

On decycling and forest numbers of Cartesian products of trees

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Abstract

The decycling number $\nabla(G)$ of a graph G is the minimum number of vertices that must be removed to eliminate all cycles in G . The forest number $f(G)$ is the maximum number of vertices that induce a forest in G . So $\nabla(G) + f(G) = |V(G)|$. For the Cartesian product $T \square T'$ of trees T and T' it is proved that $\nabla(S_n \square S_{n'}) \leq \nabla(T \square T')$, thus resolving the conjecture of Wang and Wu asserting that $f(T \square T') \leq f(S_n \square S_{n'})$. It is shown that $\nabla(T \square T') \geq |V(T)| - 1$ and the equality cases characterized. For prisms over trees, it is proved that $\nabla(T \square K_2) = \alpha'(T)$, and for arbitrary graphs G_1 and G_2 , it is proved that $\nabla(G_1 \square G_2) \geq \alpha'(G_1)\alpha'(G_2)$, where α' is the matching number.

Keywords: decycling set; decycling number; forest number; tree; Cartesian product of graphs

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1 Introduction

A *decycling set* S of a graph G is a subset of its vertices such that $V(G) - S$ is a forest. In other words, each cycle of the graph contains at least one vertex of a decycling set. From this reason, decycling sets are also known in the literature as feedback vertex sets, cf. [10, 11]. The *decycling number* $\nabla(G)$ of G is the minimum cardinality of a decycling set. The *forest number* $f(G)$ of G is the maximum number of vertices that induce a forest in G . The following equation that goes back to Erdős et al. [15] is evident:

$$\nabla(G) + f(G) = |V(G)|. \quad (1)$$

The issue of eliminating all cycles in a graph by deleting a set of vertices was first introduced in 1974 in combinatorial circuit design [19]. Soon after it has been established that determining the decycling number is NP-complete for general graphs [20]. Decycling sets have found applications across various fields, including deadlock prevention in operating systems [41, 42], constraint satisfaction problem and Bayesian inference in artificial intelligence [4], monopolies in synchronous distributed systems [34, 35], the converter placement problem in optical networks [23], and VLSI chip design by [16].

At this point, we should also mention the related concept of cycle isolation, where we also want to destroy all cycles of a given graph, the difference here is that together with a given set of vertices we remove its entire closed neighbourhood. For more information on cycle isolation see [8, 26, 45] and references therein.

In 1987, Akiyama and Watanabe [1], along with Albertson and Haas in 1998 [2], independently conjectured that if G is a planar bipartite graph, then $f(G) \geq \frac{5}{8}|V(G)|$. Motivated by this conjecture, Alon [3] proved that every bipartite graph G with an average degree $d \geq 1$ has $f(G) \geq \left(\frac{1}{2} + e^{-bd^2}\right)|V(G)|$ for some positive constant b . On the other hand, there are bipartite graphs G with an average degree $d \geq 1$ such that $f(G) < \left(\frac{1}{2} + \frac{1}{e^{b'\sqrt{d}}}\right)|V(G)|$, where b' is an absolute constant. In 2014, Conlon et al. [9] improved Alon's lower bound to $\left(\frac{1}{2} + d^{-db}\right)|V(G)|$ for $d \geq 2$. In 2017, Wang et al. [44] demonstrated that every simple bipartite planar graph G satisfies the inequality $f(G) \geq \lceil \frac{4|V(G)|+3}{7} \rceil$. More recently, one of the authors, along with other [18], proved that every balanced bipartite graph G with a minimum degree of at least $\frac{|V(G)|}{4} + 1$ has $f(G) = \frac{|V(G)|}{2} + 1$. This result was initially proposed as a conjecture in [43]. Upper and lower bounds for the forest number of a graph in relation to its order, size, and maximum degree are presented in [7, 40]. The forest number has been further studied in the context of planar graphs [12, 13, 21, 25, 29, 36], regular graphs [38, 39], (sub)cubic graphs [22, 33], line graphs [30], and graphs embedded into surfaces [31].

The notations P_n , S_n , and C_n represent the path, the star, and the cycle graph on n vertices, respectively.

The study of decycling and forest numbers of Cartesian products of graphs has a long history. Beineke and Vandell [7] investigated them on grids and hypercubes. Luccio [28] demonstrated that if $n, n' \geq 2$, then $\nabla(P_n \square P_{n'}) \geq \left\lceil \frac{(n-1)(n'-1)+1}{3} \right\rceil$. Madelaine and Stewart [32] established that $\nabla(P_n \square P_{n'}) \leq \left\lceil \frac{(n-1)(n'-1)+1}{3} \right\rceil + 2$. Lien et al. [27] followed by proving $\nabla(P_n \square P_{n'}) \leq \left\lceil \frac{(n-1)(n'-1)+1}{3} \right\rceil + 1$. Pike and Zou [37] derived $\nabla(C_n \square C_{n'}) = \left\lceil \frac{3n'}{2} \right\rceil$ when $n = 4$, and $\nabla(C_n \square C_{n'}) = \left\lceil \frac{nn'+2}{3} \right\rceil$ otherwise. Recently, and as our main motivation, Wang and Wu [43] investigated the forest number of Cartesian, direct, and lexicographic products of graphs. Among other things, in [43, Conjecture 5.1] they put forward the following conjecture, where S_n denotes the star $K_{1,n-1}$.

Conjecture 1.1. *If T and T' are trees of respective orders n and n' , then*

$$f(T \square T') \leq f(S_n \square S_{n'}).$$

In Section 2, we prove this conjecture. In the subsequent section, we characterize the trees that attain the equality in the conjecture. If the orders of T and T' are the same, then the equality is attained only if at least one of the factors is a star, while if the orders of T and T' are different, then the equality is attained if and only if the larger factor is a star (and the smaller factor is arbitrary). In Section 4, we consider the missing case when a star is the smaller factor and establish a sharp lower bound for the decycling number of Cartesian products of graphs based on the matching number of their factors.

To conclude the introduction, we give definition needed in this paper.

The subgraph of a graph G induced by $X \subseteq V(G)$ is denoted by $G[X]$. A *vertex cover* of G is a subset of its vertices that includes at least one endpoint of every edge. The *covering number* $\beta(G)$ of G is the minimum size of a vertex cover of G . A *matching* of G is a subset of edges such that no two edges from it share a common vertex. The *matching number* $\alpha'(G)$ of G is the maximum size of a matching set for the graph.

Let G and H be two graphs. Their *Cartesian product* $G \square H$ is a graph with vertex set $V(G) \times V(H)$, and two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The subgraph of $G \square H$ induced by the set of vertices $\{(g, h) : g \in V(G)\}$ is called a *G -layer* and denoted by G^h . Similarly, the subgraph of $G \square H$ induced by $\{(g, h) : h \in V(H)\}$ is an *H -layer* and denoted by gH . Note that $G^h \cong G$ for every $g \in V(G)$, and that ${}^gH \cong H$ for

every $h \in V(H)$. Recall that the Cartesian product operation is commutative, that is, $G \square H \cong H \square G$, where $\cong$ denotes graph isomorphism.

2 Proof of Conjecture 1.1

Conjecture 1.1 will follow from the following result.

Theorem 2.1. *If T and T' are trees of respective orders n and n' , where $n' \geq n \geq 2$, then*

$$\nabla(T \square T') \geq n - 1 = \nabla(T \square S_{n'}).$$

Proof. Let T and T' be trees $V(T) = \{v_1, \dots, v_n\}$ and $V(T') = \{u_1, \dots, u_{n'}\}$, where $n' \geq n \geq 2$. Suppose on the contrary that $\nabla(T \square T') \leq n - 2$ and let S be a decycling set in $T \square T'$ with $|S| = \nabla(T \square T') \leq n - 2$. By the pigeonhole principle there exist $i, j \in [n]$, $i \neq j$, such that $S \cap V({}^{v_i}T') = \emptyset$ and $S \cap V({}^{v_j}T') = \emptyset$, and there exist $k, \ell \in [n']$, $k \neq \ell$, such that $S \cap V(T^{u_k}) = \emptyset$ and $S \cap V(T^{u_\ell}) = \emptyset$. Consider now the subgraph X of $T \square T'$ induced by

$$V({}^{v_i}T') \cup V({}^{v_j}T') \cup V(T^{u_k}) \cup V(T^{u_\ell}).$$

X is a connected graph of order $2n + 2n' - 4$ and of size at least $2n + 2n' - 4$. It follows that X contains a cycle which is not possible because $S \cap V(X) = \emptyset$. This contradiction proves that $\nabla(T \square T') \geq n - 1$.

By the above, $\nabla(T \square S_{n'}) \geq n - 1$. Let $V(S_{n'}) = \{u_1, \dots, u_{n'}\}$, and let $\{u_1, \dots, u_{n'-1}\}$ be an independent set of $S_{n'}$. In this case, the subgraph of $T \square S_{n'}$ induced by $V(T) \times (V(S_{n'}) - \{u_{n'}\})$ is isomorphic to the disjoint union of $(n' - 1)$ copies of T . Consequently, the subgraph induced by

$$(V(T) \times (V(S_{n'}) - \{u_{n'}\})) \cup \{(v_1, u_{n'})\}$$

is a tree. It follows that $\{(v_i, u_{n'}) : 2 \leq i \leq n\}$ is a decycling set for $T \square S_{n'}$, and thus $\nabla(T \square S_{n'}) \leq n - 1$. $\square$

Theorem 2.1 in particular implies that $\nabla(T \square T') \geq n - 1 = \nabla(S_n \square S_{n'})$. Conjecture 1.1 now follows by applying (1). More precisely, since $\nabla(T \square T') \geq \nabla(S_n \square S_{n'})$, by (1) we get $|V(T \square T')| - \nabla(T \square T') \leq |V(T \square T')| - \nabla(S_n \square S_{n'})$. Thus we can conclude that

$$\begin{aligned} f(T \square T') &= |V(T \square T')| - \nabla(T \square T') \\ &\leq |V(T \square T')| - \nabla(S_n \square S_{n'}) \\ &= f(S_n \square S_{n'}), \end{aligned}$$

as desired.

Using the same line of reasoning as above, Theorem 2.1 also yields:

Corollary 2.2. *If T is a nontrivial tree, and $n' \geq |V(T)|$ an integer, then*

$$f(T \square S_{n'}) = |V(T)|n' - |V(T)| + 1.$$

3 The equality case in Theorem 2.1

In this section, we characterize the graphs that attain the equality in Theorem 2.1 and consequently in Conjecture 1.1. For this sake, the following two results are crucial.

Lemma 3.1. *If T and T' are trees of respective orders n and n' , where $n' > n \geq 2$, and $T' \not\cong S_{n'}$, then*

$$\nabla(T \square T') \geq n.$$

Proof. Let $V(T) = \{v_1, \dots, v_n\}$ and $V(T') = \{u_1, \dots, u_{n'}\}$. Suppose on the contrary that $\nabla(T \square T') \leq n - 1$ and let S be a decycling set in $T \square T'$ with $|S| = \nabla(T \square T') \leq n - 1$. We distinguish the following two cases.

Case 1: $\exists i, j \in [n], i \neq j$, such that $S \cap V({}^{v_i}T') = \emptyset$ and $S \cap V({}^{v_j}T') = \emptyset$.

Since $n' > n$, there also exist $k, \ell \in [n']$, $k \neq \ell$, such that $S \cap V(T'^{u_k}) = \emptyset$ and $S \cap V(T'^{u_\ell}) = \emptyset$. As in the proof of Theorem 2.1 we now consider the subgraph of $T \square T'$ induced by $V({}^{v_i}T') \cup V({}^{v_j}T') \cup V(T'^{u_k}) \cup V(T'^{u_\ell})$ to arrive at a contradiction.

Case 2: $S \cap V({}^{v_i}T') = \emptyset$, and $|S \cap V({}^{v_j}T')| = 1$ for every $j \in [n], j \neq i$.

Since T' is not a star, it contains an induced path on four vertices, say $u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}$. Let further v_ℓ be a neighbor of v_i in T . Let Y be the subgraph of $T \square T'$ induced by the vertices

$$\{v_i, v_\ell\} \times \{u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}\}.$$

The subgraph Y is isomorphic to $P_2 \square P_4$ and since $|S \cap V(Y)| \leq 1$ we can conclude that S is not a decycling set. $\square$

Lemma 3.2. *If T and T' are trees, both of order $n \geq 4$, and none of T and T' is a star, then*

$$\nabla(T \square T') \geq n.$$

Proof. Let $V(T) = \{v_1, \dots, v_n\}$, $V(T') = \{u_1, \dots, u_n\}$, and suppose for the sake of contradiction that $\nabla(T \square T') \leq n - 1$. Let S be decycling set in $T \square T'$ with $|S| = \nabla(T \square T') \leq n - 1$. If there exist $i, j \in [n], i \neq j$, such that $S \cap V({}^{v_i}T') = \emptyset$

and $S \cap V({}^{v_j}T') = \emptyset$, and there exist $k, \ell \in [n]$, $k \neq \ell$, such that $S \cap V(T^{u_k}) = \emptyset$ and $S \cap V(T^{u_\ell}) = \emptyset$, then we get a contradiction as in the proof of Theorem 2.1.

By the commutativity of the Cartesian product we may without loss of generality assume that in the remaining case there exists an integer $i \in [n]$ such that $S \cap V({}^{v_i}T') = \emptyset$, and $|S \cap V({}^{v_j}T')| = 1$ for every $j \in [n]$, $j \neq i$. Then we get a contradiction along the same lines as in Case 2 of the proof of Lemma 3.1. $\square$

The equality case in Theorem 2.1 now reads as follows.

Theorem 3.3. *Let T and T' be trees of respective orders n and n' , where $2 \leq n \leq n'$. Then the following holds.*

- (i) *If $n = n'$, then $\nabla(T \square T') = n - 1$ if and only if at least one of $T \cong S_n$ and $T' \cong S_{n'}$ holds.*
- (ii) *If $n < n'$, then $\nabla(T \square T') = n - 1$ if and only if $T' \cong S_{n'}$ (and T is arbitrary).*

Proof. Combine Theorem 2.1, Lemmas 3.1 and 3.2, and the facts $\nabla(P_2 \square P_2) = 1$ and $\nabla(P_3 \square P_3) = 2$. $\square$

Using Theorem 2.1, Lemmas 3.1 and 3.2, and (1), we also deduce the following.

Corollary 3.4. *If T and T' are trees of respective orders n and n' , where $2 \leq n \leq n'$, then $f(T \square T') \leq nn' - n + 1$. Moreover, the following holds.*

- (i) *If $n = n'$, then $f(T \square T') = nn' - n + 1$ if and only if at least one of $T \cong S_n$ and $T' \cong S_{n'}$ holds.*
- (ii) *If $n < n'$, then $f(T \square T') = nn' - n + 1$ if and only if $T' \cong S_{n'}$ (and T is arbitrary).*

4 Additional results

We have already proven that if T is a tree of order $n \geq 2$, and if $n' \geq n$, then $\nabla(T \square S_{n'}) = n - 1$. In the case when the star has order smaller than T , the following holds.

Theorem 4.1. *If $2 \leq n' < n$, and $T \not\cong S_n$ is a tree of order n , then*

$$n' \leq \nabla(T \square S_{n'}) \leq n - 1.$$

Proof. The lower bound follows by Lemma 3.1. To establish the upper bound, let $V(T) = \{v_1, \dots, v_n\}$ and $V(S_{n'}) = \{u_1, \dots, u_{n'}\}$, where $u_{n'}$ is the central vertex of $S_{n'}$. Just as in the last part of the proof of Theorem 2.1, we deduce that the set $\{(v_i, u_{n'}) : 2 \leq i \leq n\}$ is a decycling set in $T \square S_{n'}$. Consequently, $\nabla(T \square S_{n'}) \leq n - 1$. $\square$

Corollary 4.2. *If $T \not\cong S_{n+1}$ is a nontrivial tree of order $n + 1$, then*

$$\nabla(S_n \square T) = n \quad \text{and} \quad f(S_n \square T) = n^2.$$

In the following theorem, we establish a relationship between the decycling number of Cartesian products and the matching number of their factors. For its proof recall that the famous König-Egerváry theorem [14, 24] asserts that if G is a bipartite graph, then $\beta(G) = \alpha'(G)$.

Theorem 4.3. *If G_1 and G_2 are graphs, then*

$$\nabla(G_1 \square G_2) \geq \alpha'(G_1)\alpha'(G_2).$$

Moreover, if T is a tree, then $\nabla(T \square P_2) = \alpha'(T)$.

Proof. To prove the first assertion, consider maximum matchings M_1 and M_2 of G_1 and G_2 , respectively. Let V_i , $i \in [2]$, be the set of endvertices of the edges from M_i . Then the subgraph of $G_1 \square G_2$ induced by the vertices $V_1 \times V_2$ is isomorphic to the disjoint union of $\alpha'(G_1)\alpha'(G_2)$ copies of C_4 . It follows that any decycling set S in $G_1 \square G_2$ must contain at least $\alpha'(G_1)\alpha'(G_2)$ elements, that is, $\nabla(G_1 \square G_2) \geq \alpha'(G_1)\alpha'(G_2)$.

Let T be a tree. Then by the already proved inequality we have $\nabla(T \square P_2) \geq \alpha'(T)$. To prove that $\nabla(T \square P_2) \leq \alpha'(T)$, consider a smallest vertex cover $W = \{w_1, \dots, w_{\beta(T)}\}$ of T . By the definition of a vertex cover, $T[V(T) - W]$ is an edgeless graph, and by König-Egerváry theorem, $|W| = \alpha'(T)$. Let $V(P_2) = \{a, b\}$ and set $\Omega = \{(a, w_i) : i \in [\beta(T)]\}$. Since $T[V(T) - W]$ is an edgeless graph, the subgraph of $T \square P_2$ induced by $V(T \square P_2) \setminus \Omega$ is a tree. Therefore, $\nabla(T \square P_2) \leq |W| = \alpha'(T)$. $\square$

5 Concluding remarks

The main contribution of this paper is to demonstrate Conjecture 1.1 asserting that if T and T' are trees of respective orders n and n' , then $f(T \square T') \leq f(S_n \square S_{n'})$, and to characterize graphs that attain the inequality. In [43, Conjecture 5.1], it was also conjectured that

$$f(P_n \square P_{n'}) \leq f(T \square T')$$

holds. This remains open. In partial support of the fact that this conjecture is difficult to verify, we note that $\nabla(P_n \square P_{n'})$ (equivalently $f(P_n \square P_{n'})$) is not known yet in all the cases. The decycling number of the Cartesian product of paths has been investigated in [6], for the summary of this research see Section 3 in the survey [5]. Later, progress on $\nabla(P_n \square P_{n'})$ has been made in [27, 28, 32], however, to the best of our knowledge, the general solution is not yet known.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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