorable.*

Proof. As $\Delta(S_p^n) = p$, we have $\chi_i(S_p^n) \geq p$.

Let $\mathcal{C}$ be the edge clique cover of $\widehat{S}_p^n$ consisting of all the cliques of order p of $\widehat{S}_p^n$ which are obtained from the cliques of order p of S_p^n after contacting the edges of S_p^n which lie in no such clique. See Fig. 3 where the described clique-edge cover of $\widehat{S}_3^3$ is shown. By the way this cover is constructed, we infer that $\mathcal{C}$ is a sparse clique-edge cover. Hence we can consider $(\widehat{S}_p^n)^{\mathcal{C}}$ and again by the construction we see that $(\widehat{S}_p^n)^{\mathcal{C}} \cong S_p^n$, see Fig. 3 again. Applying Lemma 2.1 we then get

$$\chi_i(S_p^n) = \chi_i((\widehat{S}_p^n)^{\mathcal{C}}) \leq \chi(\widehat{S}_p^n) = p\,,$$

where the last equality is a result due to Jakovac proved in [18]. This proves the first assertion of the theorem.

To prove the second assertion, note that the vertex set of S_p^n partitions into p^{n-1} cliques of cardinality p. Combining this with $\chi_i(S_p^n) = p$, we infer that each color class of any $\chi_i(S_p^n)$-coloring has a non-empty intersection with each such clique of S_p^n. Since each color class is an open packing, we infer $\rho^{\rm o}(S_p^n) \geq p^{n-1}$. On the other hand, note that if P is an open packing of a graph, then no two vertices of P can lie in a triangle in G. Therefore, when $p \geq 3$, we have $\rho^{o}(S_p^n) \leq p^{n-1}$. Hence, $\rho^{o}(S_p^n) = p^{n-1}$, and moreover, every color class of any $\chi_i(S_p^n)$-coloring has $\rho^{o}(S_p^n)$ vertices. This proves the second assertion. □

We next give a short proof of the following result which was proved first by Hinz and Parisse [17] by a lengthy argument.

Corollary 3.2. *If $p \geq 2$ and $n \geq 2$, then S_p^n is a Class 1 graph.*

Proof. The case $p = 2$ is clear since $S_2^n \cong P_{2^n}$. So, assume in the rest that $p \geq 3$. By Theorem 3.1 we have $\chi_i(S_p^n) = p = \Delta(S_p^n)$, which by Lemma 2.2 immediately gives the conclusion. □

Gravier, Kovše, and Parreau [13] defined generalized Sierpiński graphs S_p^n as follows. Let G be an arbitrary graph. Then the *generalized Sierpiński graph* S_G^n is the graph with the vertex set $V(G)^n$, where two vertices $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are adjacent if there exists an $i \in [n]$ such that $u_j = v_j$ for $j < i$, $u_i v_i \in E(G)$, and $u_j = v_i$ and $v_j = u_i$ for $j > i$. See [22] where the packing coloring of generalized Sierpiński graphs was investigated.

Clearly, $\chi_i(S_G^n) \leq |V(G)|$ follows from Theorem 3.1 and the fact that S_G^n is a spanning subgraph of S_p^n, where $p = |V(G)|$. The next result shows that this bound need not be sharp. In other words, Theorem 3.1 does not have a counterpart in generalized Sierpiński graphs.

Proposition 3.3. *If $n \geq 2$, then $\chi_i(S_{C_4}^n) = 3$ and $S_{C_4}^n$ is not perfect injectively colorable.*

Proof. Let $n \geq 2$. Then $\Delta(S_{C_4}^n) = 3$, which implies that $\chi_i(S_{C_4}^n) \geq 3$. To see that $\chi_i(S_{C_4}^n) \leq 3$ consider the labeling of $S_{C_4}^2$ as presented in the left-hand side of Fig. 4.

The labeling of $S_{C_4}^2$ from Fig. 4 is easily checked to be injective. Now we iteratively proceed as indicated in the figure, that is, we four times use the labeling of $S_{C_4}^2$ to get a labeling of $S_{C_4}^3$. Based on the distribution of the color 3, it is straightforward to check that also the labeling of $S_{C_4}^3$ is injective. The process can be repeated to get the desired conclusion.

Finally, notice that $S_{C_4}^n$ is not perfect injectively colorable because $|V(S_{C_4}^n)| = 4^n$, which is not divisible by $\chi_i(S_{C_4}^n) = 3$ for $n \geq 2$. □

Using similar approach as in the proof of Proposition 3.3 one might deduce that if $n \geq 2$ and $k \geq 5$, then $\chi_i(S_{C_k}^n) = 3$. Moreover, one might also deduce that among all these generalized Sierpiński graphs over cycles only $S_{C_6}^n$ is perfect injectively colorable.

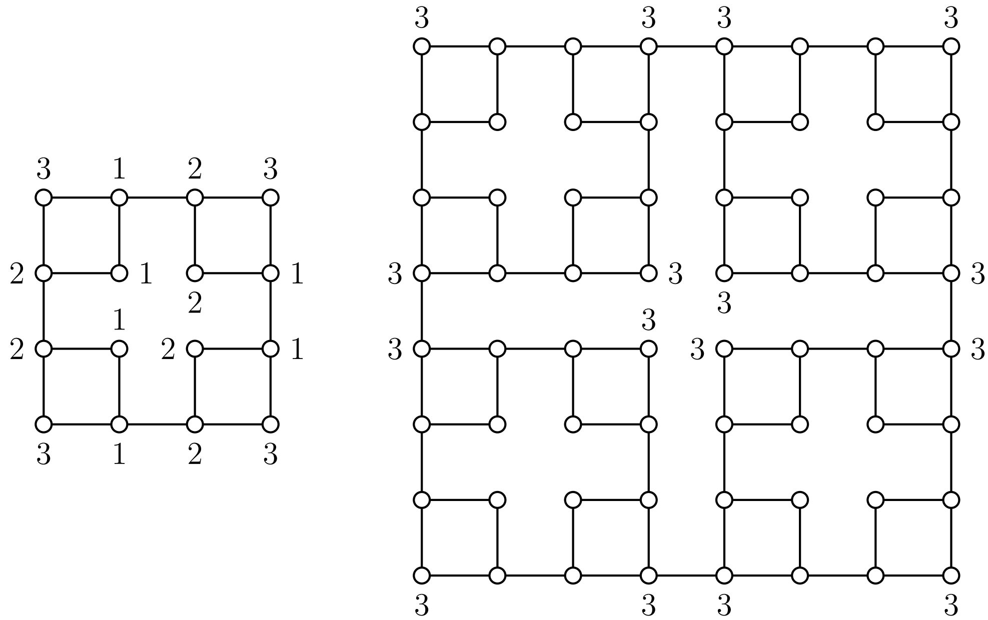

Figure 4: An injective coloring of $S^2_{C_4}$ (left) and its lift up to an injective coloring of $S^3_{C_4}$ (right)

4 Rooted product graphs

A *rooted graph* is a graph in which one vertex is labeled in a special way to distinguish it from other vertices. This vertex is called the *root* of the graph. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. Let $\mathcal{H}$ be a sequence of n rooted graphs $H_1, \ldots, H_n$. The *rooted product graph* $G(\mathcal{H})$ is the graph obtained by identifying the root of H_i with v_i (see [12]). We here consider the particular case of rooted product graphs where $\mathcal{H}$ consists of n isomorphic rooted graphs [32]. More formally, assuming that the root of H is v, we define the rooted product graph $G \circ_v H = (V, E)$, where $V = V(G) \times V(H)$ and

$$E = \left(\bigcup_{i=1}^{n} \{(v_i, h)(v_i, h') \mid hh' \in E(H)\} \right) \bigcup \Big\{ (v_i, v)(v_j, v) \mid v_i v_j \in E(G) \Big\}.$$

We remark that rooted product graphs can be seen as an instance of the operation called Sierpiński product introduced in [23], and denoted by $G \otimes_f H$, where $f\colon V(G) \to V(H)$ is a function. The graph $G \otimes_f H$ has vertex set $V(G) \times V(H)$ and two vertices $(g, h), (g', h') \in V(G \otimes_f H)$ are adjacent if (i) $g = g'$ and $hh' \in E(H)$, or (ii) $gg' \in E(G)$, $h = f(g')$ and $h' = f(g)$. In this sense, it can be readily seen that a rooted product graph $G \circ_v H$ represents a Sierpiński product $G \otimes_f H$, where f is a constant function in the product, i.e., $f(u) = v$ for any $u \in V(G)$.

It is somehow natural to think that the injective chromatic number of a rooted product graph relates to that of the factors of the product. Indeed, the following basic bounds can be easily deduced for any graph G and any rooted graph H with root v.

$$\max\{\chi_i(G), \chi_i(H)\} \leq \chi_i(G \circ_v H) \leq \chi_i(G) + \chi_i(H). \tag{1}$$

Both bounds above are realizable. However, not all possible values between these bounds are reached. We next focus on these facts and show that $\chi_i(G \circ_v H)$ achieves only six values from the interval $[\max\{\chi_i(G), \chi_i(H)\}, \ldots, \chi_i(G) + \chi_i(H)]$.

From now on, in order to facilitate our exposition, given an integer $k \in [n]$, by F_k we represent the subgraph of $G \circ_v H$ induced by vertices in $V(G) \cup V(H_k)$.

To show that only six values from the interval $[\max\{\chi_i(G), \chi_i(H)\}, \ldots, \chi_i(G) + \chi_i(H)]$ (according to the bounds from (1)) can be realized, we exhibit a closed formula for the injective chromatic number of rooted product graphs. To do so, we proceed with a series of lemmas that are giving the values of the injective chromatic number of F_k, under some conditions happening in G and in H. For the sake of convenience, by assigning/giving a color to a vertex subset S of a graph G we mean assigning such color to all vertices in S.

We first remark that the root $v \in V(H)$ is identified with $v_k \in V(G)$. Hence, for instance $d_G(v) = d_G(v_k)$. Throughout the remainder of this section, we consider g as a $\chi_i(G)$-coloring, and in this sense, $\{U_1, \ldots, U_{\chi_i(G)}\}$ as the vertex partition of $V(G)$ into open packings associated with g. We may assume that g assigns the color i to U_i for each $i \in [\chi_i(G)]$. Now, in order to consider an injective coloring of F_k for each $k \in [n]$, in concordance with the $\chi_i(G)$-coloring g, we may also assume that $N_G(v) \subseteq U_1 \cup \ldots \cup U_{d_G(v)}$. Moreover, by simplicity we write $H = H_k$ in the proofs of the following lemmas, as $H_k \cong H$.

We first observe that such subgraph F_k satisfies that

$$\chi_i(F_k) \geq \max\left\{\chi_i(G), \chi_i(H), d_G(v) + d_H(v)\right\} \tag{2}$$

as both G and H are subgraphs of F_k and $\chi_i(F_k) \geq \Delta(F_k) \geq d_G(v) + d_H(v)$. The proof of this is based on a case-by-case analysis. The general procedure is to extend g to F_k in such a way that the restriction of the resulting function to $V(H)$ turns out to be a $\chi_i(H)$-coloring using the colors in $[\chi_i(G)]$ along with the least number of colors not in $[\chi_i(G)]$.

Our first lemma regarding the injective chromatic number of the graph F_k reads as follows.

Lemma 4.1. *If $g(v) \notin [d_G(v)]$ and $h(v) \in h\big(N_H(v)\big)$ for some $\chi_i(H)$-function $h = (V_1, \ldots, V_{\chi_i(H)})$, then*

$$\chi_i(F_k) = \max\big\{\chi_i(G), \chi_i(H), d_G(v) + d_H(v)\big\}.$$

Proof. We first observe that $\chi_i(G) \geq d_G(v)+1$ since $g(v) \notin [d_G(v)]$ (i.e., v and all its neighbors have different colors). We need to distinguish two cases depending on the behaviors of $\chi_i(G)$ and $\chi_i(H)$. Without loss of generality, for the graph $H = H_k$, assume that $N_H(v) \subseteq V_1 \cup \ldots \cup V_{d_H(v)}$ and that $v \in V_1$. Also, for the graph G, assume that $v \in U_{d_G(v)+1}$.

Case 1. $\chi_i(G) \geq d_H(v) + d_G(v)$.
If $\chi_i(H) = d_H(v)$, then we extend g to F_k by respectively assigning the colors $d_G(v)+1, \ldots, d_G(v)+d_H(v)$ to $V_1, \ldots, V_{d_H(v)}$. Note that the resulting function g_1 is an injective coloring of F_k with $\chi_i(G)$ colors. Therefore, $\chi_i(F_k) \leq \chi_i(G)$, which is indeed an equality since G is a subgraph of F_k. Now, if $\chi_i(H) > d_H(v)$, then we proceed as follows. If $\eta = \chi_i(G) - d_G(v) - d_H(v) \geq \chi_i(H) - d_H(v) = \varphi$, then we extend g_1 to a new function g_2, by respectively giving the colors $d_G(v)+d_H(v)+1, \ldots, d_G(v)+\chi_i(H)$ to $V_{d_H(v)+1}, \ldots, V_{\chi_i(H)}$. It is readily seen that g_2 is an injective coloring of F_k with $\chi_i(G)$ colors. This similarly results in the equality $\chi_i(F_k) = \chi_i(G)$. Hence, assume $\eta < \varphi$ and consider two cases depending on η.

Subcase 1.1. $\eta = 0$.
If $\varphi \leq d_G(v)$, then we assign φ colors from $[d_G(v)]$ to V_i for $i = d_H(v)+1, \ldots, \chi_i(H)$. This gives us an injective coloring of F_k using $\chi_i(G)$ colors, and hence $\chi_i(F_k) = \chi_i(G)$. If $\varphi > d_G(v)$, then we assign the colors from $[d_G(v)]$ to V_i with $i = d_H(v) + 1, \ldots, d_H(v) + d_G(v)$. In addition, we need $\chi_i(H) - d_G(v) - d_H(v)$ new colors for the rest of open packings in H. This results in an injective coloring of F_k with $\chi_i(H) - d_G(v) - d_H(v) + \chi_i(G) = \chi_i(H)$ colors, and therefore $\chi_i(F_k) = \chi_i(H)$.

Subcase 1.2. $\eta > 0$.
Let g_2 be an extension of g_1 that respectively assigns the values $d_G(v) + d_H(v) + 1, \ldots, \chi_i(G)$ to $V_{d_H(v)+1}, \ldots, V_{\chi_i(G)-d_G(v)}$. In view of this, $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{\chi_i(H)}$ are open packings of H which have not been injectively colored by g_2. We need to consider two possibilities.

Subcase 1.2.1. $\chi_i(G) \geq \chi_i(H)$.
It follows that $\zeta = \chi_i(H) - \chi_i(G) + d_G(v) \leq d_G(v)$. In such a situation, g_2 can be extended to F_k by assigning the colors $1, \ldots, \zeta$ to the rest of open packings in H. Note that the resulting function is an injective coloring of F_k with $\chi_i(G)$ colors, and hence $\chi_i(F_k) = \chi_i(G)$.

Subcase 1.2.2. $\chi_i(G) < \chi_i(H)$.
This shows that $\zeta > d_G(v)$. In such a situation, we first assign the color i to

$V_{\chi_i(G)-d_G(v)+i}$ for each $i \in [d_G(v)]$. We next assign $\chi_i(H) - \chi_i(G)$ new colors to the rest of open packings in H. This results in an injective coloring of F_k with $\chi_i(H)$ colors, and hence $\chi_i(F_k) = \chi_i(H)$.

Case 2. $\chi_i(G) < d_H(v) + d_G(v)$.
We extend g to g_1 by respectively assigning the colors $d_G(v) + 1, \ldots, \chi_i(G)$ to $V_1, \ldots, V_{\chi_i(G)-d_G(v)}$, as well as, we assign $d_H(v) - \chi_i(G) + d_G(v)$ new colors to $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{d_H(v)}$. If $\chi_i(H) = d_H(v)$, then g_1 turns out to be an injective coloring of F_k with $d_G(v) + d_H(v)$ colors. Therefore, $\chi_i(F_k) = d_H(v) + d_G(v)$. Suppose now that $\chi_i(H) > d_H(v)$. We distinguish two more possibilities.
Subcase 2.1. $\chi_i(H) \leq d_G(v) + d_H(v)$.
In such situation, the function g_1 can be extended to F_k by respectively assigning the colors $1, \ldots, \chi_i(H) - d_H(v)$ to $V_{d_H(v)+1}, \ldots, V_{\chi_i(H)}$. This gives us an injective coloring of F_k with $d_H(v) + d_G(v)$ colors. So, $\chi_i(F_k) = d_H(v) + d_G(v)$.
Subcase 2.2. $\chi_i(H) > d_G(v) + d_H(v)$.
Now, in order to extend g_1 to F_k, we respectively assign the colors $1, \ldots, d_G(v)$ to $V_{d_H(v)+1}, \ldots, V_{d_H(v)+d_G(v)}$, as well as, $\chi_i(H) - d_G(v) - d_H(v)$ new colors to the rest of open packings in H. This leads to an injective coloring of F_k with $\chi_i(H)$ colors, and hence $\chi_i(F_k) = \chi_i(H)$.

One might think that in order to complete our proof, some other cases like for instance $\chi_i(H) \leq d_H(v) + d_G(v)$ or $\chi_i(H) > d_H(v) + d_G(v)$ need to be considered. However, they are indeed implicitly checked in the two cases above (with the corresponding subcases).

In conclusion, we have proved that $\chi_i(F_k) \in \{\chi_i(G), \chi_i(H), d_G(v)+d_H(v)\}$. This leads to $\chi_i(F_k) = \max\{\chi_i(G), \chi_i(H), d_G(v) + d_H(v)\}$ due to (2). $\square$

From this point on, three similar lemmas to the one above shall be proved. These lemmas are considering the remaining cases regarding the inclusion or not of $g(v)$ and $h(v)$ in $[d_G(v)]$ and $h\big(N_H(v)\big)$ (for some/each $\chi_i(H)$-function h), respectively. Some of the arguments are similar to the ones in the proof of Lemma 4.1.

Lemma 4.2. *If* $g(v) \notin [d_G(v)]$ *and* $h(v) \notin h\big(N_H(v)\big)$ *for each* $\chi_i(H)$*-function* h*, then*

$$\chi_i(F_k) \in \big\{\chi_i(G), \chi_i(H), d_G(v) + d_H(v), d_G(v) + d_H(v) + 1\big\}.$$

Proof. The assumptions imply that $\chi_i(G) \geq d_G(v) + 1$ and $\chi_i(H) \geq d_H(v) + 1$. We need to distinguish two cases depending on $\chi_i(G)$, $d_G(v)$ and $d_H(v)$. For the sake of simplicity, we assume $N_H(v) \subseteq V_1 \cup \ldots \cup V_{d_H(v)}$, $v \in V_{d_H(v)+1}$ and $v \in U_{d_G(v)+1}$.

Case 1. $\chi_i(G) \geq d_G(v) + d_H(v) + 1$.
If $\chi_i(H) = d_H(v) + 1$, then the extension g_1 of g that respectively assigns the colors $d_G(v) + 1, d_G(v) + 2, \ldots, d_G(v) + d_H(v) + 1$ to $V_{d_H(v)+1}, V_1, \ldots, V_{d_H(v)}$ defines an injective coloring of F_k using $\chi_i(G)$ colors. This leads to $\chi_i(F_k) = \chi_i(G)$. Suppose now that $\chi_i(H) > d_H(v) + 1$. If

$$\mu = \chi_i(G) - d_G(v) - d_H(v) - 1 \geq \chi_i(H) - d_H(v) - 1 = \xi,$$

then an extension g_2 of g_1 assigning the color i to $V_{i-d_G(v)}$, for each $i = d_G(v) + d_H(v) + 2, \ldots, d_G(v) + \chi_i(H)$, defines an injective coloring of F with $\chi_i(G)$ colors. Hence, $\chi_i(F_k) = \chi_i(G)$. Letting $\mu < \xi$, we need to consider two more possibilities.

Subcase 1.1. $\mu = 0$.
First note that the ξ open packings $V_{d_H(v)+2}, \ldots, V_{\chi_i(H)}$ of H have not been colored under g_1. If $\xi \leq d_G(v)$, then we respectively assign the colors $1, \ldots, \xi$ to $V_{d_H(v)+2}, \ldots, V_{\chi_i(H)}$. Note that the resulting function is an injective coloring with $\chi_i(G)$ colors. So, $\chi_i(F_k) = \chi_i(G)$. If $\xi > d_G(v)$, then we first respectively assign the colors $1, \ldots, d_G(v)$ to $V_{d_H(v)+2}, \ldots, V_{d_H(v)+d_G(v)+1}$. Next, we assign $\xi - d_G(v)$ new colors to the rest of the open packings of H. The resulting function turns out to be an injective coloring of F_k using

$$\chi_i(G) + \xi - d_G(v) = \chi_i(G) + \chi_i(H) - d_H(v) - 1 - d_G(v) = \chi_i(H)$$

colors. So, we deduce that $\chi_i(F_k) \leq \chi_i(H)$, which means $\chi_i(F_k) = \chi_i(H)$ in view of the inequality (2).

Subcase 1.2. $\mu > 0$.
As an extension of g_1, we first assign the color i to $V_{i-d_G(v)}$ when $i \in \{d_G(v)+d_H(v)+2, \ldots, \chi_i(G)\}$. In this situation, the $\xi - \mu$ open packings $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{\chi_i(H)}$ have not been colored under g_1. We now distinguish two possibilities.

Subcase 1.2.1. $\chi_i(G) \geq \chi_i(H)$. This implies that $\xi - \mu \leq d_G(v)$. So, by respectively assigning the colors $1, \ldots, \xi - \mu$ to $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{\chi_i(H)}$, we obtain an injective coloring of F_k with $\chi_i(G)$ colors. Therefore, $\chi_i(F_k) = \chi_i(G)$.

Subcase 1.2.2. $\chi_i(G) < \chi_i(H)$.
This shows that $\xi - \mu > d_G(v)$. In this situation, we respectively assign the values $1, \ldots, d_G(v)$ to $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{\chi_i(G)}$. Also, we assign $\chi_i(H) - \chi_i(G)$ new colors to the rest of open packings in H. Note that the resulting coloring of F_k is injective, and that it uses $\chi_i(H)$ colors. Hence, $\chi_i(F_k) = \chi_i(H)$.

Case 2. $\chi_i(G) < d_H(v) + d_G(v) + 1$.
Consider first that $\chi_i(H) = d_H(v) + 1$. In such a situation, let g_1 be an extension of g that respectively assigns the colors $d_G(v) + 1, d_G(v) + 2, \ldots, \chi_i(G)$ to $V_{d_H(v)+1}, V_1, \ldots, V_{\chi_i(G)-d_G(v)-1}$, as well as, $\varphi = \chi_i(H) - \chi_i(G) + d_G(v)$ new colors

to $V_{\chi_i(G)-d_G(v)}, \dots, V_{\chi_i(H)}$. This defines an injective coloring of F_k with $\varphi + \chi_i(G) = d_G(v) + d_H(v) + 1$ colors, and hence $\chi_i(F_k) \leq d_G(v) + d_H(v) + 1$. This shows that $\chi_i(F) \in \{d_G(v) + d_H(v), d_G(v) + d_H(v) + 1\}$ due to (2). Suppose now that $\chi_i(H) > d_H(v) + 1$. We need to consider two possibilities.

Subcase 2.1. $\chi_i(G) \geq \chi_i(H)$.
Due to the initial inequality of Case 2, we get $d_G(v) + d_H(v) + 1 > \chi_i(H)$. Therefore, $d_G(v) > \chi_i(H) - d_H(v) - 1 = \psi$. Hence, g_1 can be extended to F_k by respectively assigning the colors $1, \dots, \psi$ to $V_{d_H(v)+2}, \dots, V_{\chi_i(H)}$. This gives an injective coloring of F_k with $d_G(v) + d_H(v) + 1$ colors. Consequently, $\chi_i(F_k) \in \{d_G(v) + d_H(v), d_G(v) + d_H(v) + 1\}$.

Subcase 2.2. $\chi_i(G) < \chi_i(H)$.
If $\psi \leq d_G(v)$, then we have the same conclusion as in Subcase 2.1. So, let $\psi > d_G(v)$. Let g_2 be an extension of g_1 that assigns the color i to $V_{i+d_H(v)+1}$ for each $i \in [d_G(v)]$. In such a situation, we give $\psi - d_G(v)$ new colors to the rest of the open packings in H. This process leads to an injective coloring of F_k with at most $d_G(v) + d_H(v) + 1 + \psi - d_G(v)$ colors. Therefore, $\chi_i(F_k) \leq \chi_i(H)$. This implies that $\chi_i(F_k) = \chi_i(H)$. □

Lemma 4.3. *Let $g(v) \in [d_G(v)]$ and $h(v) \in h\big(N_H(v)\big)$ for some $\chi_i(H)$-function $h = (V_1, \dots, V_{\chi_i(H)})$. Then,*

$$\chi_i(F_k) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G) + 1, \chi_i(H) + 1, d_G(v) + d_H(v)\big\}.$$

Proof. With the assumptions given in the statement of the lemma, we may assume that $N_H(v) \subseteq V_1 \cup \ldots \cup V_{d_H(v)}$, $v \in V_1$ and $v \in U_1$. We distinguish two cases depending on $\chi_i(G)$, $d_G(v)$ and $d_H(v)$.

Case 1. $\chi_i(G) \geq d_H(v) + d_G(v)$.
Let $\chi_i(H) = d_H(v)$. In such a situation, assume g_1 respectively assigns the colors $d_G(v) + 1, \dots, d_G(v) + d_H(v)$ to $V_1 \setminus \{v\}, V_2, \dots, V_{d_H(v)}$. This results in the existence of an injective coloring of F_k with $\chi_i(G)$ colors, and hence $\chi_i(F_k) = \chi_i(G)$. Now let $\chi_i(H) > d_H(v)$. If $\vartheta = \chi_i(G) - d_G(v) - d_H(v) \geq \chi_i(H) - d_H(v) = \epsilon$, then we consider g_1 is extended to g_2 by assigning i to $V_{i-d_G(v)}$ for each $i = d_G(v) + d_H(v) + 1, \dots, d_G(v) + \chi_i(H)$. This defines an injective coloring of F_k using $\chi_i(G)$ colors. Hence, $\chi_i(F_k) = \chi_i(G)$. Letting $\vartheta < \epsilon$ we need to consider two more cases depending on ϑ.

Subcase 1.1. $\vartheta = 0$.
If $\epsilon \leq d_G(v) - 1$, then in order to extend g_1, we respectively assign the colors $2, \dots, \epsilon + 1$ to $V_{d_H(v)+1}, \dots, V_{\chi_i(H)}$. Note that the resulting function is an injective coloring of F_k using $\chi_i(G)$ colors, and hence $\chi_i(F_k) = \chi_i(G)$. Suppose now that

$\epsilon \geq d_G(v)$. As an extension of g_1, we first respectively assign the colors $2, \dots, d_G(v)$ to $V_{d_H(v)+1}, \dots, V_{d_H(v)+d_G(v)-1}$. We next give $\epsilon - d_G(v) + 1$ new colors to the rest of open packings in H. The resulting function is an injective coloring of F_k using

$$\chi_i(G) + \epsilon - d_G(v) + 1 = \chi_i(G) + \chi_i(H) - d_H(v) - d_G(v) + 1 = \chi_i(H) + 1 \tag{3}$$

colors (since $\chi_i(G) - d_G(v) - d_H(v) = \vartheta = 0$). We infer, in this case, that $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H) + 1\}$ due to (2).

Subcase 1.2. $\vartheta > 0$.
Let g_2 be an extension of g_1 such that the color i is assigned to $V_{i-d_G(v)}$ for each $i = d_G(v)+d_H(v)+1, \dots, \chi_i(G)$. In such a situation, $\varsigma = \chi_i(H)-\chi_i(G)+d_G(v) \geq 1$ open packings in H have not been colored under g_2. We consider two cases depending on $\chi_i(G)$ and $\chi_i(H)$.

Subcase 1.2.1. $\chi_i(G) \geq \chi_i(H)$.
This shows that $\varsigma \leq d_G(v)$. Let $\varsigma \leq d_G(v) - 1$. In such a situation, g_2 can be extended to g_3 by assigning ς colors from $\{2, \dots, d_G(v)\}$ to the rest of open packings in H. The resulting coloring is injective and uses $\chi_i(G)$ colors. So, $\chi_i(F_k) = \chi_i(G)$. If $\varsigma = d_G(v)$, then g_3 can be extended to F_k by assigning a new color to the last open packing in H. Therefore, $\chi_i(F_k) \in \{\chi_i(G), \chi_i(G) + 1\}$.

Subcase 1.2.2. $\chi_i(H) > \chi_i(G)$.
We then have $\varsigma > d_G(v)$. In this situation, let g_3 be an extension of g_2 that respectively assigsn the colors $2, \dots, d_G(v)$ to $V_{\chi_i(G)-d_G(v)+1}, \dots, V_{\chi_i(G)-1}$. We now assign $\chi_i(H) - \chi_i(G) + 1$ new colors to the remaining open packings in H. This leads to the existence of an injective coloring of F_k with $\chi_i(H) + 1$ colors. Therefore, $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H) + 1\}$.

Case 2. $\chi_i(G) < d_H(v) + d_G(v)$.
Let first $\sigma = \chi_i(G) - d_G(v) = 0$. Assume now that $\chi_i(H) = d_H(v)$. Let g_1 be an extension of g which respectively assigns the colors $d_G(v) + 1, d_G(v) + 2, \dots, d_H(v)$ to $V_1 \setminus \{v\}, V_2, \dots, V_{d_H(v)}$. Hence, g_1 is an injective coloring of F_k with $d_G(v)+d_H(v)$ colors, and so $\chi_i(F_k) = d_G(v) + d_H(v)$. Suppose now that $\chi_i(H) > d_H(v)$. If $\chi_i(H) < d_G(v) + d_H(v)$, then g_1 can be extended by assigning $\chi_i(H) - d_H(v)$ colors from $\{2, \dots, d_G(v)\}$ to the rest of open packings in H. This gives an injective coloring of F_k using $d_G(v) + d_H(v)$ colors, and therefore $\chi_i(F_k) = d_G(v) + d_H(v)$. Now let $\chi_i(H) \geq d_G(v) + d_H(v)$. In such a situation, as an extension of g_1, we first respectively give $2, \dots, d_G(v)$ colors to $V_{d_H(v)+1}, \dots, V_{d_G(v)+d_H(v)-1}$. We next assign $\chi_i(H) - d_G(v) - d_H(v) + 1$ new colors to the rest of open packings in H. This leads to an injective coloring of F_k with $\chi_i(H) + 1$ colors, and therefore $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H) + 1\}$.

Consider now that $\sigma > 0$. Note that g can be extended to a function g_1 by respectively assigning $d_G(v)+1, d_G(v)+2, \ldots, \chi_i(G)$ to $V_1 \setminus \{v\}, V_2, \ldots, V_\sigma$, as well as, $d_H(v)-\sigma$ new colors to $V_{\sigma+1}, \ldots, V_{d_H(v)}$. If $\chi_i(H) = d_H(v)$, then this defines an injective coloring of F_k using $\chi_i(G) + d_H(v) - \sigma = d_G(v) + d_H(v)$ colors. Therefore, $\chi_i(F_k) = d_G(v) + d_H(v)$. So, let $\chi_i(H) > d_H(v)$. Again, we need to consider two more possibilities.

Subcase 2.1. $\chi_i(H) < d_G(v) + d_H(v)$.
In view of this, let g_2 be an extension of g_1 to F_k by giving $\chi_i(H) - d_H(v)$ colors from $\{2, \ldots, d_G(v)\}$ to the rest of open packings in H. This process injectively colors F_k by $d_G(v) + d_H(v)$ colors. So, we again have $\chi_i(F_k) = d_G(v) + d_H(v)$.

Subcase 2.2. $\chi_i(H) \geq d_G(v) + d_H(v)$.
Respectively assigning the colors $2, \ldots, d_G(v)$ to $V_{d_H(v)+1}, \ldots, V_{d_H(v)+d_G(v)-1}$, as well as, $\chi_i(H) - d_G(v) - d_H(v) + 1$ new colors to the rest of open packings in H, we obtain an extension of g_1 to F_k. It is easy to see that the resulting function is an injective coloring using $\chi_i(H) + 1$ colors. Therefore, $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H)+1\}$. □

Lemma 4.4. *Let $g(v) \in [d_G(v)]$ and $h(v) \notin h\big(N_H(v)\big)$ for each $\chi_i(H)$-function h. Then,*

$$\chi_i(F_k) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G)+1, \chi_i(H)+1, d_G(v)+d_H(v), d_G(v)+d_H(v)+1\big\}.$$

Proof. We first observe, by the assumption given in the statement of the lemma, that $\chi_i(H) \geq d_H(v)+1$. For the sake of simplicity, we let $N_H(v) \subseteq V_1 \cup \ldots \cup V_{d_H(v)}$, $v \in V_{d_H(v)+1}$ and $v \in U_1$. We again need to distinguish two possibilities depending on $\chi_i(G)$, $d_G(v)$ and $d_H(v)$.

Case 1. $\chi_i(G) \geq d_H(v) + d_G(v) + 1$.
Assume first that $\chi_i(H) = d_H(v)+1$. The extension g_1, of g, that respectively assigns $d_G(v)+1, \ldots, d_G(v)+d_H(v)+1$ to $V_1, \ldots, V_{d_H(v)+1} \setminus \{v\}$ is an injective coloring of F_k with $\chi_i(G)$ colors. Thus, $\chi_i(F_k) = \chi_i(G)$ (note that if $V_{d_H(v)+1} \setminus \{v\} = \emptyset$, then the color $d_G(v)+d_H(v)+1$ is not used in H). So, let $\chi_i(H) > d_H(v)+1$. Again, we need to consider two more cases.

Subcase 1.2. $\lambda = \chi_i(G) - d_G(v) - d_H(v) - 1 \geq \chi_i(H) - d_H(v) - 1 = \varepsilon$.
We observe that a function g_2 defined, as an extension of g_1, by assigning the color i to $V_{i-d_G(v)}$ for every $i = d_G(v)+d_H(v)+2, \ldots, \chi_i(H)+d_G(v)$ is an injective coloring of F_k with $\chi_i(G)$ colors. Therefore, $\chi_i(F_k) = \chi_i(G)$.

Subcase 2.2. $\lambda < \varepsilon$.
Suppose first that $\lambda = 0$. There exist two possibilities depending on $\chi_i(G)$ and $\chi_i(H)$.

Subcase 2.2.1. $\chi_i(G) \geq \chi_i(H)$.
This implies that $\varepsilon \leq d_G(v)$ by taking $\lambda = 0$ into account. If $\varepsilon \leq d_G(v) - 1$, then an extension of g_1 that assigns ε colors from $\{2, \ldots, d_G(v)\}$ to $V_{d_H(v)+2}, \ldots, V_{\chi_i(H)}$ gives an injective coloring of F_k with $\chi_i(G)$ colors. Therefore, $\chi_i(F_k) = \chi_i(G)$. Now let $\varepsilon = d_G(v)$. In such a situation, we first respectively assign $2, \ldots, d_G(v)$ colors to $V_{d_H(v)+2}, \ldots, V_{\chi_i(H)-1}$, and a new color to $V_{\chi_i(H)}$. The resulting function is an injective coloring of F_k with $\chi_i(G)+1$ colors. Therefore, $\chi_i(F_k) \in \{\chi_i(G), \chi_i(G)+1\}$.

Subcase 2.2.1. $\chi_i(H) > \chi_i(G)$.
We then have $\varepsilon > d_G(v)$ since $\lambda = 0$. Let g_2 be an extension of g_1 that respectively assigns $2, \ldots, d_G(v)$ to $V_{d_H(v)+2}, \ldots, V_{d_G(v)+d_H(v)}$. Notice that $\chi_i(H) - d_G(v) - d_H(v)$ open packings in H have not received colors under g_2. In such a situation, we obtain an injective coloring of F_k with $\chi_i(H) + 1$ colors by assigning $\chi_i(H) - d_G(v) - d_H(v)$ new colors to the remaining open packings. Hence, $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H) + 1\}$.

Assume now that $\lambda > 0$. Let g_2 be an extension of g_1 that assigns the color i to $V_{i-d_G(v)}$ when $i \in \{d_G(v)+d_H(v)+2, \ldots, \chi_i(G)\}$. We note that $\upsilon = \chi_i(H) - \chi_i(G) + d_G(v)$ open packings in H have not received colors under g_2. If $\chi_i(H) \geq \chi_i(G)$, then we respectively assign the colors $2, \ldots, d_G(v)$ to $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{\chi_i(G)-1}$. Also, we give $\chi_i(H) - \chi_i(G) + 1$ new colors to the rest of open packings in H. This leads to an injective coloring of F_k using $\chi_i(H)+1$ colors, and hence $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H)+1\}$.

On the other hand, if $\chi_i(G) > \chi_i(H)$, then $\upsilon < d_G(v)$. In such a case, g_2 can be extended to F_k by assigning υ colors from $\{2, \ldots, d_G(v)\}$ to the rest of open packings in H. This defines an injective coloring of F_k with $\chi_i(G)$ colors, and therefore $\chi_i(F_k) = \chi_i(G)$.

Case 2. $\chi_i(G) < d_H(v) + d_G(v) + 1$.
We need to distinguish two more possibilities depending on $\chi_i(G) - d_G(v)$.

Subcase 2.1. $\chi_i(G) = d_G(v)$.
If $\chi_i(H) = d_H(v) + 1$, then assume g_1 respectively assigns the colors $d_G(v) + 1, \ldots, d_G(v) + d_H(v) + 1$ to $V_1, \ldots, V_{d_H(v)+1} \setminus \{v\}$ (if $V_{d_H(v)+1} \setminus \{v\} = \emptyset$, then the color $d_G(v) + d_H(v) + 1$ is not used in H). This gives an injective coloring of F_k using at most $d_G(v) + d_H(v) + 1$ colors. This shows that $\chi_i(F_k) \in \{d_G(v)+d_H(v), d_G(v)+d_H(v)+1\}$. Let $\chi_i(H) > d_H(v)+1$. If $\varepsilon < d_G(v)$, then g_1 can be extended as an injective coloring of F_k by assigning ε colors from $\{2, \ldots, d_G(v)\}$ to $V_{d_H(v)+2}, \ldots, V_{\chi_i(H)}$. Therefore, $\chi_i(F_k) \in \{d_G(v) + d_H(v), d_G(v) + d_H(v) + 1\}$. So, we let $\varepsilon \geq d_G(v)$. Let g_2 be an extension of g_1 that respectively assigns the colors $2, \ldots, d_G(v)$ to $V_{d_H(v)+2}, \ldots, V_{d_G(v)+d_H(v)}$. We now give $\chi_i(H) - d_G(v) - d_H(v)$ new colors to the rest of open packings in H. This process ends with an injective coloring of F_k using $\chi_i(H) + 1$ colors, and hence $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H) + 1\}$.

Subcase 2.1. $\chi_i(G) > d_G(v)$.

Let $\chi_i(H) = d_H(v) + 1$. Assume g_1 is an extension of g that respectively assigns $d_G(v)+1, \ldots, \chi_i(G)$ to $V_1, \ldots, V_{\chi_i(G)-d_G(v)}$. We then give at most $\chi_i(H) - \chi_i(G) + d_G(v)$ new colors to $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{d_H(v)+1} \setminus \{v\}$ (trivially, $V_{d_H(v)+1} \setminus \{v\}$ does not receive any color if $V_{d_H(v)+1} = \{v\}$). This results in an injective coloring of F_k with at most $d_G(v)+d_H(v)+1$ colors. Therefore, $\chi_i(F_k) \in \{d_G(v)+d_H(v), d_G(v)+d_H(v)+1\}$.

Let $\chi_i(H) > d_H(v) + 1$. Consider now that g_2 extends g_1 by giving $d_H(v) + 1 - \chi_i(G)+d_G(v)$ new colors to $V_{\chi_i(G)-d_G(v)+1}, \ldots, V_{d_H(v)+1} \setminus \{v\}$ (note that $V_{d_H(v)+1} \setminus \{v\}$ does not receive any color if $V_{d_H(v)+1} = \{v\}$). In such a situation, ε open packings in H have not received colors under g_2. If $\varepsilon < d_G(v)$, then we assign ε colors from $\{2, \ldots, d_G(v)\}$ to the rest of open packings in H. This defines an injective coloring of F_k with at most $d_G(v) + d_H(v) + 1$ colors. Hence, $\chi_i(F_k) \in \{d_G(v) + d_H(v), d_G(v) + d_H(v) + 1\}$. So, we let $\varepsilon \geq d_G(v)$. In this situation, we extend g_2 by respectively assigning the colors $2, \ldots, d_G(v)$ to $V_{d_H(v)+2}, \ldots, V_{d_G(v)+d_H(v)}$, as well as, $\chi_i(H) - d_G(v) - d_H(v)$ new colors to the rest of open packings in H. This leads to the existence of an injective coloring of F_k with at most $\chi_i(H) + 1$ colors. Thus, $\chi_i(F_k) \in \{\chi_i(H), \chi_i(H) + 1\}$. This completes the proof. □

Altogether, Lemmas 4.1–4.4 imply that

$$\chi_i(F_k) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G)+1, \chi_i(H)+1, d_G(v)+d_H(v), d_G(v)+d_H(v)+1\big\} \quad (4)$$

for each $k \in [n]$ and any graphs G and H with $v \in V(H)$.

For every $k \in [n]$, by renaming the colors assigned to $V(H_k) \setminus \{v_k\}$ if necessary, we may assume that the optimal injective coloring of F_k uses the colors from $[\chi_i(F_k)]$. Recall that such an injective coloring uses the colors from $[\chi_i(G)]$ in G.

We observe that $V(G \circ_v H) = \bigcup_{k=1}^{n} V(F_k)$ and that $V(F_i) \cap V(F_j) = V(G)$ for every distinct $i, j \in [n]$. Assume in the rest that $F \in \{F_1, \ldots, F_n\}$ has the property that

$$\chi_i(F) = \max_{k\in[n]}\{\chi_i(F_k)\}.$$

With these notations in mind, we prove the following simple but useful lemma.

Lemma 4.5. $\chi_i(F) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G) + 1, \chi_i(G) + 1, \Delta(G) + d_H(v), \Delta(G) + d_H(v) + 1\big\}$.

Proof. Since $F = F_k$ for some $k \in [n]$, we have

$$\chi_i(F) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G) + 1, \chi_i(H) + 1, d_G(v_k) + d_H(v), d_G(v_k) + d_H(v) + 1\big\}.$$

by (4). Moreover, for each vertex v_j in G of maximum degree, (4) implies that

$$\chi_i(F_j) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G)+1, \chi_i(H)+1, \Delta(G)+d_H(v), \Delta(G)+d_H(v)+1\big\} = M_\Delta.$$

Suppose to the contrary that $\chi_i(F) \notin M_\Delta$. This necessarily implies that $\chi_i(F) \geq \chi_i(G) + 2$ and that $\chi_i(F) \geq \chi_i(H) + 2$. If $\chi_i(F) = d_G(v_k) + d_H(v)$, then $d_G(v_k) + d_H(v) = \chi_i(F) \geq \chi_i(F_j) \geq \Delta(G) + d_H(v)$. This necessarily implies that $d_G(v_k) = \Delta(G)$, contradicting the supposition $\chi_i(F) \notin M_\Delta$. Therefore, $\chi_i(F) = d_G(v_k) + d_H(v) + 1$. Similarly, we have $d_G(v_k) + d_H(v) + 1 = \chi_i(F) \geq \chi_i(F_j) \geq \Delta(G) + d_H(v)$. Hence, $d_G(v_k) + 1 \in \{\Delta(G), \Delta(G)+1\}$, contradicting $\chi_i(F) \notin M_\Delta$. So, the statement of the lemma holds. □

We are now in a position to prove the main result of this section.

Theorem 4.6. *For any graph G and any graph H with root $v \in V(H)$,*

$$\chi_i(G \circ_v H) \in \big\{\chi_i(G), \chi_i(H), \chi_i(G)+1, \chi_i(H)+1, \Delta(G)+d_H(v), \Delta(G)+d_H(v)+1\big\}.$$

Proof. Note that $\chi_i(G \circ_v H) \geq \chi_i(F)$ as $F = F_k$ is a subgraph of $G \circ_v H$. Let f_j be a $\chi_i(F_j)$-coloring for each $j \in [n]$, as constructed along the proofs of Lemmas 4.1–4.4. We now define f on $V(G \circ_v H)$, as an extension of g, by $f(x) = f_j(x)$ when $x \in V(F_j)$. Notice that f is well-defined because $f_i(x) = f_j(x)$ for each $x \in V(G)$ and every distinct $i, j \in [n]$.

Suppose to the contrary that there exist distinct vertices $x, y, z \in V(G \circ_v H)$ such that $y, z \in N_{G \circ_v H}(x)$ and $f(y) = f(z)$. Since the restrictions of f to $V(G)$ and $V(H_j)$, are injective colorings for each $j \in [n]$, it follows that neither "$y, z \in V(G)$" nor "$y, z \in V(H_j)$ for some $j \in [n]$" happens. So, without loss of generality, we may assume that $z \in V(G)$ and $y \in V(H_j)$ for some $j \in [n]$. By the structure, this necessarily implies that $x = v = v_j$. This contradicts the fact that f_j is an injective coloring of F_j. So, we deduce that f is an injective coloring of $G \circ_v H$.

Recall that for every $j \in [n]$, f_j assigns the colors in $[\chi_i(F_j)]$ so as to injectively color F_j. Due to this fact, we observe that f assigns $\chi_i(F)$ colors to $V(G \circ_v H)$, and hence $\chi_i(G \circ_v H) \leq \chi_i(F)$. This leads to the desired inequality $\chi_i(G \circ_v H) = \chi_i(F) \in \{\chi_i(G), \chi_i(H), \chi_i(G)+1, \chi_i(H)+1, \Delta(G)+d_H(v), \Delta(G)+d_H(v)+1\}$ by Lemma 4.5. □

It can be readily seen that the six possible values for $\chi_i(G \circ_v H)$ given in Theorem 4.6 can indeed be presented in the following way.

Corollary 4.7. *For any graph G and any graph H with root $v \in V(H)$,*

$$\max\{\chi_i(G), \chi_i(H), \Delta(G)+d_H(v)\} \leq \chi_i(G \circ_v H) \leq \max\{\chi_i(G), \chi_i(H), \Delta(G)+d_H(v)\}+1.$$

In what follows, we show that $\chi_i(G \circ_v H)$ can indeed reach each of the six values appearing in the closed formula of Theorem 4.6, depending on our choice for G and H. Suppose first that $G \cong K_n$ on $n \geq 3$ vertices and let H be obtained from K_m, with $m \geq 3$, by joining a new vertex v to only one vertex of K_m. It is then readily checked that $\chi_i(G \circ_v H) = n = \chi_i(G)$ if $n \geq m$, and that $\chi_i(G \circ_v H) = m = \chi_i(H)$ if $m > n$. Let $G \cong K_{1,a}$ and $H \cong K_{1,b}$ for some integers $a, b \geq 1$. It is then clear that $\chi_i(G \circ_v H) = a + b = \Delta(G) + d_H(v)$, in which v is the center of H. This in particular shows that $\chi_i(G \circ_v H) = a + b = \chi_i(G) + 1$ when $b = 1$. Let $G \cong C_{4n}$ and $H \cong K_m$ for some integers $n \geq 1$ and $m \geq 3$. Recall that $\chi_i(C_{4n}) = 2$. We then have $\chi_i(G \circ_v H) = m + 1 = \chi_i(H) + 1$. Finally, let $G \cong K_n$ and $H \cong K_m$ for some integers $m, n \geq 3$, in which v is any vertex of K_m. It is easy to see that $\chi_i(G \circ_v H) = n + m - 1 = \Delta(G) + d_H(v) + 1$.

4.1 Corona products viewed as rooted products

Let G and H be graphs where $V(G) = \{v_1, \ldots, v_n\}$. The *corona product* $G \odot H$ of the graphs G and H is obtained from the disjoint union of G and n disjoint copies of H, say $H_1, \ldots, H_n$, such that $v_i \in V(G)$ is adjacent to all vertices of H_i for each $i \in [n]$. Recall that the *join* of graphs G and H, written $G \vee H$, is a graph obtained from the disjoint union G and H by adding the edges $\{gh \mid g \in V(G) \text{ and } h \in V(H)\}$.

As an immediate consequence of Theorem 4.6, we obtain the closed formula for the injective chromatic number of corona product graphs given in [30]. To do so, we need some routine observations. Let G and H have no isolated vertices. Moreover, we may assume that they are connected. We observe that $G \odot H$ is isomorphic to $G \circ_v (K_1 \vee H)$, in which the root v is the unique vertex of K_1. Due to this and the fact that $\chi_i(K_1 \vee H) = |V(H)| + 1 = d_{K_1 \vee H}(v) + 1$, Theorem 4.6 implies that $\chi_i(G \odot H)$ belongs to the set

$$\big\{\chi_i(G), |V(H)|+1, \chi_i(G)+1, |V(H)|+2, \Delta(G)+|V(H)|, \Delta(G)+|V(H)|+1\big\}. \quad (5)$$

On the other hand, it is a routine matter to see that $\chi_i(K_2 \odot H) = |V(H)| + 1 = \Delta(K_2) + |V(H)|$. In view of this, we may assume that $\Delta(G) \geq 2$. Since $\chi_i(G \odot H) \geq \Delta(G \odot H) = \Delta(G) + |V(H)|$, it follows that $|V(H)| + 1$ can be excluded from the set in (5). By a similar fashion, $|V(H)| + 2$ can also be excluded when $\Delta(G) \geq 2$.

We observe, in view of Lemmas 4.1–4.4, that the equality $\chi_i(G \odot H) = \chi_i(G) + 1$ may only occur in Lemma 4.4 (note that $h(v) \notin h\big(N_{K_1 \vee H}(v)\big)$, for each $\chi_i(K_1 \vee H)$-function h, since H has no isolated vertices). Suppose now that $\chi_i(G \odot H) = \chi_i(G)+1$. By the proof of Lemma 4.4, it only happens when $\chi_i(K_1 \vee H) > |V(H)|+1$, which is a contradiction.

Corollary 4.8. ([30]) *For any graphs G and H with no isolated vertices,*

$$\chi_i(G \odot H) \in \big\{\chi_i(G), |V(H)| + \Delta(G), |V(H)| + \Delta(G) + 1\big\}.$$

5 Kneser graphs

For positive integers n and r, where $n \geq 2r$, the *Kneser graph* $K(n,r)$ has the r-subsets of an n-set as its vertices and two vertices are adjacent in $K(n,r)$ if the corresponding sets are disjoint. Kneser graphs are among the most studies classes of graphs, since the two classical results concerning their independence and chromatic numbers were proved roughly half a century ago [11, 24]. In two recent papers [2, 9], the 2-packing numbers of Kneser graphs were studied, and we will use some results from these papers for finding the open packing numbers and discussing their perfect injectively colorability.

It is well known and easy to see that $\mathrm{diam}(K(n,r)) = 2$ if and only if $n \geq 3r-1$. This immediately gives $\rho_2(K(n,r)) = 1$ if and only if $n \geq 3r-1$. Now, we invoke the result about perfect injectively colorable graphs with diameter 2 from [4].

Proposition 5.1. ([4, Proposition 13]) *If G is a graph with $\mathrm{diam}(G) = 2$, then G is a perfect injectively colorable graph if and only if either each edge of G lies in a triangle, or there exists a perfect matching M in G such that no edge of M lies in a triangle.*

If $n \geq 3r$, then every edge of $K(n,r)$ clearly lies in a triangle, hence by Proposition 5.1, it is a perfect injectively colorable graph. Now, if $n = 3r-1$, we claim that $K(n,r)$ has a perfect matching. Indeed, one can see this by using the recent result from [27] that all Kneser graphs with the sole exception of the Petersen graph are Hamiltonian, and the fact that $\binom{3r-1}{r}$ is an even number. Thus, since $K(n,r)$ has no triangles, we infer by Proposition 5.1 again, that $K(n,r)$ is a perfect injectively colorable graph. We state the obtained remarks as follows.

Observation 5.2. *If $n \geq 3r-1$, then $K(n,r)$ is a perfect injectively colorable graph.*

In [9], the authors studied the Kneser graphs $K(n,r)$ with $n = 3r-2$, which are in a sense the closest to diameter-2 Kneser graphs. They obtained the exact values of the 2-packing number for all these Kneser graphs as follows:

$$\rho_2\big(K(3r-2,r)\big) = \begin{cases} 7 & \text{if } r = 3, \\ 5 & \text{if } r = 4, \\ 3 & \text{if } r \geq 5. \end{cases} \tag{6}$$

Let S be an open packing of $K(3r-2,r)$, and suppose that S is not a 2-packing. Therefore, there exist vertices u and v in S, which are adjacent in $K(n,r)$. Without loss of generality, let $u = [r]$ and $v = \{r+1,\ldots,2r\}$. Since $\rho(G) \leq \rho^o(G)$ for all graphs G, the equality (6) implies that there exists a vertex $w \in S \setminus \{u,v\}$. Clearly,

$d(w,u) > 2$ and $d(w,v) > 2$. In particular, $w \cap u \neq \emptyset$. Suppose that $|w \cap u| \geq 2$. Then $|w \cup u| \leq 2r-2$. Since $n = 3r-2$, we infer that there exists a vertex x which is adjacent to both w and u, a contradiction to $d(w,u) > 2$. Therefore, $|w \cap u| = 1$, and by a similar argument $|w \cap v| = 1$. This yields $\{2r+1, \ldots, 3r-2\} \subset w$. Hence, if w' is any other vertex in S, we infer that $|w \cap w'| \geq r-2$. In the case $r > 3$, this gives $|w \cap w'| \geq 2$ implying $|w \cup w'| \leq r+2$, yet this yields that w and w' have a common neighbor, which is impossible. We have thus shown that every maximum open packing is a 2-packing in $K(3r-2,r)$ for $r > 3$, and hence $\rho^o\big(K(3r-2,r)\big) = \rho_2\big(K(3r-2,r)\big)$ in this case.

If $r = 3$, then $\rho^o\big(K(3r-2,r)\big) \geq 7$ by (6). Notice that every vertex in $S \setminus \{u,v\}$ is of the form $w = \{a,b,7\}$, in which $a \in u$ and $b \in v$. Moreover, $|w \cap w'| = 1$ for any two vertices $w, w' \in S \setminus \{u,v\}$ as $d(w,w') > 2$. Hence, we can add at most three vertices to $u = \{1,2,3\}$ and $v = \{4,5,6\}$ in order to get an open packing. This contradicts that fact that $\rho^o\big(K(3r-2,r)\big) \geq 7$. In fact, we have proved that every maximum open packing in $K(3r-2,r)$ is a 2-packing. In particular, we have

$$\rho^o\big(K(3r-2,r)\big) = \begin{cases} 7 & \text{if } r = 3, \\ 5 & \text{if } r = 4, \\ 3 & \text{if } r \geq 5. \end{cases} \tag{7}$$

by the equality (6).

Note that Observation 5.2 is in a sense best possible as there exists a Kneser graph $K(n,r)$ with $n = 3r-2$, for some positive integer r, which is not perfect injectively colorable.

Proposition 5.3. *Kneser graph $K(7,3)$ is not perfect injectively colorable.*

Proof. Note that a maximum 2-packing P in $K(7,3)$ consists of seven 3-subsets of the set $[7]$, where each $i \in [7]$ appears in exactly three of these seven subsets. Thus, P corresponds to the Fano plane.

Suppose to the contrary that $K(7,3)$ admits an injective coloring such that each color class is a maximum open packing. Hence, all color classes are of cardinality 7 by (7), and by the remark preceding the proposition, we infer that every color class is a 2-packing of cardinality 7. Therefore, there exists a 2-distance coloring of $K(7,3)$ with 5 colors such that each color class has 7 vertices. In particular, we infer $\chi_2(K(7,3) = 5$. This is a contradiction as the fact that one cannot partition $V\big(K(7,3)\big)$ into five Fano planes goes back to Cayley [6]. Therefore, $K(7,3)$ is not a perfect injectively colorable graph. $\square$

We remark that the exact value $\chi_2(K(7,3)) = 6$ was proved in [20],

6 Concluding remarks

Edge clique covers have been extensively investigated so far, see the survey [31]. On the other hand, the concept of sparse edge clique covers, which turned out to be very useful for our purpose, seems to be a new notion. We believe that such notion deserves an independent interest.

In Section 3 we have briefly considered the generalized Sierpiński graphs over cycles, that is, the graphs $S^n_{C_k}$. The results presented indicate that an investigation of the injective chromatic number of generalized Sierpiński graphs deserves attention, in particular describing those that are perfect injectively colorable. Notice that this task also requires the study of the open packing number of generalized Sierpiński graphs.

In Section 4 we have considered the rooted product graphs that can be seen as an instance of the operation called Sierpiński product (see [23]). In this sense, it is of interest to continue investigating the injective chromatic number of other Sierpiński products. In addition, the open packing number of such graphs is worthy of attention.

In Section 5, we have shown that Kneser graphs $K(n, r)$ are perfect injectively colorable as soon as $n \geq 3r-1$, and that $K(7, 3)$ is not a perfect injectively colorable graph. The latter graph is the only Kneser graph for which we know that it is not perfect injectively colorable, and it would be interesting to determine for which $r > 3$ graphs $K(3r-2, r)$ are (not) perfect injectively colorable. The same question can be posed for the odd graphs (Kneser graphs of the form $K(2r+1, r)$) and Kneser graphs in general.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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