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Total k-coalition: bounds, exact values and an application to double coalition

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Let $G = \big(V(G), E(G)\big)$ be a graph with minimum degree k. A subset $S \subseteq V(G)$ is called a total k-dominating set if every vertex in G has at least k neighbors in S. Two disjoint sets $A, B \subset V(G)$ form a total k-coalition in G if none of them is a total k-dominating set in G but their union $A \cup B$ is a total k-dominating set. A vertex partition $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ of G is a total k-coalition partition if each set V_i forms a total k-coalition with another set V_j. The total k-coalition number $\mathrm{TC}_k(G)$ of G equals the maximum cardinality of a total k-coalition partition of G. In this paper, the above-mentioned concepts are investigated from combinatorial points of view. Several sharp lower and upper bounds on $\mathrm{TC}_k(G)$ are proved, where the main emphasis is given on the invariant when $k = 2$. As a consequence, the exact values of $\mathrm{TC}_2(G)$ when G is a cubic graph or a 4-regular graph are obtained. By using similar methods, an open question posed by Henning and Mojdeh regarding double coalition is answered. Moreover, $\mathrm{TC}_3(G)$ is determined when G is a cubic graph.

Keywords: total k-coalition; total k-domination; regular graph; double coalition

1 Introduction

Coalition in graphs was introduced by Haynes et al. (2020), and was studied in a number of subsequent papers Alikhani et al. (2024c); Bakhshesh et al. (2023); Haynes et al. (2021, 2023a,b,c). While the concept of coalition in graphs arises from (standard) domination in graphs, several authors studied variants of coalition that are related to other domination-type concepts. For instance, total coalition Alikhani et al. (2024a); Barát and Blázsik (2024); Henning and Jogan (2024), independent coalition Alikhani et al. (2025); Samadzadeh and Mojdeh (2024), paired coalition Samadzadeh et al. (2025), connected coalition Alikhani et al. (2024b) and k-coalition Jafari et al. (2025) correspond to total, independent, paired, connected and k-domination in graphs, respectively. In addition, transversal coalition in hypergraphs was introduced recently Henning and Yeo (2025) presenting a coalition version of transversals in hypergraphs.

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In this paper, we present a coalition counterpart to total k-domination Bermudo et al. (2019); Henning and Kazemi (2010); Kazemi (2012); Kulli (1991). The latter concept is also known under the names k-tuple total domination Henning and Kazemi (2010); Henning and Yeo (2010) and total k-tuple domination Bonomo et al. (2018). Total k-domination was studied from various perspectives, while the main focus was on the parameter when $k = 2$. An interplay between strong transversals in hypergraphs and total 2-domination served as a tool for obtaining sharp upper bounds on the total 2-domination number in general graphs and in cubic graphs Henning and Yeo (2010, 2013).

In Section 2, we establish notation and provide main definitions used in the paper. We also prove that a total k-coalition partition exists for all graphs with minimum degree at least k. In Section 3, several sharp bounds on the total k-coalition number are proved where our emphasis is given on $k = 2$. We prove that $\delta(G) - k + 2 \leq \mathrm{TC}_k(G) \leq n(G) - k + 1$ holds for any graph G with minimum degree at least k, where both bounds are sharp, and we characterize the graphs attaining the upper bound. One of our main results is the bound $\mathrm{TC}_2(G) \leq \lfloor\frac{\delta}{2}\rfloor(\Delta - 2\lfloor\frac{\delta}{2}\rfloor + 1) + \lceil\frac{\delta}{2}\rceil$, which holds for all graphs G with minimum degree $\delta \geq 2$ and maximum degree $\Delta \geq 4\lfloor\frac{\delta}{2}\rfloor - 2$, and we construct a family of graphs attaining the bound for every even minimum degree $\delta \geq 2$. In addition, this enables us to make use of some techniques that are more effective in relation to small values of k. In particular, as a consequence of this approach, we give the exact values of TC_2 in the case of cubic graphs and 4-regular graphs, and determine $\mathrm{TC}_3(G)$ for any cubic graph G.

Henning and Mojdeh (2025) studied double coalition in graphs, which can be considered as a closed variant of total 2-coalition. Indeed, if in the definition of total 2-coalition one replaces open neighborhoods with closed neighborhoods, we get the definition of double coalition. The corresponding invariant of G is the *double coalition number*, denoted $\mathrm{DC}(G)$. It was proved in Henning and Mojdeh (2025) that $\mathrm{DC}(G) = 4$ for any cubic graph G. Based on this fact and some other pieces of evidence, Henning and Mojdeh asked if $\mathrm{DC}(G) \leq 1 + \Delta(G)$ holds for any graph G with $\delta(G) = 3$. Some of the methods developed in Section 3 can be applied to this question. In fact, we give a negative answer to it by presenting an infinite family of graphs G of minimum degree 3 for which $\mathrm{DC}(G)$ is arbitrarily greater than $1+\Delta(G)$. In addition, we provide an upper bound for $\mathrm{DC}(G)$, which is expressed as a function of $\delta(G)$ and $\Delta(G)$, and is sharp for arbitrarily large $\delta(G)$ and $\Delta(G)$; the bound is proved for $\Delta(G) \geq 4\lceil\frac{\delta(G)}{2}\rceil - 3$.

2 Preliminaries

Throughout this paper, we consider G as a finite, connected and simple graph with vertex set $V(G)$ and edge set $E(G)$. We use West (1996) as a reference for terminology and notation which are not explicitly defined here. The (*open*) *neighborhood* of a vertex v is denoted by $N_G(v)$, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. When G will be clear from the context, we may simplify the notation to $N(v)$ and $N[v]$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set of vertices $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) if every vertex in $V(G) \setminus S$ $\big($resp. $V(G)\big)$ has a neighbor is S.

The study of a natural generalization of total domination was initiated by Kulli (1991) as follows. For $k \geq 1$ and a graph G of minimum degree at least k, a subset $S \subseteq V(G)$ is a *total k-dominating set* if $|N(v) \cap S| \geq k$ for all $v \in V(G)$. This concept was later investigated by Henning and Kazemi (2010) under the name *k-tuple total domination*. A vertex partition of such a graph into total k-dominating sets is called the *total k-domatic partition* of G. (Since $V(G)$ is a total k-dominating set of a graph G with minimum degree at least k, such a partition exists in G.) The *total k-domatic number* of G, denoted by

$d_{\times k,t}(G)$, is the maximum cardinality taken over all total k-domatic partitions of G Sheikholeslami and Volkmann (2014).

A *total k-coalition* in a graph G with $\delta(G) \geq k$ consists of two disjoint sets $U, V \subseteq V(G)$, such that neither U nor V is a total k-dominating set, but the union $U \cup V$ is a total k-dominating set in G. We say that V is a *partner* of U (and U is a partner of V). A *total k-coalition partition* in G is a vertex partition $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ such that every set V_i forms a total k-coalition with another set V_j. The *total k-coalition number* $\mathrm{TC}_k(G)$ equals the maximum cardinality taken over all total k-coalition partitions in G.[(i)]

Harary and Haynes (2000) initiated the study of tuple domination, which is conceptually close to total k-domination. In fact, for a graph G with $\delta(G) \geq k-1$, a *k-tuple dominating set* is defined by making use of closed neighborhood instead of open neighborhood in the definition of a total k-dominating set. Note that a 2-tuple dominating set is usually referred to as a *double dominating set*. In view of this, similarly to the concept of total 2-coalition, a *double coalition partition* can be defined for all graphs with minimum degree at least 1. Henning and Mojdeh (2025) investigated double coalition in graphs.

By an $\eta(G)$-partition, where $\eta \in \{\mathrm{TC}, \mathrm{DC}, d_{\times k,t}\}$, we mean an η-partition of G of largest cardinality.

First, we show that total k-coalition partitions exist for all graphs of minimum degree at least k.

Proposition 2.1 *Any graph G of minimum degree at least k has a total k-coalition partition.*

Proof: Let $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ be a $d_{\times k,t}(G)$-partition. Then $|\Omega| = d_{\times k,t}(G)$. We may assume that $V_1, \ldots, V_{|\Omega|-1}$ are minimal total k-dominating sets. Otherwise, we replace them with minimal total k-dominating sets $V_1' \subseteq V_1, \ldots, V_{|\Omega|-1}' \subseteq V_{|\Omega|-1}$ respectively, and replace $V_{|\Omega|}$ with $V_{|\Omega|} \cup \big(\cup_{i=1}^{|\Omega|-1} (V_i \setminus V_i')\big)$. Let $\{V_{i,1}, V_{i,2}\}$ be any partition of V_i for each $i \in [|\Omega|-1]$. It is then clear, by definitions, that $V_{i,1}$ and $V_{i,2}$ form a total k-coalition in G. If $V_{|\Omega|}$ turns out to be a minimal total k-dominating set, then $\Theta = \{V_{i,1}, V_{i,2}\}_{i=1}^{|\Omega|}$ is a total k-coalition partition in G, in which $\{V_{|\Omega|,1}, V_{|\Omega|,2}\}$ is any partition of $V_{|\Omega|}$. Otherwise, we replace $V_{|\Omega|}$ with a minimal total k-dominating set $V_{|\Omega|}' \subseteq V_{|\Omega|}$ and set $V_{|\Omega|}'' = V_{|\Omega|} \setminus V_{|\Omega|}'$. Notice that $V_{|\Omega|}''$ is not a total k-dominating set in G as Ω is a $d_{\times k,t}(G)$-partition. Let $\{V_{|\Omega|,1}', V_{|\Omega|,2}'\}$ be any partition of $V_{|\Omega|}'$. If $V_{|\Omega|}''$ forms a total k-coalition with $V_{|\Omega|,1}'$ or $V_{|\Omega|,2}'$, then $\{V_{i,1}, V_{i,2}\}_{i=1}^{|\Omega|-1} \cup \{V_{|\Omega|,1}', V_{|\Omega|,2}', V_{|\Omega|}''\}$ will be a total k-coalition partition. So, we assume that neither $V_{|\Omega|,1}'$ nor $V_{|\Omega|,2}'$ forms a total k-coalition with $V_{|\Omega|}''$. In such a case, $V_{|\Omega|,1}' \cup V_{|\Omega|}''$ is a total k-coalition partner of $V_{|\Omega|,2}'$. Therefore, $\{V_{i,1}, V_{i,2}\}_{i=1}^{|\Omega|-1} \cup \{V_{|\Omega|,1}' \cup V_{|\Omega|}'', V_{|\Omega|,2}'\}$ is a desired partition. □

Invoking the proof of Proposition 2.1, we deduce that $\mathrm{TC}_k(G) \geq 2d_{\times k,t}(G)$ for any graph G with $\delta(G) \geq k$.

3 Total k-coalition with an emphasis on $k = 2$

First, we present general lower and upper bounds on the total k-coalition number of a graph in terms of minimum and maximum degrees, respectively. Then, with emphasis on $k = 2$, we give two upper bounds on $\mathrm{TC}_2(G)$ in terms of both minimum and maximum degrees, and show that they are sharp by exhibiting an infinite family of graphs, which is illustrated in Example 3.9.

(i) One of the referees informed us that the total k-coalition in graphs for $k = 2$ was already studied under the name *double total coalition* in Golmohammadi et al. (2024). The special cases of the present Propositions 2.1 and 3.2, and of Lemma 3.3 for $k = 2$ can be found in Golmohammadi et al. (2024).

Theorem 3.1 *For any graph G with minimum degree at least k, $\mathrm{TC}_k(G) \geq \delta(G) - k + 2$. This bound is sharp.*

Proof: Let $v \in V(G)$ be a vertex of minimum degree and let $N(v) = \{v_1, \ldots, v_{\delta(G)}\}$. We set $V' = V(G) \setminus \{v_1, \ldots, v_{\delta(G)-k+1}\}$ and $V_i = \{v_i\}$ for $i \in [\delta(G) - k + 1]$. Obviously, no set V_i is a total k-dominating set in G. Moreover, since v has precisely $k-1$ neighbors in V', it follows that V' is not a total k-dominating set in G either. Let V_i be any set where $i \in [\delta(G) - k + 1]$ and u be any vertex in G. On the other hand, because $|N(u) \cap \big(V(G) \setminus (V' \cup V_i)\big)| \leq \delta(G) - k \leq \deg_G(u) - k$, it follows that u is adjacent to at least k vertices in $V' \cup V_i$. Hence, V_i and V' form a total k-coalition in G for each $i \in [\delta(G) - k + 1]$. The above discussion shows that $\Omega = \big\{V_1, \ldots, V_{\delta(G)-k+1}, V'\big\}$ is a total k-coalition partition in G. Thus, $\mathrm{TC}_k(G) \geq |\Omega| = \delta(G) - k + 2$.

To see that the bound is sharp, consider the cycle C_n and the complete graph K_n, for which we have $\mathrm{TC}_k(C_n) = 2$ (for $k = 2$) and $\mathrm{TC}_k(K_n) = n - k + 1$ (for each positive integer k with $n \geq k + 1$). □

Note that no two disjoint subsets $A, B \subseteq V(G)$ with $|A \cup B| \leq k$ form a total k-coalition in any graph G with $\delta(G) \geq k$. This leads to the clear upper bound $\mathrm{TC}_k(G) \leq |V(G)| - k + 1$. However, an infinite family of nontrivial graphs attain this upper bound. Recall that $G \vee H$ denotes the join of graphs G and H.

Proposition 3.2 *Let G be a graph of order n with $\delta(G) \geq k \geq 2$. Then, $\mathrm{TC}_k(G) \leq n - k + 1$ holds with equality if and only if $G \cong K_k \vee G'$, where G' is any graph of order $n - k$.*

Proof: Suppose first that $\mathrm{TC}_k(G) = n - k + 1$ and that $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ is a $\mathrm{TC}_k(G)$-partition. Then $|\Omega| = n - k + 1$. Letting V_1 form a total k-coalition with V_2, we get

$$n = |V_1 \cup V_2| + \textstyle\sum_{i=3}^{|\Omega|} |V_i| \geq (k + 1) + |\Omega| - 2.$$

Due to this, the equality $|\Omega| = n - k + 1$ guarantees that

(i) $|V_1 \cup V_2| = k + 1$ and $|V_i| = 1$ for each $i \in [|\Omega|] \setminus \{1, 2\}$,

(ii) $G[V_1 \cup V_2] \cong K_{k+1}$, and

(iii) every singleton set $|V_i|$, for $i \in [|\Omega|] \setminus \{1, 2\}$, forms a total k-coalition with V_1 or V_2.

By taking the above statements into account, without loss of generality, we may assume that $|V_1| = k$ and $|V_2| = 1$. Therefore, $\Omega = \{V_1\} \cup \big\{\{v\} \mid v \in V(G) \setminus V_1\big\}$, in which $\{v\}$ forms a total k-coalition with V_1 for each $v \in V(G) \setminus V_1$. In particular, $vx \in E(G)$ for all vertices $x \in V_1$ and $v \in V(G) \setminus V_1$. Therefore, $G \cong K_k \vee G'$, where $G' = G[V(G) \setminus V_1]$.

Conversely, assume that $G \cong K_k \vee G'$, in which G' is any graph of order $n - k$. It is then easy to see that $\{V(K_k)\} \cup \big\{\{v\} \mid v \in V(G')\big\}$ is a total k-coalition partition in G of cardinality $n - k + 1$, and hence $\mathrm{TC}_k(G) \geq n - k + 1$. This leads to the desired equality due to the upper bound $\mathrm{TC}_k(G) \leq n - k + 1$. □

The following lemma will turn out to be useful in several places of this paper.

Lemma 3.3 *Let G be a graph with minimum degree at least k and let Ω be a $\mathrm{TC}_k(G)$-partition. If $A \in \Omega$, then A forms a total k-coalition with at most $\Delta(G) - k + 1$ sets in Ω.*

Proof: Since A is not a total k-dominating set in G, there exists a vertex v such that $|N(v) \cap A| \leq k-1$. Let A form a total k-coalition with $A_1, \ldots, A_t \in \Omega$. By definition and since A does not totally k-dominate v, it follows that v has at least k neighbors in $A \cup A_t$ and $|N(v) \cap A_i| \geq 1$ for each $i \in [t]$. Then

$$\Delta(G) \geq |N(v)| \geq |N(v) \cap A_1| + \cdots + |N(v) \cap A_{t-1}| + |N(v) \cap (A \cup A_t)| \geq t - 1 + k\,,$$

which proves the result. □

Associated with any total k-coalition partition $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ in a graph G with $\delta(G) \geq k$, the *total k-coalition graph* $\mathrm{TC}_k\,\mathrm{G}(G, \Omega)$ has the set of vertices Ω, in which two vertices V_i and V_j are adjacent if they form a total k-coalition in G. Recall that $\alpha(G)$ and $\beta(G)$ denote the independence number and the vertex cover number of G, respectively.

Lemma 3.4 *Let G be a graph of minimum degree at least 2. If Ω is a total 2-coalition partition of G, then the following statements hold.*

(*i*) $\Delta\big(\mathrm{TC}_2\, G(G, \Omega)\big) \leq \Delta(G) - 1$, *and*

(*ii*) $\beta\big(\mathrm{TC}_2\, G(G, \Omega)\big) \leq \delta(G) - 1$.

Proof: (i) The statement follows from Lemma 3.3 with $k = 2$.

(ii) Let v be a vertex of minimum degree in G. We set $\Omega' = \{A \in \Omega \mid N(v) \cap A \neq \emptyset\}$. Obviously, $|\Omega'| \leq \delta(G)$. Suppose that $|\Omega'| = \delta(G)$. This shows that every set in Ω' contains precisely one vertex from $N(v)$. In such a situation, v has at most one neighbor in $A \cup (\cup_{S \in \Omega \setminus \Omega'} S)$ for each $A \in \Omega'$. This shows that no two sets in $\mathcal{I}_A = \{A\} \cup \{S \mid S \in \Omega \setminus \Omega'\}$ form a total 2-coalition in G, in which A is any set in Ω'. Equivalently, $\mathcal{I}_A$ is an independent set in $\mathrm{TC}_2\, G(G, \Omega)$, and hence $\alpha\big(\mathrm{TC}_2\, G(G, \Omega)\big) \geq |\mathcal{I}_A| = |\Omega| - \delta(G) + 1$. Using the equality $\alpha(H) + \beta(H) = |V(H)|$, which holds for each graph H (the Gallai Theorem), we infer that $\beta\big(\mathrm{TC}_2\, G(G, \Omega)\big) \leq \delta(G) - 1$. Moreover, if $|\Omega'| < \delta(G)$, then no two sets in $\{S \mid S \in \Omega \setminus \Omega'\}$ form a total 2-coalition in G. In a similar fashion, the inequality $\beta\big(\mathrm{TC}_2\, G(G, \Omega)\big) \leq \delta(G) - 1$ is obtained. □

Apart from bounding the total 2-coalition number of a graph, the following result together with Theorem 3.1 will enable us to obtain the exact value of this parameter for cubic graphs and for 4-regular graphs.

Theorem 3.5 *If G is a graph with $\delta(G) \geq 2$, then*

$$\mathrm{TC}_2(G) \leq \max\Big\{\Delta(G), \Big\lfloor \tfrac{\delta(G)}{2} \Big\rfloor (\Delta(G) - 4) + \delta(G)\Big\}.$$

Moreover, this bound is sharp.

Proof: Let Ω be a $\mathrm{TC}_2(G)$-partition. If $\delta(G) = 2$, then $\beta\big(\mathrm{TC}_2\, G(G, \Omega)\big) = 1$ by Lemma 3.4(ii). Therefore, $\mathrm{TC}_2\, G(G, \Omega)$ has a universal vertex V, that is, $V \in \Omega$ forms a total 2-coalition with any other set in Ω. So, we have $\mathrm{TC}_2(G) \leq \Delta(G)$ in view of Lemma 3.3.

Now let $\delta(G) = 3$ and let v (resp. u) be a vertex of maximum (resp. minimum) degree in G. We are going to show that $\mathrm{TC}_2(G) \leq \Delta(G)$ also in this case. Suppose that $\mathrm{TC}_2(G) > \Delta(G)$ and that $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ is a $\mathrm{TC}_2(G)$-partition. We distinguish two cases depending on the behavior of the sets in Ω.

Case 1. $|N(v) \cap V_i| \leq 1$, $i \in [|\Omega|]$.
Since no two vertices in $N(v)$ belong to the same set in Ω, we may assume that $N(v) \subseteq V_1 \cup \cdots \cup V_{\Delta(G)}$. Note that $V_{\Delta(G)+1}$ forms a total 2-coalition with V_j for some $j \in [|\Omega|]$. Since $|N(v) \cap V_j| \leq 1$ and because $|N(v) \cap (V_{\Delta(G)+1} \cup V_j)| \geq 2$, it follows that v has at least one neighbor in $V_{\Delta(G)+1}$, which contradicts the fact that $\deg_G(v) = \Delta(G)$.

Case 2. There are at least two vertices in $N(v)$ that belong to the same set in Ω.
In such a situation, we assume without loss of generality that $N(v) \subseteq V_1 \cup \cdots \cup V_p$ for some $p \in [\Delta(G)-1]$ and that V_i has at least one vertex in $N(v)$ for each $i \in [p]$. We may assume without loss of generality that $|N(v) \cap V_1| \geq 2, \ldots, |N(v) \cap V_t| \geq 2$ for some $t \in [p]$ and that $|N(v) \cap V_i| \leq 1$ for the remaining sets V_i in Ω. The following claim will turn out to be useful.

Claim 1. *For each index $i > t$, the set V_i forms a total 2-coalition with a set V_j for some $j \in [t]$.*

Proof of Claim 1. We first consider an arbitrary index $i > p$. Suppose V_i forms a total 2-coalition with V_j for some $j > t$. Then, the resulting inequality $|N(v) \cap (V_i \cup V_j)| \geq 2$ and the fact that $|N(v) \cap V_j| \leq 1$ imply that v has at least one neighbor in V_i, in contradiction with the equality $\deg_G(v) = \Delta(G)$. Therefore, V_i is a total 2-coalition partner of V_j for some $j \in [t]$. In particular, we may assume that $V_{\Delta(G)}$ forms a total 2-coalition with V_1.

We now consider any index $i \in [p] \setminus [t]$. Note that V_i does not form a total 2-coalition with any set V_j, with $j > p$, as proved above. Suppose that V_i is a total 2-coalition partner of V_j for some $j \in [p] \setminus [t]$. This in particular implies that $|N(u) \cap (V_i \cup V_j)| \geq 2$. On the other hand, $|N(u) \cap (V_1 \cup V_{\Delta(G)})| \geq 2$ as V_1 and $V_{\Delta(G)}$ form a total 2-coalition in G. Therefore, $3 = |N(u)| \geq |N(u) \cap (V_i \cup V_j)| + |N(u) \cap (V_1 \cup V_{\Delta(G)})| \geq 4$, a contradiction. Thus, V_i is a total 2-coalition partner of V_j for some $j \in [t]$, proving the claim. ($\Box$)

Invoking Claim 1, we assume without loss of generality that V_{t+1} forms a total 2-coalition with V_1. On the other hand, there exists a set $V_j \in \Omega$ that does not form a total 2-coalition with V_1, because V_1 is a total 2-coalition partner of at most $\Delta(G) - 1$ sets in Ω due to Lemma 3.3. We need to consider two more possibilities.

Subcase 2.1. V_j forms a total 2-coalition with V_r for some $r \in [|\Omega|] \setminus \{1, t+1\}$.
In this case we have $3 = |N(u)| \geq |N(u) \cap (V_1 \cup V_{t+1})| + |N(u) \cap (V_j \cup V_r)| \geq 4$, which is impossible.

Subcase 2.2. V_j forms a total 2-coalition with V_{t+1}.
If V_1 is a total 2-coalition partner of a set V_r for some $r \in [|\Omega|] \setminus \{t+1\}$, then $3 = |N(u)| \geq |N(u) \cap (V_1 \cup V_r)| + |N(u) \cap (V_j \cup V_{t+1})| \geq 4$, a contradiction. Therefore, we infer from the above argument that every set in $\Omega \setminus \{V_1\}$ forms a total 2-coalition with V_{t+1}. Hence, V_{t+1} forms a total 2-coalition with at least $\Delta(G)$ sets in Ω, contradicting Lemma 3.3 with $k = 2$.

By the above, we have proved that

$$\mathrm{TC}_2(G) \leq \Delta(G) \tag{1}$$

when $\delta(G) \in \{2, 3\}$.

Claim 2. $\mathrm{TC}_2(G) = 4$ *for any 4-regular graph G.*

Proof of Claim 2. Note that we already have $\mathrm{TC}_2(G) \geq 4$ by Theorem 3.1. Let $u \in V(G)$ and let $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ be a $\mathrm{TC}_2(G)$-partition. Suppose that $|N(u) \cap V_i| \leq 1$ for each $i \in [|\Omega|]$. So, we may assume without loss of generality that $N(u) \subseteq \cup_{i=1}^{4} V_i$ and that $|N(u) \cap V_i| = 1$ for each $i \in [4]$.

If $|\Omega| \geq 5$, then let V_5 form a total 2-coalition with V_j for some $j \in [|\Omega|]$. Since $|N(u) \cap V_j| \leq 1$ and $V_5 \cup V_j$ is a total 2-dominating set in G, it follows that u has at least one neighbor in V_5, contradicting the fact that $\deg_G(u) = 4$. Thus, $\mathrm{TC}_2(G) = |\Omega| \leq 4$.

Assume that there exists exactly one set $V_i \in \Omega$ such that $|N(u) \cap V_i| \geq 2$. We let, without loss of generality, $i = 1$. If $|N(u) \cap V_1| \in \{3, 4\}$, then every set in $\Omega \setminus \{V_1\}$ forms a total 2-coalition with V_1. Together with Lemma 3.3 for $k = 2$, this shows that $\mathrm{TC}_2(G) \leq 4$. Now let $|N(u) \cap V_1| = 2$. Due to this, we can assume that $|N(u) \cap V_2| = |N(u) \cap V_3| = 1$. Similarly, we deduce that every set in $\Omega \setminus \{V_1, V_2, V_3\}$ is necessarily a total 2-coalition partner of V_1 only. If V_1 forms a total 2-coalition with V_2 and V_3, respectively, then $\mathrm{TC}_2(G) \leq 4$ by Lemma 3.3. So, we assume that at least one of V_2 and V_3, say V_2, does not form a total 2-coalition with V_1. This implies that there exists a vertex x which is not totally 2-dominated by $V_1 \cup V_2$, and that V_2 forms a total 2-coalition with V_3.

Suppose that $\mathrm{TC}_2(G) \geq 5$. Since V_1 forms a total 2-coalition with V_4 and V_5, respectively, we have $|N(x) \cap (V_1 \cup V_4)| \geq 2$ and $|N(x) \cap (V_1 \cup V_5)| \geq 2$. Moreover, we have $|N(x) \cap (V_2 \cup V_3)| \geq 2$ as V_2 forms a total 2-coalition with V_3. Since $\deg_G(x) = 4$, it necessarily follows that $|N(x) \cap (V_1 \cup V_4)| = |N(x) \cap (V_1 \cup V_5)| = |N(x) \cap (V_2 \cup V_3)| = 2$. Therefore, $|N(x) \cap V_1| = 2$, in contradiction with the fact that x is not totally 2-dominated by $V_1 \cup V_2$.

Now suppose that $|N(x) \cap V_i| \geq 2$ for at least two sets $V_i \in \Omega$. Without loss of generality, we let $|N(x) \cap V_1| = |N(x) \cap V_2| = 2$. If one of the sets V_1 and V_2 forms a total 2-coalition with all other sets in Ω, then in view of Lemma 3.3, we have $\mathrm{TC}_2(G) = 4$. Suppose to the contrary that $\mathrm{TC}_2(G) > 4$. We may thus assume without loss of generality that V_1 forms a total 2-coalition with V_3 and V_4, and V_2 forms a total 2-coalition with V_5. Since V_1 is not a total 2-dominating set, there is a vertex $v \in V(G)$ such that $|N(v) \cap V_1| \leq 1$. If $N(v) \cap V_1 = \emptyset$, then $|N(v) \cap V_3| \geq 2$, and $|N(v) \cap V_4| \geq 2$. However, since $\deg_G(v) = 4$, there exists no vertex in $N(v)$ that belongs to $V_2 \cup V_5$, which is a contradiction to the fact that V_2 and V_5 form a total 2-coalition in G. The second possibility is that $|N(v) \cap V_1| = 1$. Similarly as in the previous case, there is a vertex in $N(v)$ that belongs to V_3 and a vertex in $N(v)$ that belongs to V_4. Since $\deg_G(v) = 4$, we derive $|N(v) \cap (V_2 \cup V_5)| \leq 1$, again a contradiction. This completes the proof of Claim 2. ($\square$)

In view of (1) and Claim 2, for graphs G with $\delta(G) \in \{2, 3\}$ the desired upper bound holds when $\Delta(G) \leq 4$. (Note that by invoking Claim 2 and (1), $\Delta(G) = 4$ leads to $\mathrm{TC}_2(G) = \Delta(G)$ or $\mathrm{TC}_2(G) \leq \Delta(G)$ when $\delta(G) = 4$ or $\delta(G) \in \{2, 3\}$, respectively. Moreover, if $\Delta(G) \leq 3$, then the resulting inclusion of $\delta(G)$ in $\{2, 3\}$ leads to $\mathrm{TC}_2(G) \leq \Delta(G)$ by (1).) Assume in the rest that $\Delta(G) \geq 5$. Let u be a vertex of minimum degree in G. If $|N(u) \cap V_i| \leq 1$ for each $i \in [|\Omega|]$, then $\mathrm{TC}_2(G) \leq \delta(G)$. Indeed, if there exists a set $V_i \in \Omega$ such that $N(u) \cap V_i = \emptyset$, then V_i does not form a total 2-coalition with any set in Ω, a contradiction. So, $\mathrm{TC}_2(G)$ is less than or equal to the desired upper bound.

Now assume that $|N(u) \cap V_i| \geq 2$ for some $i \in [|\Omega|]$. We may assume, without loss of generality, that $V_1, \ldots, V_p$ are the sets in Ω having at least two vertices in $N(u)$. Let $n_i = |N(u) \cap V_i|$ for each $i \in [p]$. It is clear that $p \leq \lfloor \delta(G)/2 \rfloor$. Setting

$$\Omega_i = \{V \in \Omega \mid V \text{ forms a total 2-coalition with } V_i \text{ and } N(u) \cap V = \emptyset\}$$

for each $i \in [p]$, we deduce from Lemma 3.3 that $|\Omega_i| \leq \Delta(G) - 1$.

Assume that $|\Omega_i| = \Delta(G) - 1$ for some $i \in [p]$. Since V_i is not a total 2-dominating set, there exists a vertex $x \in V(G)$ such that $|N(x) \cap V_i| \leq 1$. If $N(x) \cap V_i = \emptyset$, then x has at least two neighbors in each set V in Ω_i as V is a total 2-coalition partner of V_i. Therefore, $\deg_G(x) \geq 2|\Omega_i| = 2\Delta(G) - 2$, in contradiction with $\Delta(G) \geq 5$. So, we have $|N(x) \cap V_i| = 1$. In such a situation, the vertex x has at least one neighbor in each set $V \in \Omega_i$ because V forms a total 2-coalition with V_i. In particular, this implies that x has precisely one neighbor in every set in $\Omega_i \cup \{V_i\}$ and that $\deg_G(x) = \Delta(G)$ since $|\Omega_i \cup \{V_i\}| = \Delta(G)$. Suppose that there exists a set $U \in \Omega \setminus (\Omega_i \cup \{V_i\})$, which forms a total 2-coalition with a set $W \in \Omega$. Notice that $|N(x) \cap W| \in \{0,1\}$ if and only if $W \notin \Omega_i$ or $W \in \Omega_i$, respectively. This shows that U has at least one vertex in $N(u)$ as U and W form a total 2-coalition in G. Therefore, $\deg_G(x) \geq |N(x) \cap V_i| + |N(x) \cap (\cup_{V \in \Omega_i} V)| + |N(x) \cap U| \geq \Delta(G) + 1$, which is impossible. The above argument guarantees that $\Omega = \Omega_i \cup \{V_i\}$, and hence $\mathrm{TC}_2(G) = \Delta(G)$.

From here on, in view of the above discussion, we assume that $|\Omega_i| \leq \Delta(G) - 2$ for each $i \in [p]$. Moreover, note that

$$\Omega = \{V \in \Omega \mid N(u) \cap V \neq \emptyset\} \cup (\textstyle\bigcup_{i=1}^{p} \Omega_i).$$

We distinguish two cases depending of the behavior of the family $\{\Omega_i\}_{i=1}^{p}$.

Case A. $|\Omega_i| \leq \Delta(G) - 3$ for each $i \in [p]$.
In such a situation, since $\delta(G) - (n_1 + \cdots + n_p)$ is the number of sets in Ω that have exactly one vertex in $N(u)$, we can estimate as follows:

$$\begin{aligned}\mathrm{TC}_2(G) = |\Omega| &\leq p + p(\Delta(G) - 3) + \delta(G) - (n_1 + \cdots + n_p) \\ &\leq p(\Delta(G) - 2) + \delta(G) - 2p = p(\Delta(G) - 4) + \delta(G) \\ &\leq \lfloor \delta(G)/2 \rfloor (\Delta(G) - 4) + \delta(G).\end{aligned}$$

Case B. $|\Omega_i| = \Delta(G) - 2$ for some $i \in [p]$.
Without loss of generality, we may assume that $|\Omega_1| = \Delta(G) - 2$. Since V_1 is not a total 2-dominating set in G, there exists a vertex x such that $|N(x) \cap V_1| \leq 1$. If $N(x) \cap V_1 = \emptyset$, then the vertex x has at least two neighbors in each set in Ω_1. Hence, $\deg_G(x) \geq 2|\Omega_1| \geq 2\Delta(G) - 4$, contradicting the fact that $\Delta(G) \geq 5$. Therefore, $|N(x) \cap V_1| = 1$. Then, x has at least one neighbor in each set in Ω_1. If there exists a set $U \in \Omega_i \setminus \Omega_1$ for some $i \in [p]$, then

$$\deg_G(x) \geq |N(x) \cap V_1| + |N(x) \cap (\cup_{V \in \Omega_1} V)| + |N(x) \cap (U \cup V_i)| \geq \Delta(G) + 1,$$

a contradiction. This shows that $\Omega_i \subseteq \Omega_1$ for every $i \in [p]$. Therefore,

$$\begin{aligned}\mathrm{TC}_2(G) = |\Omega| &\leq \Delta(G) - 1 + p - 1 + \delta(G) - (n_1 + \cdots + n_p) \\ &\leq \Delta(G) + \delta(G) - p - 2 \\ &\leq \Delta(G) + \delta(G) - 3\,. \end{aligned} \tag{2}$$

By (2), the desired upper bound holds when $\delta(G) \leq 3$. So, we assume that $\delta(G) \geq 4$. Since $\Delta(G) \geq 5$, by (2) we thus have

$$\mathrm{TC}_2(G) \leq \Delta(G) + \delta(G) - 3 \leq \lfloor \delta(G)/2 \rfloor (\Delta(G) - 4) + \delta(G)\,.$$

This completes the proof of the desired upper bound. The sharpness of the bound is presented in Example 3.9. □

Using the obtained bounds, we derive the values of the total k-coalition numbers of cubic graphs for all $k \geq 2$. In this case, the necessary condition $\delta(G) \geq k$ implies that $k \in \{2, 3\}$.

Theorem 3.6 *If G is a cubic graph, then*

$$\mathrm{TC}_k(G) = \begin{cases} 3 & \text{if } k = 2, \\ 2 & \text{if } k = 3. \end{cases}$$

Proof: The equality $\mathrm{TC}_2(G) = 3$ is an immediate consequence of Theorems 3.1 and 3.5. So, we turn our attention to $k = 3$. The lower bound $\mathrm{TC}_3(G) \geq 2$ is clear from Theorem 3.1. Let Ω be a $\mathrm{TC}_3(G)$-partition. Since G is a cubic graph, it follows that for any vertex $v \in V(G)$, there exists a set $A \in \Omega$ such that $|N(v) \cap A| \leq 1$. Let B be any set in Ω that forms a total 3-coalition with A. Particularly, we have $N(v) \subseteq A \cup B$ and $|N(v) \cap B| \geq 2$. Hence, $N(v) \cap C = \emptyset$ for each $C \in \Omega \setminus \{A, B\}$. Thus, if such a set C existed, it would not form a total 3-coalition with any set in Ω, therefore there is no set in $\Omega \setminus \{A, B\}$. Hence, $\mathrm{TC}_3(G) = |\Omega| = 2$. $\square$

A graph G with $\Delta(G) = 3$ is called *subcubic*. If G is a subcubic graph with $\delta(G) \geq 2$, then $\mathrm{TC}_2(G) \in \{2, 3\}$ and both options are possible. Say, $\mathrm{TC}_2(K_{2,3}) = 3$, while if G is obtained from the cycle C_5 by adding an edge between two nonadjacent vertices, $\mathrm{TC}_2(G) = 2$. In view of these remarks, it is natural to pose the following.

Problem 1 *Characterize subcubic graphs G with $\mathrm{TC}_2(G) = 2$.*

Note that Claim 2 of the proof of Theorem 3.5 provides the following auxiliary result, which can be of independent interest.

Proposition 3.7 *If G is a 4-regular graph, then $\mathrm{TC}_2(G) = 4$.*

In the next result, we provide a different upper bound on $\mathrm{TC}_2(G)$ for graphs G with sufficiently large maximum degree. If $\delta(G) \geq 6$, the bound improves that of Theorem 3.5, yet there is no restriction on $\Delta(G)$ in Theorem 3.5.

Theorem 3.8 *If G is a graph with $\delta = \delta(G) \geq 2$ and $\Delta = \Delta(G) \geq 4\left\lfloor \frac{\delta}{2} \right\rfloor - 2$, then*

$$\mathrm{TC}_2(G) \leq \left\lfloor \frac{\delta}{2} \right\rfloor (\Delta - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1) + \left\lceil \frac{\delta}{2} \right\rceil.$$

Moreover, the bound is sharp for every even minimum degree $\delta \geq 2$.

Proof: Since by Theorem 3.5, the upper bound holds when $\delta(G) \leq 4$, we may restrict our attention to a graph G with $\delta(G) \geq 5$. Let $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ be a $\mathrm{TC}_2(G)$-partition, and let u be a vertex of minimum degree in G. If $|N(u) \cap V_i| \leq 1$ for all $i \in [|\Omega|]$, then we easily see that $\mathrm{TC}_2(G) \leq \delta(G)$, which is in turn bounded from above by $\left\lfloor \frac{\delta}{2} \right\rfloor (\Delta - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1) + \left\lceil \frac{\delta}{2} \right\rceil$, and so the statement of the theorem is proved.

Thus, we may assume that there exists an integer $s \geq 1$ such that, without loss of generality, $|N(u) \cap V_i| \geq 2$ for all $V_i \in \Omega$ with $i \in [s]$, while $|N(u) \cap V_i| \leq 1$ if $i > s$. Clearly, $s \leq \lfloor \delta/2 \rfloor$. Let $\Psi \subsetneq \Omega$ be the set of all V_j such that $V_j \cap N(u) = \emptyset$. If $\Psi = \emptyset$, then $\mathrm{TC}_2(G) = |\Omega| \leq s + (\delta - 2s) < \delta$, which is impossible. Thus, $\Psi \neq \emptyset$. Note that every $V_j \in \Psi$ forms a total 2-coalition with some V_i, where $i \in [s]$. In other words, in the graph $\mathrm{TC}_2\, G(G, \Omega)$, the vertices $V_1, \ldots, V_s$ dominate all vertices in Ψ. Let r be the smallest number of vertices in $\{V_1, \ldots, V_s\}$ that dominate all vertices of Ψ. By renaming the sets if necessary, let $V_1, \ldots, V_r$ dominate all vertices in Ψ. Clearly, $r \in [s]$.

For each $i \in [r]$, let Ω_i be the set of neighbors of V_i in the graph $\mathrm{TC}_2\,G(G,\Omega)$ that belong to Ψ. By our choice of r, we deduce that $\Omega_i \nsubseteq \cup_{j\in[r]\setminus\{i\}}\Omega_j$. Since V_i is not a total 2-dominating set of G, there exists a vertex $v \in V(G)$ such that $|V_i \cap N(v)| \le 1$. Assume that $|V_i \cap N(v)| = 1$. (The case when $|V_i \cap N(v)| = 0$ uses similar, yet simpler arguments.) Then, all sets in Ω_i must have a non-empty intersection with $N(v)$. In addition, for every set $V_j \in \{V_1,\dots,V_r\} \setminus \{V_i\}$, there exists a set $V_{j'} \in \Omega_j \setminus \cup_{t\in[r]\setminus\{j\}}\Omega_t$. Thus, there are at least two vertices in $N(v)$ that belong to $V_j \cup V_{j'}$. Altogether, we infer that $|\Omega_i| + 2r - 1 \le \deg_G(v) \le \Delta$. Thus, for all $i \in [r]$, we have

$$|\Omega_i| \le \Delta - 2r + 1. \tag{3}$$

Since $\{V_1,\dots,V_r\}$ dominates Ψ in $\mathrm{TC}_2\,G(G,\Omega)$, we deduce that

$$\mathrm{TC}_2(G) = |\Omega| \le s + \sum_{i=1}^{r}|\Omega_i| + \delta - 2s = \sum_{i=1}^{r}|\Omega_i| + \delta - s. \tag{4}$$

Combining (3) and (4) we infer that

$$\mathrm{TC}_2(G) \le r(\Delta - 2r + 1) + \delta - s. \tag{5}$$

Note that $r \le s \le \lfloor \delta/2 \rfloor$, and so the upper bound in (5) is in turn bounded from above as follows:

$$r(\Delta - 2r + 1) + \delta - s \le r(\Delta - 2r + 1) + \delta - r = f(r). \tag{6}$$

If $r = \lfloor \delta/2 \rfloor$, then we get the desired upper bound. On the other hand, since $\Delta \ge 4\lfloor \delta/2 \rfloor - 2$, it follows that f is an increasing function on $[1, \lfloor \delta/2 \rfloor - 1]$. Therefore,

$$\begin{aligned}\mathrm{TC}_2(G) &\le f(r) \le f(\left\lfloor \frac{\delta}{2} \right\rfloor - 1) = f(\left\lfloor \frac{\delta}{2} \right\rfloor) + 4\left\lfloor \frac{\delta}{2} \right\rfloor - 2 - \Delta \le f(\left\lfloor \frac{\delta}{2} \right\rfloor) \\ &= \left\lfloor \frac{\delta}{2} \right\rfloor (\Delta - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1) + \left\lceil \frac{\delta}{2} \right\rceil ,\end{aligned}$$

as desired.

The sharpness of this upper bound is illustrated in Example 3.9. □

Example 3.9 *(Sharpness of the bounds in Theorems 3.5 and 3.8) To see that the upper bounds given in Theorems 3.5 and 3.8 are sharp, we introduce the graphs $G(d,\ell)$, where $d \ge 2$ and $\ell \ge 2d-1$ are integers, as follows. For each $i \in [d]$, consider the set of vertices*

$$A_i = \{x_{i,j} : j \in [\ell]\},$$

and join a new vertex y_i to each $x_{i,j}$ so that $A_i \cup \{y_i\}$ induces a star $K_{1,\ell}$. For all $i \in [d-1]$, add to the graph the edge $x_{i,1}x_{i+1,1}$ if i is odd, and the edge $x_{i,2}x_{i+1,2}$ if i is even. Next, for each $i \in [d]$ and $j \in [\ell]$, take a copy of the complete tripartite graph $K_{d,d,d}$, and denote it by $B_{i,j}$. Choose $S_{i,j} \subset V(B_{i,j})$ with $|S_{i,j}| = 2d-1$ such that $B_{i,j} - S_{i,j}$ contains an independent set with d vertices. For all $i \in [d]$ and $j \in [\ell]$, join $x_{i,j}$ to the vertices in $S_{i,j}$. Similarly, for each $i \in [d]$ take a copy of $K_{d,d,d}$, denote it by B_i, and choose $S_i \subset V(B_i)$ with $|S_i| = 2d-1$ such that $B_i - S_i$ contains an independent set with d vertices.

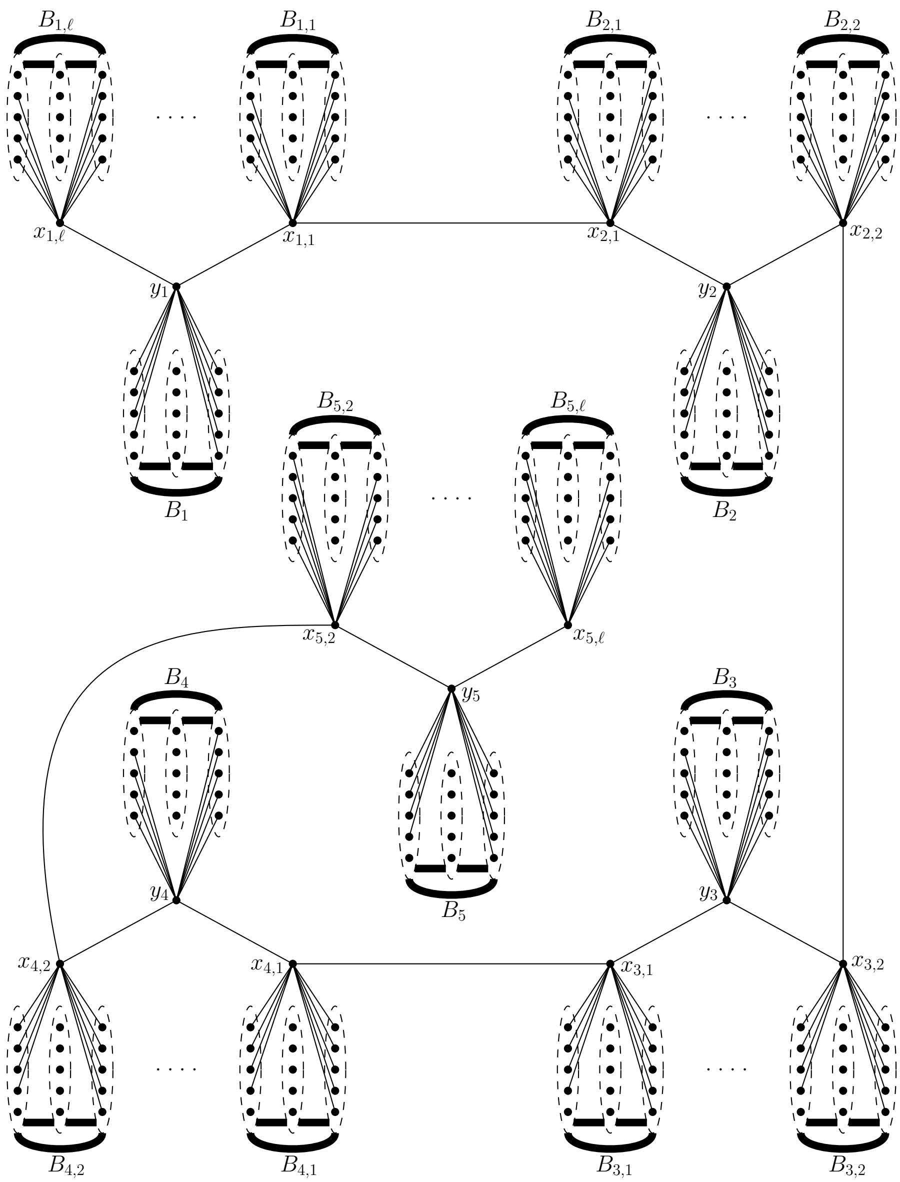

Fig. 1: The graph $G(d, \ell)$ (for $d = 5$ and $\ell \geq 9$) given in Example 3.9 with $\delta\big(G(5, \ell)\big) = 10$, $\Delta\big(G(5, \ell)\big) = 9 + \ell$, and $\mathrm{TC}_2\left(G(5, \ell)\right) = 5(\ell + 1)$. Here, each thick line/curve segment represents all possible edges between the corresponding partite sets.

For all $i \in [d]$, join y_i to the vertices in S_i. The resulting graph is connected, and we denote it by $G(d, \ell)$. The graph $G(5, \ell)$, for $\ell \geq 9$, is depicted in Fig. 1.

The degree in $G(d,\ell)$ of the vertices from the sets A_i, $B_{i,j}$ and B_i is either $2d$ or $2d+1$. On the other hand, the degree of the vertices y_i is $\ell+2d-1$. Since $\ell \geq 2$, we get $2d+1 \leq \ell+2d-1$. So,

$$\delta\big(G(d,\ell)\big) = 2d \quad \textit{and} \quad \Delta\big(G(d,\ell)\big) = \ell+2d-1,$$

which satisfy $\Delta\big(G(d,\ell)\big) \geq 4\lfloor \delta\big(G(d,\ell)\big)/2 \rfloor - 2$ as $\ell \geq 2d-1$.

Now, let us present a partition $\Omega = \{V_1,\dots,V_{|\Omega|}\}$ of $V\big(G(d,\ell)\big)$, where $|\Omega| = d(\ell+1)$, for which we will show it is a total 2-coalition partition. For all $i \in [d]$ and $j \in [\ell]$, vertices of each partite set from $B_{i,j}$ are distributed to the sets $V_1,\dots,V_d$, respectively. In addition, note that $S_{i,j}$ is chosen in such a way that $V(B_{i,j}) \setminus S_{i,j}$ contains all vertices of one partite set of $B_{i,j}$ and an additional vertex, and we may assume that this additional vertex is in the set V_i. For all $i \in [d]$, we let y_i belong to V_i. Similarly, vertices of each partite set from B_i are distributed to the sets $V_1,\dots,V_d$, respectively. In addition, since S_i is chosen in such a way that $V(B_i) \setminus S_i$ contains all vertices of one partite set of B_i and an additional vertex, we may assume that this additional vertex is put in the set V_i. Finally, for each $i \in [d]$, the vertices $x_{i,1}, x_{i,2}, \dots, x_{i,\ell} \in A_i$ are distributed into the sets

$$V_{d+(i-1)\ell+1}, V_{d+(i-1)\ell+2}, \dots, V_{d+i\ell},$$

respectively. More precisely, $V_{d+(i-1)\ell+j} = \{x_{i,j}\}$ for all $i \in [d]$ and $j \in [\ell]$. The set V_i, $i \in [d]$, is not a total 2-dominating set because y_i is adjacent to exactly one vertex of V_i. On the other hand, V_i forms a total 2-coalition with every set in $\{V_{d+(i-1)\ell+1}, V_{d+(i-1)\ell+2}, \dots, V_{d+i\ell}\}$. Hence, Ω is a total 2-coalition partition of cardinality $d(\ell+1)$. Therefore,

$$\begin{aligned}
\mathrm{TC}_2(G(d,\ell)) \quad & \geq d(\ell+1) \\
& = d\big((\ell+2d-1)-2d+1\big)+d \\
& = \left\lfloor \tfrac{\delta(G(d,\ell))}{2} \right\rfloor (\Delta(G(d,\ell)) - 2\left\lfloor \tfrac{\delta(G(d,\ell))}{2} \right\rfloor + 1) + \left\lceil \tfrac{\delta(G(d,\ell))}{2} \right\rceil \\
& \geq \mathrm{TC}_2(G(d,\ell)).
\end{aligned}$$

We infer that the graphs $G(d,\ell)$ attain the upper bound of Theorem 3.8.

It should be noted that the bound is also sharp when $\delta = 2$. In fact, the upper bound gives the exact value $\mathrm{TC}_2(C_n) = 2$ in this case.

In view of Theorem 3.1 with $k=2$, the upper bound given in Theorem 3.5 gives us the exact value of the total 2-coalition number of r-regular graphs for $r \in \{2,3,4\}$. On the other hand, for each $\ell \geq 3$, we get

$$\mathrm{TC}_2\left(G(2,\ell)\right) = 2\ell+2 = \left\lfloor \tfrac{\delta(G(2,\ell))}{2} \right\rfloor \big(\Delta\big(G(2,\ell)\big) - 4\big) + \delta\big(G(2,\ell)\big).$$

Therefore, the graphs $G(2,\ell)$ attain the upper bound of Theorem 3.5.

It is possible that Theorem 3.8 is true even if the restriction $\Delta(G) \geq 4\lfloor \delta(G)/2 \rfloor - 2$ is omitted. In spite of extensive investigations, we could not find a counterexample to that statement. In addition, we base our suspicion that the restriction can be omitted because this is true in the case when $\delta(G) \leq 5$, which follows from Theorem 3.5. Based on the above discussion, we propose the following:

Conjecture 1 *If G is a graph with $\delta = \delta(G) \geq 2$ and $\Delta = \Delta(G)$, then*

$$\mathrm{TC}_2(G) \leq \left\lfloor \frac{\delta}{2} \right\rfloor (\Delta - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1) + \left\lceil \frac{\delta}{2} \right\rceil.$$

As noted earlier, if the conjecture holds, then it is widely sharp. Notably, the family of graphs $G(d,\ell)$ from Example 3.9 shows that it is sharp for an arbitrary even $\delta \geq 2$ and any $\Delta \geq 2\delta - 2$. In addition, there are regular graphs G with even $\delta(G) = \Delta(G)$ for which the bound is sharp. Consider the complete graph K_{2p+1} for any integer $p \geq 1$. Note that

$$\mathrm{TC}_2(K_{2p+1}) = 2p = \left\lfloor \frac{\delta}{2} \right\rfloor (\Delta - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1) + \left\lceil \frac{\delta}{2} \right\rceil .$$

4 On two open problems on double coalition

Henning and Mojdeh (2025) proved that $\mathrm{DC}(G) \leq 1 + \Delta(G)$ for all graphs G with $\delta(G) \in \{1, 2\}$. They posed the following:

Question 1. If G is a graph with $\delta(G) = 3$, then is it true that $\mathrm{DC}(G) \leq 1 + \Delta(G)$?

This is indeed the case when the graph is cubic, as they proved that $\mathrm{DC}(G) = 4$ for each cubic graph G. Despite the above-mentioned pieces of evidence in support of the inequality, in what follows, we answer this question in the negative.

Moreover, by utilizing the approach developed for total 2-domination (Theorem 3.8), we present a general upper bound on the double coalition number of a graph G in terms of minimum and maximum degrees, provided that $\Delta(G) \geq 4\lceil \delta(G)/2 \rceil - 3$. Since the bound is sharp for all odd $\delta(G)$, where $\Delta(G)$ can be arbitrarily large, we see that the value of $\mathrm{DC}(G)$ can be relatively close to $\lceil \delta(G)/2 \rceil \Delta(G)$. Given a graph G and a double coalition partition Ω, the graph $\mathrm{DCG}(G, \Omega)$ is defined with vertex set Ω in which two vertices/sets are adjacent if they form a double coalition.

Theorem 4.1 *If G is a graph with $\delta = \delta(G) \geq 1$ and $\Delta = \Delta(G) \geq 4\left\lceil \frac{\delta}{2} \right\rceil - 3$, then*

$$\mathrm{DC}(G) \leq \left\lceil \frac{\delta}{2} \right\rceil (\Delta - 2\left\lceil \frac{\delta}{2} \right\rceil + 2) + 1 + \left\lfloor \frac{\delta}{2} \right\rfloor .$$

Moreover, the bound is sharp, and is attained for graphs with any odd minimum degree $\delta \geq 3$.

Proof: Let $\Omega = \{V_1, \ldots, V_{|\Omega|}\}$ be a $\mathrm{DC}(G)$-partition, and let u be a vertex of minimum degree in G. If $|N[u] \cap V_i| \leq 1$ for all $i \in [|\Omega|]$, then $\mathrm{DC}(G) \leq \delta(G) + 1$, which directly implies the statement of the theorem.

Thus, we may assume that there exists an integer $s \geq 1$ such that, without loss of generality, $|N[u] \cap V_i| \geq 2$ for all $V_i \in \Omega$ with $i \in [s]$, while $|N[u] \cap V_i| \leq 1$ if $i > s$. Clearly, $s \leq \lceil \delta/2 \rceil$. Let $\Psi \subsetneq \Omega$ be the set of all V_j such that $V_j \cap N[u] = \emptyset$. If $\Psi = \emptyset$, then $\mathrm{DC}(G) = |\Omega| \leq s + (\delta + 1 - 2s) < \delta + 1$, which is impossible. Thus, $\Psi \neq \emptyset$. Note that every $V_j \in \Psi$ forms a double coalition with some V_i, where $i \in [s]$. In other words, in the graph $\mathrm{DCG}(G, \Omega)$, the vertices $V_1, \ldots, V_s$ dominate all vertices in Ψ. Let r be the smallest number of vertices in $\{V_1, \ldots, V_s\}$ that dominate all vertices of Ψ. By renaming the sets if necessary, let $V_1, \ldots, V_r$ dominate all vertices in Ψ. Clearly, $r \in [s]$.

For each $i \in [r]$, let Ω_i be the set of neighbors of V_i in the graph $\mathrm{DCG}(G, \Omega)$ that belong to Ψ. By our choice of r, we deduce that $\Omega_i \nsubseteq \cup_{j \in [r] \setminus \{i\}} \Omega_j$. Since V_i is not a double dominating set of G, there exists a vertex $v \in V(G)$ such that $|V_i \cap N[v]| \leq 1$. Assume that $|V_i \cap N[v]| = 1$. (The case when $|V_i \cap N[v]| = 0$ uses similar, yet slightly simpler arguments.) Then, all sets in Ω_i must have a non-empty intersection with $N[v]$. In addition, for every set $V_j \in \{V_1, \ldots, V_r\} \setminus \{V_i\}$, there exists a set

$V_{j'} \in \Omega_j \setminus \cup_{t\in[r]\setminus\{j\}}\Omega_t$. Thus, there are at least two vertices in $N[v]$ that belong to $V_j \cup V_{j'}$. Altogether, we infer that $|\Omega_i| + 2r - 1 \le \deg_G(v) + 1 \le \Delta + 1$. Thus, for all $i \in [r]$, we have

$$|\Omega_i| \le \Delta - 2r + 2. \tag{7}$$

Since $\{V_1, \dots, V_r\}$ dominates Ψ in $\mathrm{DCG}(G, \Omega)$, we deduce that

$$\mathrm{DC}(G) = |\Omega| \le s + \sum_{i=1}^{r} |\Omega_i| + \delta + 1 - 2s = \sum_{i=1}^{r} |\Omega_i| + \delta + 1 - s. \tag{8}$$

Combining (7) and (8) we get

$$\mathrm{DC}(G) \le r(\Delta - 2r + 2) + \delta + 1 - s. \tag{9}$$

Note that $r \le s \le \lceil \delta/2 \rceil$, and so the upper bound in (9) is in turn bounded from above as follows:

$$r(\Delta - 2r + 2) + \delta + 1 - s \le r(\Delta - 2r + 2) + \delta + 1 - r = f(r). \tag{10}$$

If $r = \lceil \delta/2 \rceil$, then we get the desired upper bound. Now let $r < \lceil \delta/2 \rceil$. Since $\Delta \ge 4\lceil \delta/2 \rceil - 3$, it follows that f is a nondecreasing function on $[1, \lceil \delta/2 \rceil - 1]$. Therefore,

$$\begin{aligned}\mathrm{DC}(G) &\le f(r) \le f(\lceil \frac{\delta}{2} \rceil - 1) = f(\lceil \frac{\delta}{2} \rceil) + 4\lceil \frac{\delta}{2} \rceil - \Delta - 3 \le f(\lceil \frac{\delta}{2} \rceil) \\ &= \left\lceil \frac{\delta}{2} \right\rceil (\Delta - 2\left\lceil \frac{\delta}{2} \right\rceil + 2) + 1 + \left\lfloor \frac{\delta}{2} \right\rfloor ,\end{aligned}$$

as desired.

For the sharpness of this upper bound, we present the family of graphs $H(r,t)$, where $r \ge 2$ and $t \ge 4$ are integers with $t \ge 2r - 1$, as follows. For each $i \in [r]$, consider the set of vertices

$$A_i = \{x_{i,j} : j \in [t]\},$$

and join a new vertex y_i to each $x_{i,j}$ so that $A_i \cup \{y_i\}$ induces a star $K_{1,t}$. For all $i \in [r-1]$, add to the graph the edge $x_{i,1}x_{i+1,1}$ if i is odd, and the edge $x_{i,2}x_{i+1,2}$ if i is even. Next, for each $i \in [r]$ and $j \in [t]$, take a copy of the complete graph K_{2r}, and denote them by $B_{i,j}$. Join $x_{i,j}$ with all vertices from $B_{i,j}$. Similarly, for each $i \in [r]$ take a copy of K_{2r}, denote it by B_i, and join y_i to $2r-2$ vertices in B_i. The resulting graph is connected, and we denote it by $H(r,t)$. See Fig. 2 depicting $H(5,t)$ for $t \ge 9$.

The degree in $H(r,t)$ of the vertices in the sets $B_{i,j}$ is $2r$, the degree of the vertices in the sets B_i is either $2r-1$ or $2r$, and the degree of the vertices in the sets A_i is either $2r+1$ or $2r+2$. On the other hand, the degree of the vertices y_i is $2r-2+t$, which is greater than or equal to $2r+2$ as $t \ge 4$. Thus,

$$\delta\big(H(r,t)\big) = 2r - 1 \text{ and } \Delta\big(H(r,t)\big) = 2r - 2 + t.$$

Moreover, $\Delta\big(H(r,t)\big) \ge 4\lceil \delta\big(H(r,t)\big)/2 \rceil - 3$ as $t \ge 2r - 1$.

Now, let us present a partition $\Omega = \{V_1, \dots, V_{|\Omega|}\}$ of $V\big(H(r,t)\big)$, where $|\Omega| = r(t+1)$, for which we will show it is a double coalition partition. For all $i \in [r]$ and $j \in [t]$, we let $|B_{i,j} \cap V_k| = 2$ for every $k \in [r]$. For all $i \in [r]$, we let y_i belong to V_i. Next, for every $i \in [r]$, let $|B_i \cap V_k| = 2$ for all $k \in [r]$

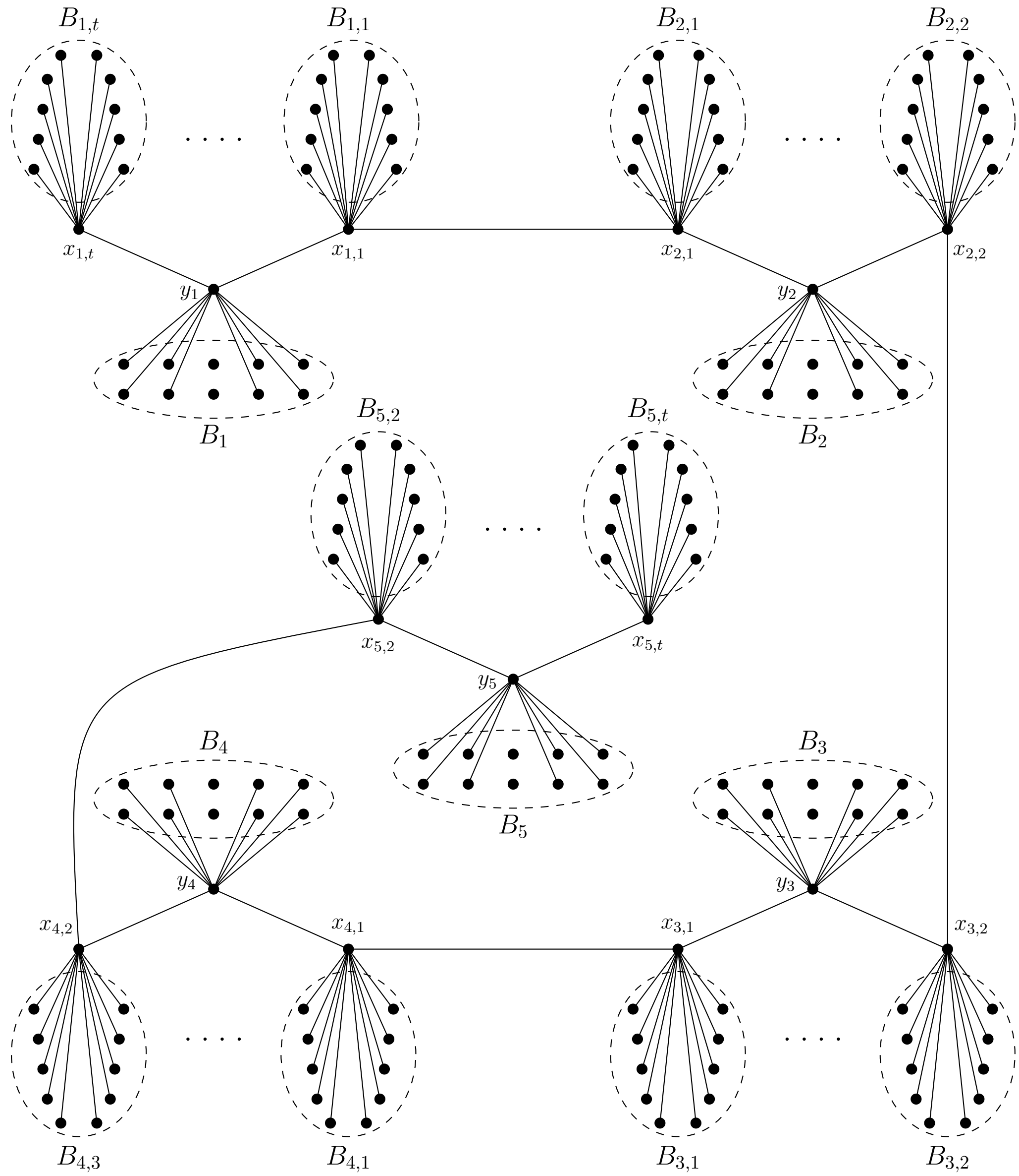

Fig. 2: The graph $H(r,t)$ $\big($for $r = 5$ and $t \geq 9\big)$ given in the proof of Theorem 4.1 with $\delta\big(H(5,t)\big) = 9$, $\Delta\big(H(5,t)\big) = 8 + t$, and $\text{DC}\,\big(H(5,t)\big) = 5(t+1)$. Vertices in each of the dashed ellipses form the clique K_{10}.

so that the two vertices of degree $2r-1$ in each B_i belong to V_i. Finally, for each $i \in [r]$ and $j \in [t]$, let $V_{r+(i-1)t+j} = \{x_{i,j}\}$. No set V_i, $i \in [r]$, is a double dominating set, but it forms a double coalition with every set in $\{V_{r+(i-1)t+1}, V_{r+(i-1)t+2}, \ldots, V_{r+it}\}$. Hence, Ω is a double coalition partition of cardinality $r(t+1)$. Therefore,

$$\begin{aligned}\mathrm{DC}\big(H(r,t)\big) &\geq r(t+1)\\ &= r\big((2r-2+t)-2r+2\big)+1+(r-1)\\ &= \left\lceil \tfrac{\delta(H(r,t))}{2}\right\rceil \big(\Delta(H(r,t))-2\left\lceil \tfrac{\delta(H(r,t))}{2}\right\rceil+2\big)+1+\left\lfloor \tfrac{\delta(H(r,t))}{2}\right\rfloor\\ &\geq \mathrm{DC}\big(H(r,t)\big).\end{aligned}$$

Hence, the graphs $H(r,t)$ attain the upper bound. □

The graphs $H(r,t)$ attain the bound in Theorem 4.1 for all $r \geq 2$ and all $t \geq 4$, such that $t \geq 2r-1$, and

$$\mathrm{DC}\big(H(r,t)\big) = \left\lceil \frac{\delta(H(r,t))}{2}\right\rceil \Big(\Delta(H(r,t))-2\left\lceil \frac{\delta(H(r,t))}{2}\right\rceil+2\Big)+1+\left\lfloor \frac{\delta(H(r,t))}{2}\right\rfloor$$

holds with $\delta\big(H(r,t)\big) = 2r-1$ and $\Delta\big(H(r,t)\big) = 2r-2+t$. Clearly, this gives the negative answer to the question "if G is a graph with $\delta(G) = 3$, then is it true that $\mathrm{DC}(G) \leq \Delta(G)+1$?" in Henning and Mojdeh (2025). Furthermore, the difference $\mathrm{DC}(H(r,t)) - (\Delta(H(r,t))+1)$ can be made arbitrarily large.

We remark that Theorem 4.1 and the family of graphs $H(r,t)$ give an incomplete, but relatively satisfying answer to the following problem posed by Henning and Mojdeh (2025):
"A natural problem is to determine a best possible upper bound on the double coalition number of a graph G in terms of its minimum degree, $\delta(G)$, and maximum degree, $\Delta(G)$ ... For sufficiently large values of δ and Δ with $\delta \leq \Delta$, it would be interesting to determine a function $f(\delta,\Delta)$ such that for every graph G with minimum degree δ and maximum degree Δ, we have $\mathrm{DC}(G) \leq f(\delta,\Delta)$ and this bound is best possible."

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