Breaking Symmetry in Graphs by Resolving Sets

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December 23, 2024

Abstract

Let $\dim(G)$ and $D(G)$ respectively denote the metric dimension and the distinguishing number of a graph G. It is proved that $D(G) \leq \dim(G) + 1$ holds for every connected graph G. Among trees, exactly paths and stars attain the bound, and among connected unicyclic graphs such graphs are t-cycles for $t \in \{3, 4, 5\}$. It is shown that for any $1 \leq n < m$, there exists a graph G with $D(G) = n$ and $\dim(G) = m$. Using the bound $D(G) \leq \dim(G) + 1$, graphs with $D(G) = n(G) - 2$ are classified.

Keywords: resolving set; metric dimension; distinguishing number; twin graph; almost asymmetric graph

AMS Math. Subj. Class. (2020): 05C12, 05C15

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1 Introduction

Let $G = (V(G), E(G))$ be a graph and $S \subseteq V(G)$. Then S is a *resolving set* if for any two vertices g_1 and g_2 of G, there exists a vertex $s \in S$ such that $\mathrm{d}_G(g_1, s) \neq \mathrm{d}_G(g_2, s)$, where d_G is the standard shortest-path distance. A smallest size resolving set of G is a *metric basis* of G, its size being the *metric dimension* $\dim(G)$ of G.

Resolving sets were introduced independently and simultaneously by Slater [33] and Harary & Melter [15] in 1975-6, and are one of the most widely studied and applied graph theory concepts. The 2023 survey [34] which presents an overview of the essential results and applications of metric dimension contains 220 references, while the survey [27] that focuses on variants of metric dimension cites 203 papers. Among these variants of metric dimension (MD) we point to strong MD [30], k-MD [12], k-metric antidimension [35], local MD [28], adjacency MD [19], non-local MD [25], edge MD [23], fractional MD [3], and mixed MD [22]. It is also extremely important that various applications of metric dimension have been found so far to solve real-world problems such as privacy in social networks, error correcting codes, locating intruders in networks, chemistry, robot navigation, pattern recognition, image processing, and coin weighing, cf. [34].

In 1977, Babai [5] presented a concept whose goal is to break symmetry in graphs by a vertex partition of graphs with smallest size. Two decades later, in 1996, the concepts was reinvented by Albertson and Collins [2] which led to a widespread attention of it. In fact, it was only a decade or so ago from now that it was noticed that the origin of the concept was Babai's paper; it thus went unnoticed for almost 40 years. Anyhow, a *distinguishing coloring* of a graph G is a vertex coloring such that there is no color preserving non-trivial automorphism of G. The minimum number of colors required for a distinguishing coloring of G is the *distinguishing number $D(G)$* of G.

Distinguishing coloring of graphs have been investigated from many perspectives and with many different techniques, the papers [1,11,17,29] form a short selection of such efforts. Distinguishing coloring can be extended to arbitrary groups acting on sets, see [4,6,8,26,37]. Moreover, the concept has been the inspiration for several related concepts such as distinguishing maps [36], proper distinguishing coloring [10], distinguishing arc-coloring [20], distinguishing index [21], and distinguishing threshold [32].

Metric dimension and distinguishing index have gone their research paths for years without paying attention to each other. The starting observation of this paper is that solving the metric dimension problem for a graph enables to break its symmetries. We note in passing that this is not the first occasion on which the concept of metric dimension has been linked with automorphism-dependent parameters, such

as in determining sets [7, 13]. We proceed as follows.

In the next section we list some definitions needed, recall a few results, and deduce a couple of new ones to be used later on. In Section 3 we first show that $D(G) \leq \dim(G) + 1$ holds for every connected graph G. After that we prove that among trees, exactly paths and stars attain the bound, while among the connected unicyclic graphs such graphs are t-cycles for $t \in \{3, 4, 5\}$. We also show that for any $1 \leq n < m$, there exists a graph G with $D(G) = n$ and $\dim(G) = m$.

Almost every graph has trivial automorphism group, hence $D(G) = 1$ for almost all graphs. In addition, if a graph has a nontrivial automorphism group, then its distinguishing number is usually small, typically 2. In view of the bound $D(G) \leq \dim(G) + 1$, it is natural to search for graphs with large distinguishing number, we do it in Section 4. In this direction we respectively classify the graphs G with $D(G) = n(G)$, $D(G) = n(G) - 1$, and $D(G) = n(G) - 2$, where $n(G)$ denotes the order of G.

2   Preliminaries

Let $G = (V(G), E(G))$ be a graph. The neighborhood of a vertex $u \in V(G)$, that is, the set of its neighbors, is denoted by $N_G(u)$. The diameter $\mathrm{diam}(G)$ of G is the largest distance between the vertices of G. The complement of G is denoted by $\overline{G}$. The join $G + H$ of graphs G and H is obtained from the disjoint union of G and H by making adjacent every vertex of G with every vertex of H. The *kite* is a graph on five vertices obtained from a $K_4 - e$ by adding a vertex and making it adjacent to one vertex of degree 2. If k is a positive integer, then $[k]$ denotes the set $\{1, \ldots, k\}$.

If $t \geq 1$, then we define

$$\mathcal{C}_t = \{G : \ G \text{ is a connected graph with } D(G) = t\},$$
$$\overline{\mathcal{C}}_t = \{G : \ G \text{ is a disconnected graph with } D(G) = t\}.$$

Lemma 2.1. *If $t \geq 1$, then $\overline{\mathcal{C}}_t = \{G : \ \overline{G} \in \mathcal{C}_t\} \setminus \mathcal{C}_t$.*

Proof. Assume first that $H \in \overline{\mathcal{C}}_t$. As H is disconnected, $\overline{H}$ is connected. Since also $D(H) = D(\overline{H}) = t$ we get $\overline{\mathcal{C}}_t \subseteq \{G : \ \overline{G} \in \mathcal{C}_t\} \setminus \mathcal{C}_t$.

Assume second that $H \in \{G : \ \overline{G} \in \mathcal{C}_t\} \setminus \mathcal{C}_t$. Since $\overline{H} \in \mathcal{C}_t$, we have $D(H) = t$. Moreover, H must be disconnected since $H \notin \mathcal{C}_t$. Thus $\{G : \ \overline{G} \in \mathcal{C}_t\} \setminus \mathcal{C}_t \subseteq \overline{\mathcal{C}}_t$. $\square$

Theorem 2.2. [9, Theorem 3] *If G is a connected graph with $n(G) \geq 2$, then $\dim(G) = n(G) - 1$ if and only if $G \cong K_{n(G)}$.*

Theorem 2.3. [9, Theorem 4] *If G is a connected graph with $n(G) \geq 4$, then $\dim(G) = n(G) - 2$ if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$.*

Proposition 2.4. *Let G be a connected graph with $\dim(G) = n(G) - 2 \geq 2$. If $\ell \geq 1$, then $D(G) = n(G) - \ell$ if and only if G is one of the following graphs:*

(a) $K_{\ell+1,\ell+1}$

(b) $K_{t,\ell}$, $t \geq \ell + 1$

(c) $K_\ell + \overline{K_t}$, $t \geq \ell$

(d) $K_t + \overline{K_\ell}$, $t \geq \ell \geq 2$

(e) $K_{\ell-1} + (K_t \cup K_1)$, $t \geq \max\{2, \ell - 1\}$ *(f) $K_t + (K_{\ell-1} \cup K_1)$, $t \geq \max\{2, \ell - 1\}$.*

Proof. Let $\ell \geq 1$. As we have assumed that $\dim(G) = n(G) - 2 \geq 2$, we need to check which of the graphs listed in Theorem 2.3 fulfill the condition $D(G) = n(G) - \ell$.

Assume first that $G = K_{s,t}$ $(s, t \geq 1)$. If $s = t$, then $D(G) = t + 1$. The assumption $D(G) = n - \ell$ implies that $t + 1 = s + t - \ell$. So, $s = t = \ell + 1$. This means that $G = K_{\ell+1,\ell+1}$. If $s \neq t$, then $D(G) = \max\{s, t\}$. Let $\max\{s, t\} = t$. Then $t = s + t - \ell$, and so $s = \ell$. Hence, $G = K_{t,\ell}$ for $t \geq \ell + 1$.

Let $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$. Then we have $D(G) = \max\{s, t\}$. If $\max\{s, t\} = t$, then $t = s + t - \ell$ and so $s = \ell$. This implies that $G = K_\ell + \overline{K_t}$. And if $\max\{s, t\} = s$, then $t = \ell$ and G is the graph as stated in (d).

Finally, suppose that $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$. Then we have $D(G) = \max\{s, t\}$. If $\max\{s, t\} = t$ ($\max\{s, t\} = s$), then $s = \ell - 1$ ($t = \ell - 1$). In such a situation, we obtain the graphs as stated in (e) and (f). $\square$

3 The connection

In this section we first show the basic connection between the distinguishing number and the metric dimension, that is, if G is a connected graph, then $D(G) \leq \dim(G) + 1$. We prove that among trees, exactly paths and stars satisfy the equality, while among connected unicyclic graphs the only such graphs are cycles on at most five vertices. We also demonstrate that for any $1 \leq n < m$, there exists a graph G with $D(G) = n$ and $\dim(G) = m$.

Proposition 3.1. *If G is a connected graph, then $D(G) \leq \dim(G) + 1$. Moreover, the bound is sharp.*

Proof. Let $S \subseteq V(G)$ be a metric basis of G, so that $|S| = \dim(G)$. Let c be a coloring of $V(G)$ which respectively assigns colors from $[\dim(G)]$ to the vertices from S, and assigns color $\dim(G) + 1$ to each vertex from $V(G) \setminus S$. We claim that c is a distinguishing coloring for G. Suppose on the contrary that there exists a non-trivial automorphism φ that preserves c. Then $\varphi(u) = u$ for each $v \in S$, hence,

since φ is non-trivial, there exist vertices $u, v \in V(G) \setminus S$, $u \neq v$, such that $f(u) = v$. Since an automorphism is a distance preserving mapping, for every $w \in S$ we have

$$d_G(u, w) = d_G(f(u), f(w)) = d_G(v, w).$$

But then S does not resolve u and v, a contradiction.

To see that the bound is sharp, consider paths P_n, $n \geq 2$, for which we have $D(P_n) = 2$ and $\dim(G) = 1$. □

The proof of Proposition 3.1 gives:

Corollary 3.2. *If G is a connected graph and $S = \{v_1, \ldots, v_n\}$ is a resolving set of G, then $\{\{v_1\}, \ldots, \{v_n\}, V(G) \setminus S\}$ is a distinguishing partition of $V(G)$.*

As observed in the proof of Proposition 3.1, paths attain the equality in Proposition 3.1. The equality is also attained by complete graphs K_n, $n \geq 2$, since $D(K_n) = n$ and $\dim(K_n) = n - 1$. A sporadic example for the equality is the wheel graph W_5 for which we have $D(W_5) = 3$ and $\dim(W_5) = 2$, see [31]. We next show that besides paths, starts are the only trees that attain the equality. To prove it, we recall the following definitions.

Let T be a tree. A vertex $u \in V(T)$ is a *leaf* if $\deg_T(u) = 1$, and a *branch* if $\deg_T(u) \geq 3$. The *terminal degree* of a branch u is the number of leaves x, such that $d_T(x, u) < d_T(x, v)$ for all branches $v \neq u$. A branch is *external* if its terminal degree is strictly positive, otherwise it is *internal*. The number of leaves and of external branches of T are respectively denoted by $\ell(T)$ and $\text{ex}(T)$. Using these definitions, the key result for the metric dimension of trees asserts that if T is a tree different from a path, then

$$\dim(T) = \ell(T) - \text{ex}(T), \tag{1}$$

see [15, 24, 33]. We further say that the branch closest to a leaf is the *ancestor* of the leaf. The leaf is *external* if its ancestor is external. Finally, the path between a leaf and its ancestor will be called a *leg*.

Theorem 3.3. *A tree T satisfies the equality $D(T) = \dim(T) + 1$ if and only if $T \in \{P_n, K_{1,n} : n \geq 2\}$.*

Proof. If $n \geq 2$, then $\dim(P_n) = 1$, $D(P_n) = 2$, $\dim(K_{1,n}) = n-1$, and $D(K_{1,n}) = n$. It remains to prove that for any tree $T \notin \{P_n, K_{1,n} : n \geq 2\}$ we have $D(T) \neq \dim(T) + 1$. Since $\dim(P_1) = D(P_1) = 1$, we may assume in the rest that $|V(T)| \geq 3$.

Assume first that $\text{ex}(T) = 1$ and let b be the unique external branch. In this case, from (1) we get $\dim(T) = \deg_T(b) - 1$. Set $d = \deg_T(b)$. Recall that T is not a star, and let p be a arbitrary leaf of T and p' its neighbor. Color leaves different

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from p with distinct colors from $[d-1]$. Assign color 2 to p', and color 1 to all the remaining vertices, including p. This is a distinguishing coloring of T, hence $D(T) \leq d-1 = \dim(T)$.

Assume second that $\mathrm{ex}(T) = k \geq 2$, and let $b_1, \ldots, b_k$ be the external branches of T. Let

$$t = \max \left\{ k, \max_{i \in [k]} \deg_T(b_i) - 1 \right\}.$$

We claim that $\dim(T) \geq t$. First, by (1) we have $\dim(T) \geq k$. Assume next that, without loss of generality, $\deg_T(b_1) = \max_{i \in [k]} \deg(b_i)$. Let $\deg_T(b_1) = d_1 + d_2$, where d_1 is the number of legs containing b_1. Then in each such leg except one, there exists a member in any metric basis of T. Moreover, in view of (1), each neighbor of b_1 not a leg contributes at least 1 to the metric dimension of T. This means that

$$\dim(T) \geq (d_1 - 1) + d_2 = \deg_T(b_1) - 1 = \max_{i \in [k]} \deg(b_i) - 1.$$

We have thus proved that $\dim(T) \geq t$. Now, for $i \in [k]$, color b_i with color i. Additionally, color the external leaves of the same ancestor b_i with distinct colors from the set $[\deg_T(b_i)]$. (Note that since there are at least two external branches, there can be no more than $\deg_T(b_i) - 1$ external leaves corresponding to the same ancestor b_i). Finally, assign color 1 to all the remaining vertices. This coloring is a distinguishing coloring which proves that $D(T) \leq t$ and we are done. $\square$

We next determine the graphs among unicyclic graphs which attain the equality in Proposition 3.1.

Theorem 3.4. *If G is a connected unicyclic graph, then $D(G) = \dim(G) + 1$ if and only if $G \in \{C_3, C_4, C_5\}$.*

Proof. Let G be a unicyclic graph and let $t_1, \ldots, t_m$ be the consecutive vertices of the unique cycle C of G. Let further T_i, $i \in [m]$, be the maximal connected subgraph of G which contains t_i and no other vertex of C. Then T_i is a tree, where it is possible that $T_i \cong K_1$. In particular, if G is a cycle, then every T_i is the one vertex graph. We now distinguish two cases.

In the first case every T_i, $i \in [m]$, is isomorphic to a path. Then every automorphism of G restricted to C is an automorphism of C. It follows that if $m \geq 6$, then $D(G) = 2$ and hence $D(G) < \dim(G) + 1$. Consider next the cases $m \in \{3, 4, 5\}$. Assume that each path T_i has at least two vertices and let t_i' be the neighbor of t_i in T_i. If $m = 3$, then assign color 1 to t_1, t_1' and t_3, and color 2 to t_2, t_2' and t_3'. If $m = 4$, then assign color 1 to t_1, t_1', t_3 and t_4', and color 2 to t_2, t_2', t_3' and t_4. And if $m = 5$, then assign color 1 to t_1, t_1', t_3, t_3', t_4' and t_5, and color 2 to t_2, t_2', t_4 and t_5'. Finally, in each of the three cases assign color 1 to all the other

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vertices. In each of the cases we have a distinguishing coloring, so $D(G) \leq 2$ and thus $D(G) \leq \dim(G)$. Note that the above argument is also applicable as soon as at least one T_i has at least two vertices. Hence we can conclude that if $m \in \{3, 4, 5\}$, then $D(G) = \dim(G) + 1$ if and only if $G \in \{C_3, C_4, C_5\}$.

In the second case at least one T_i is not a path, we may assume without loss of generality that T_1 is such a tree. Let $S = \{s_1, \ldots, s_k\}$ be a metric basis of G. Since T_1 is not a path, we have $S \cap V(T_1) \neq \emptyset$, and we may assume that $s_1 \in S \cap V(T_1)$. Color now the vertices s_i, $i \in [k] \setminus \{1\}$, with color i, and all the other vertices, that is, vertices from $(V(G) \setminus S) \cup \{s_1\}$, with color 1. If a non-trivial automorphism f preserves this coloring, then $f(s_i) = s_i$ for $i \in [k] \setminus \{1\}$ and $f(s_1) \in V(G) \setminus S$. Since t_2 and t_m are not resolved by vertices from $S \cap V(T_1)$, there exists a vertex in $S \setminus V(T_1)$, say s_2. By [14, Corollary 7] we may assume that $t_1 \notin S$, so by now it is colored 1. Change now the color of s_1 and of t_1 to 2, so that the color class of color 2 contains the vertices s_1, s_2, and t_1. This is now a distinguishing coloring using $k = \dim(G)$ colors and we can conclude that $D(G) \leq \dim(G)$. $\qquad \square$

To conclude the section we demonstrate the following result which complements Proposition 3.1.

Proposition 3.5. *For any $1 \leq n < m$, there exists a graph G with $D(G) = n$ and $\dim(G) = m$.*

Proof. If $k \geq 3$, then let T_k be the tree obtained from $K_{1,k}$ by respectively subdividing its edges $0, 1, \ldots, k - 1$ times. Then T_k is asymmetric, so that $D(T_k) = 1$. On the other hand, from (1) we get $\dim(T_k) = k - 1$. This settles the case $n = 1$.

Assume now that $2 \leq n < m$. Let $G_{n,m}$ be the graph obtained from the disjoint union of T_{m-n+2} and K_n by adding the edges between the maximum degree vertex of T_{m-n+2} and all the vertices of K_n. Then $D(G_{n,m}) = n$ and $\dim(G_{n,m}) = m$. $\qquad \square$

4 Graphs G with $D(G)$ close to $n(G)$

In this section we classify graphs G such that $D(G) \in \{n(G), n(G) - 1, n(G) - 2\}$. First, combining Proposition 3.1 with Theorem 2.2 and having Lemma 2.1 in mind we get:

Corollary 4.1. *If G is a graph, then $D(G) = n(G)$ if and only if $G \in \{K_{n(G)}, \overline{K}_{n(G)}\}$.*

Using the classification of graphs G with $\dim(G) = n(G) - 2$ from Theorem 2.3 and its application in Proposition 2.4, the graphs with $\dim(G) = n(G) - 1$ are the following.

Theorem 4.2. *If G is a graph, then $D(G) = n(G) - 1$ if and only if G is one of C_4, $2K_2$, $K_{t,1}$, and $K_t \cup K_1$, where $t \geq 2$.*

Proof. Assume first that G is a connected graph. By Proposition 3.1, $\dim(G) \in \{n(G) - 2, n(G) - 1\}$. Moreover, by Theorem 2.2 and Corollary 4.1, it more specifically holds that $\dim(G) \neq n(G) - 1$, that is, $\dim(G) = n(G) - 2$. To determine such graphs, set $\ell = 1$ in Proposition 2.4. Then C_4 and the graphs $K_{t,1}$, $t \geq 2$, are obtained from Proposition 2.4 (a) and (b). Since $\ell = 1$, G is not equal to any graph in Proposition 2.4 (d), (e) and (f). The graphs $2K_2$, and $K_t \cup K_1$, $t \geq 2$, are then deduced by Lemma 2.1. $\qquad\square$

In the rest of the section we classify the graphs with $\dim(G) = n(G) - 2$. To do so, we use Proposition 3.1 together with the forthcoming Theorems 4.3 and 4.12 in which graphs of metric dimension relevant to us and of given diameter are described. To recall Theorems 4.3, some preparation is needed.

Vertices u and v of a graph G are *twins* if $\mathrm{N}_G(v) \setminus \{u\} = \mathrm{N}_G(u) \setminus \{u\}$. Define the relation $\equiv$ on $V(G) \times V(G)$ by setting $u \equiv v$ if $u = v$, or if u and v are twins. Then $\equiv$ is an equivalence relation. For a vertex $v \in V(G)$ we denote its equivalence class by v^*, that is,

$$v^* = \{u \in V(G) : \ u \equiv v\}.$$

The *twin graph* G^* of G is the quotient graph with respect to the relation $\equiv$, that is,

$$V(G^*) = \{v^* : \ v \in V(G)\} \quad \text{and} \quad E(G^*) = \{v^* u^* : \ uv \in E(G)\}.$$

A twin equivalence class v^* of G induces an edgeless graph or a complete graph. This subgraph is denoted by $G[v^*]$. We will further say that v^* is of

- *type (1)*, if $G[v^*] \cong K_1$,

- *type (K)*, if $G[v^*] \cong K_2$, where $r \geq 2$,

- *type (N)*, if $G[v^*] \cong \overline{K_r}$, where $r \geq 2$.

Furthermore, we say that $v^* \in G^*$ is of *type (1K)* if v^* is of type (1) or (K), of *type (1N)* if v^* is of type (1) or (N), of *type (KN)* if v^* is of type (K) or (N), and of *type (1KN)* if v^* is of type (1), (K) or (N). For more details about twin graphs see [16,18]. Now we can recall the following result.

Theorem 4.3. [18, Theorem 1] *Let G be a connected graph with $\mathrm{diam}(G) = 2$. Then $\dim(G) = n(G) - 3$ if and only if G^* is one of the following graphs.*

$\mathbf{G_1}$. *$G^* = K_3$ and has at most one vertex of type (1K);*

G$_2$. $G^* = P_3$ and one of the following cases holds:

    (a) The degree-2 vertex is of type (N), and one of the leaves is of type (K) and the other is of any type;

    (b) One of the leaves is of type (K), the other is of type (KN) and the degree-2 vertex is of any type;

G$_3$. G^* is a triangle with a pendant edge, one of the degree-2 vertices is of type (N), the other is of type (1K), and the leaf is of type (1N). Moreover, a degree-2 vertex of type (K) yields the leaf and the degree-3 vertex are not of type (N);

G$_4$. $G^* = C_5$, and each vertex is of type (1);

G$_5$. G^* is a C_5 with a chord, adjacent degree-2 vertices are of type (1), the other vertices are of type (1K);

G$_6$. $G^* = K_1 + P_4$, the degree-4 vertex is of any type, the others are of type (1K). Furthermore, two non-adjacent vertices are not of type (K), and two adjacent vertices are not of different types (K) and (N);

G$_7$. G^* is a kite with a pendant edge adjacent to a degree-3 vertex, the leaf is of type (1), the degree-4 and degree-3 vertices are type (1K), one of the degree-2 vertices is of type (K) and the other is of type (1);

G$_8$. G^* is a kite, one of the degree-2 vertices is of type (K), the other is of type (1), one of the degree-3 vertices is of type (N), and the other is of type (1K);

G$_9$. $G^* = C_4$, two adjacent vertices are of type (K), the others are of type (1);

G$_{10}$. $G^* = C_4 + K_1$, two degree-3 adjacent vertices are of type (K), degree-4 vertex is of type (1K), others are of type (1).

Observation 4.4. *Let m be a positive integer, G a connected graph, and $V(G^*) = \{v_1^*, \ldots, v_{n(G^*)}^*\}$. If $D(G) = |v_i^*|$ for some $i \in [n(G^*)]$ and $\Sigma_{\substack{j \in n(G^*) \\ j \neq i}} |v_j^*| \neq m$, then $D(G) \neq n(G) - m$.*

    We say that a graph G is *almost asymmetric* if there is no non-trivial automorphism that maps a vertex from one twin equivalence class of G to a vertex of another twin equivalence class of. Clearly, if G is almost asymmetric, then $D(G) = \max_{v^* \in V(G^*)} \{|v^*| : \; v^* \in V(G^*)\}$. From Observation we get:

Corollary 4.5. *Let m be a positive integer and G a connected, almost asymmetric graph. If $n(G) - \max_{v^* \in V(G^*)} \{|v^*| : \; v^* \in V(G^*)\} \neq m$, then $D(G) \neq n(G) - m$.*

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In the following six lemmas we investigate the distinguishing number of the graphs from Theorem 4.3.

Lemma 4.6. *If G^* is a $\mathbf{G}_1$ graph, then $D(G) = n(G) - 2$ if and only if G is $K_{1,2,2}$ or $K_{1,1,t}$, $t \geq 2$.*

Proof. Assume first that v_1^* is a vertex of G^* of type (1). Then the other two vertices v_2^* and v_3^* of G^* are of type (N). If $|v_2^*| \neq |v_3^*|$, then $D(G) = \max\{|v_2^*|, |v_3^*|\}$. Let $\max\{|v_2^*|, |v_3^*|\} = |v_2^*|$. Hence $D(G) = n(G) - 2$ implies that $|v_3^*| = 1$, and we get the graphs $K_{1,1,t}$, $t \geq 2$. If $|v_2^*| = |v_3^*|$, then $D(G) = |v_2^*| + 1$. The equation $|v_2^*| + 1 = |v_2^*| + |v_3^*| + 1 - 2$ results in $|v_3^*| = 2$, and we obtain $K_{1,2,2}$.

Let v_1^* be of type (K). If $|v_2^*| \neq |v_3^*|$, then $D(G) = \max\{|v_1^*|, |v_2^*|, |v_3^*|\}$. Without loss of generality, we may assume that $\max\{|v_1^*|, |v_2^*|, |v_3^*|\} = |v_1^*|$. The assumption $D(G) = n(G) - 2$ yields $|v_2^*| + |v_3^*| = 2$, which is impossible. If $|v_2^*| = |v_3^*|$, then $D(G) = \max\{|v_1^*|, |v_2^*| + 1\}$. If $D(G) = |v_1^*|$, then the size of v_2^* will be equal to 1, which is impossible. If $D(G) = |v_2^*| + 1$, then $|v_1^*| + |v_3^*| = 3$, which is again not impossible.

Assume that v_1^* is of type (N). If $|v_1^*| \neq |v_2^*| \neq |v_3^*| \neq |v_1^*|$, then $D(G) = \max\{|v_1^*|, |v_2^*|, |v_3^*|\}$. Let $\max\{|v_1^*|, |v_2^*|, |v_3^*|\} = |v_1^*|$. Hence, $|v_2^*| + |v_3^*| = 2$, which is impossible. If $|v_1^*| = |v_2^*| \neq |v_3^*|$, then $D(G) = \max\{|v_1^*| + 1, |v_3^*|\}$. If $D(G) = |v_3^*|$, then $|v_1^*| = 1$, which contradicts the definition of type (N). If $D(G) = |v_1^*| + 1$, then $D(G) \neq n - 2$. If $|v_1^*| = |v_2^*| = |v_3^*|$, then $D(G) = |v_1^*| + 1$ and $|v_2^*| + |v_3^*| = 3$, which is impossible. $\square$

Lemma 4.7. *If G^* is a $\mathbf{G}_2$ graph, then $D(G) \neq n(G) - 2$.*

Proof. Suppose on the contrary that $D(G) = n(G) - 2$.

Assume first that G^* is a $\mathbf{G}_2(a)$ graph. Let v_1^*, v_2^* and v_3^* be the vertices of types (K), (N), and any type, respectively. Suppose first that v_3^* is of type (1). Thus $D(G) = \max\{|v_1^*|, |v_2^*|\}$. If $\max\{|v_1^*|, |v_2^*|\} = |v_1^*|$, then $|v_1^*| = |v_1^*| + |v_2^*| - 1$ and so $|v_2^*| = 1$, a contradiction. If $\max\{|v_1^*|, |v_2^*|\} = |v_2^*|$, then using a similar argument as before, $|v_1^*| = 1$, a contradiction. Let v_3^* be of type (N). Then $D(G) = \max\{|v_1^*|, |v_2^*|, |v_3^*|\}$. Without loss of generality, we may assume that $\max\{|v_1^*|, |v_2^*|, |v_3^*|\} = |v_1^*|$. So, $|v_2^*| + |v_3^*| = 2$, which is a contradiction. Assume that v_3^* is of type (K). Clearly, $|v_1^*| = |v_3^*|$. So, $D(G) = \max\{|v_1^*| + 1, |v_2^*|\}$. If $\max\{|v_1^*| + 1, |v_2^*|\} = |v_2^*|$, then it is impossible as before. If $\max\{|v_1^*| + 1, |v_2^*|\} = |v_1^*| + 1$, then $|v_2^*| + |v_3^*| = 3$, a contradiction.

Assume second that G^* is a $\mathbf{G}_2(b)$ graph. Let v_1^*, v_2^* and v_3^* be the vertices of types (K), any type, and (NK), respectively. If v_3^* is of type (N), then $|v_1^*| = 1$, which is impossible. Let v_3^* be of type (K). If $|v_1^*| \neq |v_3^*|$, then we reach the same contradiction as before. Thus $|v_1^*| = |v_3^*|$ and so $D(G) = \max\{|v_1^*| + 1, |v_2^*|\}$. One

can check that $D(G) \neq |v_2^*|$. So, $D(G) = |v_1^*|+1$. This concludes that $|v_2^*|+|v_3^*| = 3$, a contradiction. $\square$

Lemma 4.8. *If G^* is a $\mathbf{G}_3$ graph, then $D(G) \neq n(G) - 2$.*

Proof. Let v_1^*, v_2^*, v_3^*, and v_4^* be the degree-2 vertex of type (1K), the degree-2 vertex of type (N), the degree-3 vertex, and the leaf, respectively. Suppose that $D(G) = n(G) - 2$. Then in all cases, G is almost asymmetric. Hence, $D(G) = \max\{|v_1^*|, |v_2^*|, |v_3^*|, |v_4^*|\}$. Then from $D(G) = n - 2$ it follows that some of $|v_i^*|$'s must be equal to 1 or 0, which is impossible. $\square$

Lemma 4.9. *If G^* is a $\mathbf{G}_5$ graph, then $D(G) \neq n(G) - 2$.*

Proof. Suppose that $D(G) = n(G) - 2$. Let v_1^* and v_2^* be the degree-3 vertices, v_3^* be the degree-2 vertex that is adjacent to v_1^* and v_2^*, and v_4^* and v_5^* be the adjacent degree-2 vertices. Let $|v_1^*| = |v_2^*| = 1$. If $|v_3^*| = 1$, then $D(G) = 2$. If $|v_3^*| \neq 1$, then $D(G) = |v_3^*| = n(G) - 4$, a contradiction. If $|v_1^*| = |v_2^*| \neq 1$ or $|v_1^*| \neq |v_2^*|$, then $D(G) = \max\{|v_1^*|, |v_2^*|, |v_3^*|\}$. Therefore, in all cases $D(G) \neq n(G) - 2$. $\square$

Lemma 4.10. *If G^* is a $\mathbf{G}_6$ graph, then $D(G) \neq n(G) - 2$.*

Proof. Let v_1^* and v_4^* be the degree-2 vertices, v_3^* and v_5^* be the degree-3 vertices, and v_2^* be the degree-4 vertex. Suppose that $D(G) = n(G) - 2$. The only non-trivial automorphism of G^* is the automorphism which fixes v_2^*, swaps v_1^* with v_4^*, and swaps v_3^* with v_5^*. Since the non-adjacent vertices are not of type (K), if $|v_1^*| = |v_4^*|$, then $|v_1^*| = 1$. Now, if $|v_3^*| = |v_5^*| = 1$, then $D(G) = \max\{2, |v_2^*|\}$. If $D(G) = 2$, then some vertices of G^* have size 0 which is clearly not possible. If $D(G) = |v_2^*|$, then $|v_2^*| + 2 = |v_2^*|$, a contradiction. If $|v_3^*| = |v_5^*| \neq 1$, then $D(G) = \max\{|v_3^*|, |v_2^*|\}$. In both cases $D(G) = |v_3^*|$ and $D(G) = |v_2^*|$, so $D(G) \neq n(G) - 2$. If $|v_3^*| \neq |v_5^*|$ or $|v_1^*| \neq |v_4^*|$, then G is an almost asymmetric graph and Corollary 4.5 yields the conclusion. $\square$

Lemma 4.11. *If G^* is one of the graphs $\mathbf{G}_7$, $\mathbf{G}_8$, $\mathbf{G}_9$, $\mathbf{G}_{10}$, then $D(G) \neq n(G) - 2$.*

Proof. $\mathbf{G}_7$ is an almost asymmetric graph. Setting $m = 2$ in Corollary 4.5 we can conclude that $D(G) \neq n(G) - m$.

If $G^* \in \{\mathbf{G}_8, \mathbf{G}_9\}$, then $D(G) = \max\limits_{v^* \in V(G^*)} \{|v^*| : \ v^* \in G^*\}$. So, $D(G) \neq n(G) - 2$.

Assume finally that G^* is a $\mathbf{G}_{10}$ graph. If the two degree-3 adjacent vertices of type (K) have different sizes, then G is an almost asymmetric graph and the result is immediate by Corollary 4.5. If the two degree-3 adjacent vertices of type (K) have the same size, then $D(G) = n(G) - 2$ implies that some vertices of G^* have size 0 which is clearly not possible. $\square$

Before presenting the main result of this section, we need to recall the following theorem and deduce from it another lemma. Here, $\alpha(G^*)$ denotes the number of vertices of G^* of type (KN).

Theorem 4.12. [16, Theorem 2.14] *If G is a connected graph with $d = \mathrm{diam}(G) \geq 3$, then $\dim(G) = n(G) - d$ if and only if G^* is one of the following graphs.*

(a) $G^* = P_{d+1}$ *and one of the following cases holds:*

    (a1) $\alpha(G^*) \leq 1$;

    (a2) $\alpha(G^*) = 2$, *the two vertices of G^* not of type (1) are adjacent, and if one is a leaf of type (K), then the other is also of type (K);*

    (a3) $\alpha(G^*) = 2$, *the two vertices of G^* not of type (1) are at distance 2 and both are of type (N); or*

    (a4) $\alpha(G^*) = 3$ *and there is a vertex of type (N) or (K) adjacent to two vertices of type (N).*

(b) $G^* = P_{d+1,k}$ *(the path $(u_0^*, u_1^*, \ldots, u_d^*)$ with one extra vertex adjacent to u_{k-1}^*) for some integer $k \in [3, d-1]$, the degree-3 vertex u_{k-1}^* of G^* is of any type, each neighbour of u_{k-1}^* is of type (1N), and every other vertex is of type (1).*

(c) $G^* = P'_{d+1,k}$ *(the path $(u_0^*, u_1^*, \ldots, u_d^*)$ with one extra vertex adjacent to u_{k-1}^* and u_k^*) for some integer $k \in [2, d-1]$, the three vertices in the cycle are of type (1K), and every other vertex is of type (1).*

Lemma 4.13. *Let G be a connected graph with $d = \mathrm{diam}(G) \in \{3, 4\}$. If $\dim(G) = n(G) - d$, then $D(G) = n(G) - 2$ if and only if $G = P_4$.*

Proof. The graph G^* is one of the graphs described in Theorem 4.12.

In the case (a1), assume first that $d = 3$. If $\alpha(G^*) = 0$, then $G = P_4$ and $D(G) = n(G) - 2$. In what follows, we will see that $G = P_4$ is the only graph with $D(G) = n(G) - 2$ satisfying the assumptions of this lemma. If $\alpha(G^*) = 1$, then there are two non-isomorphic positions for placing a vertex of type (K) or (N) in P_4. In all cases, the distinguishing number is equal to $n - 3$. Assume next that $d = 4$. If $\alpha(G^*) = 0$, then $G = P_5$ and $D(G) = n(G) - 3$. If $\alpha(G^*) = 1$, then in all cases we have $D(G) = n(G) - 4$.

In all the other cases, that is, (a2), (a3), (a4), (b), and (c), there are no graphs with distinguishing number $n(G) - 2$, so we can proceed by using Corollary 4.5 for the graph G^* with $n(G^*) \geq 5$, and techniques similar to those used in the proofs of the previous lemmas for the graph G^* with $n(G^*) = 4$. $\qquad\square$

We have now arived to the main result of this section.

Theorem 4.14. *If G be a graph with $n(G) \geq 4$, then $D(G) = n(G) - 2$ if and only if G is one of the following graphs:*

(1) C_5

(2) P_4

(3) $K_{1,2,2}$

(4) $2K_2 \cup K_1$

(5) $K_{3,3}$

(6) $2K_3$

(7) $K_{t,2}$, $t \geq 3$

(8) $K_t \cup K_2$, $t \geq 3$

(9) $K_2 + \overline{K_t}, t \geq 2$

(10) $K_t \cup 2K_1$, $t \geq 2$

(11) $K_t + \overline{K_2}$, $t \geq 2$

(12) $\overline{K_t} \cup K_2$, $t \geq 2$

(13) $K_1 + (K_t \cup K_1)$, $t \geq 2$

(14) $K_{t,1} \cup K_1$, $t \geq 2$.

Proof. Let G be a connected graph with $D(G) = n(G) - 2$. Proposition 3.1, Theorem 2.2, and Corollary 4.1 imply that it suffices to check the graphs which have metric dimension equal to $n(G) - 2$ or to $n(G) - 3$.

Assume first that $\dim(G) = n(G) - 2$ and set $\ell = 2$ in Proposition 2.4. The graphs in (a), (b), (c), (d), and (e) in Proposition 2.4 are the graphs in (5), (7), (9), (11), and (13), respectively. Note that the graph in (f) in Proposition 2.4 appears in (11).

Assume second that $\dim(G) = n(G) - 3$. Since $\dim(G) \leq n(G) - \operatorname{diam}(G)$, we have $\operatorname{diam}(G) \leq 3$. If $\operatorname{diam}(G) = 2$, then the graphs in (1), (3) and (9) follow directly from Theorem 4.3 and Lemmas 4.6–4.11. If $\operatorname{diam}(G) = 3$, then Theorem 4.12 and Lemma 4.13 imply that G is the graph in (2). If G is disconnected, then by Lemma 2.1, G has the form described in (4), (6), ..., (14) as the complements of graphs in (3), (5), ..., (13), respectively. (Note here that C_5 and P_4 are self-complementary graphs). $\square$

To conclude the paper we note that recently graphs G with $\dim(G) = n(G) - 4$ have been described [38]. Using the above approach we believe that it is also possible to classify the graphs G with $D(G) = n(G) - 3$.

Acknowledgments

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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