Revisiting d-distance (independent) domination in trees and in bipartite graphs

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Abstract

The d-distance p-packing domination number $\gamma_d^p(G)$ of G is the minimum size of a set of vertices of G which is both a d-distance dominating set and a p-packing. In 1994, Beineke and Henning conjectured that if $d \geq 1$ and T is a tree of order $n \geq d+1$, then $\gamma_d^1(T) \leq \frac{n}{d+1}$. They supported the conjecture by proving it for $d \in \{1, 2, 3\}$. In this paper, it is proved that $\gamma_d^1(G) \leq \frac{n}{d+1}$ holds for any bipartite graph G of order $n \geq d+1$, and any $d \geq 1$. Trees T for which $\gamma_d^1(T) = \frac{n}{d+1}$ holds are characterized. It is also proved that if T has ℓ leaves, then $\gamma_d^1(T) \leq \frac{n-\ell}{d}$ (provided that $n - \ell \geq d$), and $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ (provided that $n \geq d$). The latter result extends Favaron's theorem from 1992 asserting that $\gamma_1^1(T) \leq \frac{n+\ell}{3}$. In both cases, trees that attain the equality are characterized and relevant conclusions for the d-distance domination number of trees derived.

Keywords: d-distance dominating set; p-packing set; tree; bipartite graph

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1 Introduction

Let $G = (V(G), E(G))$ be a graph, $S \subseteq V(G)$, let d and p be nonnegative integers, and let $d(\cdot\,,\cdot)$ denote the standard shortest-path distance. Then S is a *d-distance dominating set* of G if for every vertex $u \in V(G) \setminus S$ there exists a vertex $w \in S$ such that $d(u, w) \leq d$, and S is a *p-packing* of G if $d(w, w') \geq p + 1$ for every two different vertices $w, w' \in S$. The *d-distance p-packing domination number* $\gamma_d^p(G)$ of G is the minimum size of a set S which is at the same time d-distance dominating set and p-packing. (If for some parameters d and p such a set does not exist, set $\gamma_d^p(G) = \infty$.)

The d-distance p-packing domination number was introduced by Beineke and Henning [1] under the name *(p, d)-domination number* and with the notation $i_{p,d}(G)$. With the intention of placing it within the trends of contemporary graph domination theory, the notation $\gamma_d^p(G)$ was recently proposed in [2] and we follow it here. In [2] it is proved that for every two fixed integers d and p with $2 \leq d$ and $0 \leq p \leq 2d - 1$, the decision problem whether $\gamma_d^p(G) \leq k$ holds is NP-complete for bipartite planar graphs. Several bounds on $\gamma_d^p(T)$, where T is a tree on n vertices with ℓ leaves and s support vertices are also proved, including $\gamma_2^0(T) \geq \frac{n-\ell-s+4}{5}$ and $\gamma_d^2(T) \leq \frac{n-2\sqrt{n}+d+1}{d}$, $d \geq 2$. These results improve or extend earlier results from the literature.

In this paper, our focus is on the invariants γ_d^0 and γ_d^1. For the first one we will simplify the notation to γ_d because it has been investigated under the name of *d-distance domination number* of G with the notation $\gamma_d(G)$, see the survey [8]. We also refer to [3] for algorithmic aspects. For the total version of this concept see [4]. The second invariant γ_d^1 deals with d-distance dominating sets which are 1-packings. Note that a set of vertices is a 1-packing if and only if it is an independent set, hence in this case we will say that $\gamma_d^1(G)$ is the *d-distance independent domination number* of G, cf. [5, 7, 8].

Meierling and Volkmann [10], and independently Raczek, Lemańska, and Cyman [12], proved that if $d \geq 1$, and T is a tree on n vertices and with ℓ leaves, then $\gamma_d(T) \geq \frac{n-d\ell+2d}{2d+1}$. On the other hand, Meir and Moon [11] proved that if $d \geq 1$ and T is a tree of order $n \geq d + 1$, then $\gamma_d(T) \leq \frac{n}{d+1}$. About twenty years later, in 1991, Topp and Volkmann [13] gave a complete characterization of the graphs G with $\gamma_d(G) = \frac{n}{d+1}$. In 1994, Beineke and Henning [1] proved that if $d \in \{1, 2, 3\}$ and T is a tree of order $n \geq d + 1$, then $\gamma_d^1(T) \leq \frac{n}{d+1}$. Moreover, they closed their paper with the following:

Conjecture 1.1. [1] *If $d \geq 1$ and T is a tree of order $n \geq d + 1$, then $\gamma_d^1(T) \leq \frac{n}{d+1}$.*

We point out here that in the book's chapter [8], Conjecture 1.1 is stated as [8, Theorem 71] with the explanation that the above-mentioned bound on $\gamma_d(T)$ due to Meir and Moon [11] is proved in such a way, that the d-distance dominating set is also independent. Anyhow, in the next section we prove that the bound holds for all bipartite graphs. In Section 3 we then characterize trees T of order n for which $\gamma_d^1(T) = \frac{n}{d+1}$ holds. In Section 4, we prove that if T has ℓ leaves, then $\gamma_d^1(T) \leq \frac{n-\ell}{d}$ (provided that $n - \ell \geq d$), and $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ (provided that $n \geq d$). In both cases, the trees that attain the equality are characterized. Using the fact that $\gamma_d(T) \leq \gamma_d^1(T)$, we also derive analogous bounds for $\gamma_d(T)$ and characterize trees attaining those bounds. In particular, if T is a tree with ℓ leaves and of order $n \geq d + \ell$, then

$$\gamma_d(T) \leq \gamma_d^1(T) \leq \begin{cases} \frac{n-\ell}{d}, & \text{if } n < (d+1)\ell, \\ \frac{n}{d+1}, & \text{if } n = (d+1)\ell, \\ \frac{n+\ell}{d+2}, & \text{if } n > (d+1)\ell, \end{cases}$$

and the upper bounds are best possible. We conclude the paper with a conjecture.

In the rest of the introduction additional definitions necessary for understanding the rest of the paper are given. For a positive integer n we will use the convention $[n] = \{1, \ldots, n\}$. Let G be a graph. The degree of $u \in V(G)$ is denoted by $\deg_G(u)$ or $\deg(u)$ for short. Further, $\mathrm{diam}(G)$ is the diameter of G and $L(G)$ is the set of its leaves, that is, vertices of degree 1. We call a d-distance p-packing dominating set of G of size $\gamma_d^p(G)$ a *$\gamma_d^p(G)$-set*. When G is clear from the context, we may shorten it to *γ_d^p-set*. A *double star $D_{r,s}$* is a tree with exactly two vertices that are not leaves, with one adjacent to $r \geq 1$ leaves and the other to $s \geq 1$ leaves. When we say that a *path P is attached to a vertex v* of a graph G, we mean that P is disjoint from G and that we add an edge between v and an end vertex of P.

2 Bounding γ_d^1 for bipartite graphs

For the main result of this section, we first prove the following.

Theorem 2.1. *If $d \geq 1$ is an integer and G is a connected bipartite graph of order at least $d+1$, then $V(G)$ can be partitioned into $d+1$ d-distance independent dominating sets.*

Proof. Set $Z = \mathrm{diam}(G)$.

If $Z \leq d$, then each vertex is a d-distance dominating set of G. Since G is bipartite, a required partition of $V(G)$ can be constructed by considering a bipartition (X, Y) of G and partitioning X and Y into $d+1$ parts appropriately. Hence assume in the rest that $Z \geq d+1$.

Let P be a diametrical path of G, let x and y be its end-vertices, and root G at x. Let L_i, $0 \leq i \leq Z$, be the distance levels with respect to x, that is, $L_i = \{u \in V(G) : \ d(x, u) = i\}$. Consider now the sets

$$S_i = \bigcup_{k \geq 0} L_{k(d+1)+i}, \quad i \in \{0, 1, \ldots, d\}.$$

We claim that $\{S_0, S_1, \ldots, S_d\}$ is a partition of $V(G)$ as stated in the theorem.

Since distance levels of a bipartite graph form independent sets and as $d \geq 1$, each set S_i is independent. Hence it remains to prove that these sets are d-distance dominating sets.

Let u be an arbitrary vertex of G and assume that $u \in L_s$, where $0 \leq s \leq Z$. If $s \geq d$, then there exists a path of length d between u and a vertex from L_{s-d}. This already implies that u is d-distance dominated by each of the sets S_i, $i \in \{0, 1, \ldots, d\}$. Hence assume in the rest that $s < d$. Then by a parallel argument, u is d-distance dominated by each of the sets S_i, $i \in \{0, 1, \ldots, s\}$. It remains to verify that u is d-distance dominated by each of the sets S_i, $i \in \{s+1, \ldots, d\}$. For this sake consider an arbitrary, fixed $t \in \{s+1, \ldots, d\}$. Let Q be a shortest u,y-path and recall that by our assumption, $d(u, y) \leq Z$. Since every edge of G connects two vertices from consecutive distance levels L_i, the path Q necessarily contains a vertex $w \in L_t$. We claim that $d(u, w) \leq d$. Suppose on the contrary that $d(u, w) > d$. Since Q is a shortest path, $d(w, y) \geq Z - t$. Using these facts together with $t \leq d$, we get

$$Z \leq d + (Z - t) < d(u, w) + d(w, y) = d(u, y) \leq Z,$$

which is not possible. We can conclude that $d(u, w) \leq d$. This means that u is d-distance dominated by S_t and we are done. □

In connection with Theorem 2.1 we add that Zelinka [14] proved that if $d \geq 1$ and G is a connected graph of order at least $d+1$, then $V(G)$ can be partitioned into $d+1$ disjoint d-distance dominating sets. In this general case, however, the partition need not be into independent sets.

The following is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let $d \geq 1$ be an integer. If G is a bipartite graph of order $n \geq d+1$, then $\gamma_d^1(G) \leq \frac{n}{d+1}$.*

Corollary 2.2 generalizes [8, Theorem 71]. On the other hand, the upper bound in Corollary 2.2 may not hold if G is not bipartite. For example, for $n \geq d+2$ and $k \geq 2$, let $G_{n,k,d}$ be the complete graph K_n with k copies of P_d attached to each vertex. Clearly, $|V(G_{n,k,d})| = n(dk+1)$. While $\gamma_d(G_{n,k,d}) = n$, a d-distance independent domination needs much more vertices, and it is not hard to deduce that $\gamma_d^1(G_{n,k,d}) = 1 + (n-1)k$. As $n \geq d+2$ and $k \geq 2$, we infer that $\gamma_d^1(G_{n,k,d}) > \frac{|V(G_{n,k,d})|}{d+1}$.

3 Trees that attain equality in Corollary 2.2

Let $d \geq 1$ be an integer. The *P_d-corona $H \circ P_d$* of a graph H is the graph obtained from H and $|V(H)|$ disjoint copies of P_d, by attaching a copy of P_d to each vertex of H, see Fig. 1.

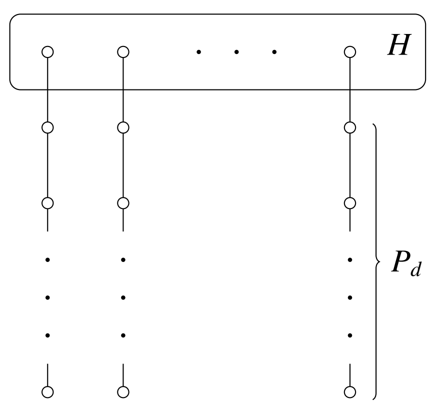

Figure 1: The P_d-corona $H \circ P_d$ of a graph H.

If $d \geq 2$, then let $\mathcal{B}_d$ be the family of P_d-coronas of bipartite graphs, that is,

$$\mathcal{B}_d = \{H \circ P_d : H \text{ is a bipartite graph}\}.$$

Note that $P_{d+1} \in \mathcal{B}_d$. Observe also that each $G \in \mathcal{B}_d$, where $G = H \circ P_d$, is a bipartite graph with $|V(G)| = (d+1)|V(H)|$. The following proposition shows that the upper bound in Corollary 2.2 is best possible.

Proposition 3.1. *If $G \in \mathcal{B}_d$ is of order n, then $\gamma_d^1(G) = \frac{n}{d+1}$.*

Proof. Let $G = H \circ P_d$ for some bipartite graph H. By the definition of $H \circ P_d$, the set $L(G)$ is a d-distance independent dominating set of G. Thus, $\gamma_d^1(G) \leq |L(G)| = |V(H)| = \frac{n}{d+1}$.

Conversely, for each $u \in V(H)$, let G_u be the subgraph of G induced by u and the vertices of the copy of P_d attached to u. Clearly, $G_u \cong P_{d+1}$. If D is a $\gamma_d^1(G)$-set, then $|D \cap V(G_u)| \geq 1$. Thus, $\gamma_d^1(G) = |D| \geq |V(H)| = \frac{n}{d+1}$. □

Note that if G is a connected bipartite graph of order $n = d + 1$, then $\gamma_d^1(G) = 1 = \frac{n}{d+1}$, and $\gamma_d^1(C_{2d+2}) = 2 = \frac{2d+2}{d+1}$. Moreover, if $d = 1$, then $\gamma_1^1(K_{r,r}) = r = \frac{r+r}{2} = \frac{n}{2}$. In 2004, Ma and Chen gave an equivalent description of the bipartite graphs G of order n with $\gamma_1^1(G) = \frac{n}{2}$, see [9, Theorem 1]. They also proved an explicit characterization of such a family for the case of trees. To state the result, let ζ_1 be a family of trees defined by the following recursive construction.

(i) $K_2 \in \zeta_1$.

(ii) If $T' \in \zeta_1$, and T is obtained by joining the center of a new copy of $K_{1,t}$ ($t \geq 1$) to a support vertex v of T' and adding $t - 1$ leaves at v, then $T \in \zeta_1$.

See Fig. 2 for an illustration.[1]

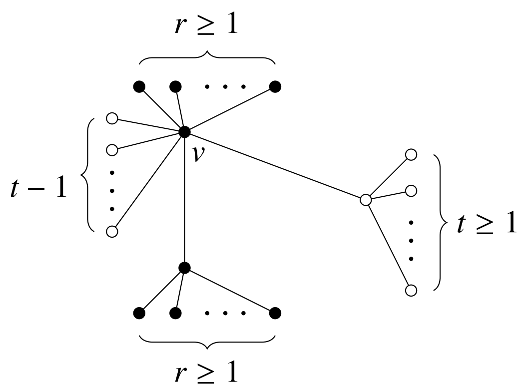

Figure 2: A tree T from the family ζ_1, where T' is the double star induced by the black vertices.

The result of Ma and Chen for trees now reads as follows.

Theorem 3.2. ([9, Corollary 1]) *If T is a tree of order n, then $\gamma_1^1(T) = \frac{n}{2}$ if and only if $T \in \zeta_1$.*

We shall focus on the general case for $d \geq 2$, and give a complete characterization of the trees achieving equality in the upper bound of Corollary 2.2. Set

$$\mathcal{T}_d = \{T^* \circ P_d : T^* \text{ is a non-trivial tree}\}.$$

Note that $\mathcal{T}_d$ does not contain the path P_{d+1}. Since $\mathcal{T}_d \subset \mathcal{B}_d$, and by Proposition 3.1, $\gamma_d^1(T) = \frac{n}{d+1}$ for each tree $T \in \mathcal{T}_d$ of order n. Moreover, if T is a tree of order $n = d + 1$, then we also have $\gamma_d^1(T) = 1 = \frac{n}{d+1}$.

[1]For the definition of ζ_1, we note that both vertices of a K_2 are support vertices. Then, (ii) can be applied to K_2, and this step results in the double star $D_{r,r}$ for every $r \geq 1$. It shows that the family ζ_1 is the same as $\{K_2\} \cup \zeta$ in [9].

In a tree T and for a vertex $v \in V(T)$, let $L(v)$ be the set of leaves of T that are neighbors of v in T. Root T at some vertex. Let T_v be the subtree induced in T by v and its descendants, and let $T - T_v = T - V(T_v)$. A vertex of T is called a *P_d-support vertex* if it is attached to a copy of P_d. For each $H \in \mathcal{T}_d$, every vertex of H^* is a P_d-support vertex of H, where $H = H^* \circ P_d$ for some non-trivial tree H^*. In particular, a P_1-support vertex of T is just a support vertex of T. A vertex of T is a *(P_i, P_j)-support vertex* if both a copy of P_i and a copy of P_j are attached to it. In particular, a (P_i, P_i)-support vertex has at least two copies of P_i attached. The *d-subdivision* of T is the tree obtained from T by subdividing each edge d-times. Then the 1-subdivision of T is just the subdivision of T.

Before proving the announced characterization of trees of order n with $\gamma_d^1(T) = \frac{n}{d+1}$, we state the following lemma which will also be used in the subsequent section.

Lemma 3.3. *Let $d \geq 2$ and let T be a tree with $s = \mathrm{diam}(T) \geq 2d+1$. Suppose that $P := v_1 v_2 \ldots v_{s+1}$ is a diametrical path in T and the tree is rooted at v_{s+1}. If there is no P_{d+1}-support vertex and no (P_i, P_j)-support vertex in T with $i \in [d-1]$ and $j \in [d]$, then the following statements hold.*

(i) If $k \in \{2, \ldots, d\} \cup \{s-d+2, \ldots, s\}$, then $\deg(v_k) = 2$.

(ii) If $k \in \{d+1, s-d+1\}$, then $\deg(v_k) \geq 3$.

(iii) For every $v \in V(T)$, if v is the only vertex with $\deg_T(v) \geq 3$ in the subtree T_v, then T_v is isomorphic to the $(d-1)$-subdivision of a star $K_{1,t}$ with $t \geq 2$.

(iv) The subtree $T_{v_{d+1}}$ is isomorphic to the $(d-1)$-subdivision of a star $K_{1,t}$ with $t \geq 2$.

(v) If $s = 2d+1$, then T is obtained by taking the $(d-1)$-subdivisions of two stars K_{1,t_1} with $t_1 \geq 2$ and K_{1,t_2} with $t_2 \geq 2$, and adding an edge between the centers.

Proof. (i)–(ii) According to the conditions, there is no (P_1, P_1)-support vertex in T. That is, every vertex of T is adjacent to at most one leaf, and in particular, $\deg(v_2) = 2$. Further, if $d \geq 3$, then $\deg(v_3) = 2$, since otherwise v_3 would be a (P_i, P_2)-support vertex with $1 \leq i \leq 2 \leq d-1$ contradicting the condition. Similarly, $\deg(v_k) = 2$ holds for all $k \in \{2, \ldots, d\}$. By symmetry, the same is true for v_k if $k \in \{s-d+2, \ldots, s\}$. This proves (i). The assumption that there is no pendant P_{d+1} in T directly implies (ii).

(iii) If $\deg_T(v) \geq 3$ and $\deg_{T_v}(u) \leq 2$ for every further vertex u from $V(T_v)$, then at least two pendant paths are attached to v. Then, by the conditions in the lemma, every path attached is isomorphic to P_d.

(iv)–(v) As P is a diametrical path, a vertex $u \in V(T_{v_{d+1}})$ different from v_{d+1} cannot be a P_d-support vertex. Part (iii) then implies (iv). If we re-root T at the vertex v_1, the same property holds for the subtree induced by v_{s-d+1} and its descendants in the re-rooted tree. This directly implies (v) for the case of $s = 2d+1$. □

Theorem 3.4. *If $d \geq 2$ and T is a tree of order n, then $\gamma_d^1(T) = \frac{n}{d+1}$ holds if and only if $n = d+1$ or $T \in \mathcal{T}_d$.*

Proof. If T is of order $n = d + 1$, then, clearly, $\gamma_d^1(T) = 1 = \frac{n}{d+1}$, and if $T \in \mathcal{T}_d$, then by Proposition 3.1, we have $\gamma_d^1(T) = \frac{n}{d+1}$. The proof of the necessity is by induction on n. If $\gamma_d^1(T) = \frac{n}{d+1}$, then $n = (d+1)q$ for some integer $q \geq 1$. If $q = 1$, then $n = d + 1$. So, we may assume that $q \geq 2$ and $n \geq 2(d+1)$. If $\mathrm{diam}(T) \leq 2d$, then $\gamma_d^1(T) = 1 < \frac{2(d+1)}{d+1} \leq \frac{n}{d+1}$. In the continuation, we assume that $\mathrm{diam}(T) \geq 2d + 1$ and $\gamma_d^1(T) = \frac{n}{d+1}$.

Claim A. *If $i \in [d-1]$ and $j \in [d]$, then there is no (P_i, P_j)-support vertex in T.*

Proof. Suppose, to the contrary, that v is a (P_i, P_j)-support vertex in T and $i \leq j$. Let $P' := x_1x_2\ldots x_i$ and $P'' := y_1y_2\ldots y_j$ be two copies of P_i and P_j attached to v in T, where $x_iv, y_jv \in E(T)$. Note that $d(x_1, v) = i \leq j = d(y_1, v)$. Consider $T' = T - V(P')$. Then $n' = |V(T')| = n - i \geq 2(d+1) - (d-1) = d + 3$. Let D' be a $\gamma_d^1(T')$-set. If $v \in D'$, then D' is also a d-distance independent dominating set of T. If $v \notin D'$, then $|D'| = \gamma_d^1(T')$ implies $|D' \cap V(P'')| \leq 1$. For the subcase $|D' \cap V(P'')| = 1$, we may assume that $y_j \in D'$. Then $d(x_k, y_j) \leq d$ for each $k \in [i]$. For the subcase $|D' \cap V(P'')| = 0$, in order to d-distance dominate y_1 in T', there exists a vertex $u \in D'$ such that $d_{T'}(u, y_1) \leq d$. Since $i \leq j$, it holds that $d_T(x_k, u) \leq d_T(u, y_1) = d_{T'}(u, y_1) \leq d$ for each $k \in [i]$. Thus, D' is always a d-distance independent dominating set of T. Corollary 2.2 then implies

$$\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'}{d+1} < \frac{n}{d+1},$$

which contradicts the assumption $\gamma_d^1(T) = \frac{n}{d+1}$. This proves Claim A. (□)

Claim B. *If T has a P_{d+1}-support vertex v, then $T \in \mathcal{T}_d$.*

Proof. Let $P' := x_1x_2\ldots x_{d+1}$ be a copy of P_{d+1} attached to v, where $x_{d+1}v \in E(T)$. Then $\deg(x_k) = 2$ for all $k \in [d+1] \setminus \{1\}$ and $\deg(x_1) = 1$. Consider $T' = T - V(P')$. Then $n' = |V(T')| = n - (d+1) \geq 2(d+1) - d - 1 = d + 1$. Let D' be a $\gamma_d^1(T')$-set. Then $D' \cup \{x_1\}$ is a d-distance independent dominating set of T. By Corollary 2.2,

$$\gamma_d^1(T) \leq |D'| + 1 = \gamma_d^1(T') + 1 \leq \frac{n'}{d+1} + 1 = \frac{n-(d+1)}{d+1} + 1 = \frac{n}{d+1},$$

and the equality holds if and only if $\gamma_d^1(T) = \gamma_d^1(T') + 1$ and $\gamma_d^1(T') = \frac{n'}{d+1}$. The induction hypothesis therefore implies $n' = d + 1$ or $T' \in \mathcal{T}_d$.

Suppose $n' = d + 1$. Then $n = 2(d+1)$ and T is the tree obtained from a copy of P_{d+1} and a tree T' of order $d + 1$ by joining x_{d+1} to a vertex v of T'. Note that $\mathrm{diam}(T') \leq d$ with equality if and only if $T' \cong P_{d+1}$. Unless $T' \cong P_{d+1}$ and v is a leaf of T', $\{x_{d+1}\}$ is a d-distance independent dominating set of T, implying that $\gamma_d^1(T) = 1 < \frac{2(d+1)}{d+1} = \frac{n}{d+1}$, a contradiction. For the exception, we observe $T \cong P_{2(d+1)} \in \mathcal{T}_d$.

Suppose $T' \in \mathcal{T}_d$. Let $T' = T'_* \circ P_d$ for some non-trivial tree T'_*. If $v \in V(T'_*)$, then $T = T^* \circ P_d \in \mathcal{T}_d$, where T^* is the tree obtained from T'_* by adding a new vertex x_{d+1} and the edge $x_{d+1}v$ to it. If $v \notin V(T'_*)$, then let u_1 be the P_d-support vertex of T'_* such that the attached copy of P_d contains v. Since $|V(T'_*)| \geq 2$, there exists a neighbor $u_2 \in V(T'_*)$ of u_1. Let u'_1 and u'_2

be the leaves of T' corresponding to u_1 and u_2, respectively. Note that $v = u'_1$ is possible, and $D = (L(T') \setminus \{u'_1, u'_2\}) \cup \{x_{d+1}, u_2\}$ is a d-distance independent dominating set of T. Thus,

$$\gamma_d^1(T) \leq |D| = |L(T')| = |V(T'_*)| = \frac{n'}{d+1} < \frac{n}{d+1}$$

that contradicts our assumption on T and finishes the proof of Claim B. $(\square)$

Claim B shows that if $\gamma_d^1(T) = \frac{n}{d+1}$ and T contains a pendant path P_{d+1}, then $T \in \mathcal{T}_d$. The remaining part of the proof verifies that there is no tree T with $|V(T)| > d+1$ and $\gamma_d^1(T) = \frac{n}{d+1}$ that does not contain a pendant P_{d+1}. From now on, we suppose that there is no P_{d+1}-support vertex in T and that $\gamma_d^1(T) = \frac{n}{d+1}$.

Let $s = \text{diam}(T) \geq 2d+1$ and $P := v_1 v_2 \ldots v_{s+1}$ be a diametrical path in T. Then $\deg(v_1) = \deg(v_{s+1}) = 1$. Root T at v_{s+1}. Our assumption on the non-existence of P_{d+1}-support vertices and Claim A imply that the properties stated in Lemma 3.3 (i)–(v) are valid for T.

If $s = \text{diam}(T) = 2d+1$ then, by Lemma 3.3 (v), the tree T can be obtained from the $(d-1)$-subdivisions of two stars K_{1,t_1} and K_{1,t_2} with $t_1 \geq t_2 \geq 2$ by joining the centers with an edge. Then $N(v_{d+2})$ is a d-distance independent dominating set of T. Since $d \geq 2$, it gives the following contradiction:

$$\gamma_d^1(T) \leq |N(v_{d+2})| = t_2 + 1 = \frac{(t_2+1)d + t_2 + 1}{d+1} < \frac{2dt_2 + 2}{d+1} \leq \frac{d(t_1+t_2)+2}{d+1} = \frac{n}{d+1}.$$

So, we may assume that $\text{diam}(T) \geq 2d+2$ and $n \geq 2d+3$. Regarding v_{d+2}, we divide the rest of the proof into two cases and prove that in both we get a contradiction.

Case 1. Each vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ is of degree at least 3.

By Lemma 3.3 (iii) and since P is a diametrical path, for each $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$, the subtree T_v is isomorphic to the $(d-1)$-subdivision of a star K_{1,t_v} with $t_v \geq 2$. Clearly, $T_{v_{d+1}}$ is contained in $T_{v_{d+2}}$, and therefore, $|V(T_{v_{d+2}})| \geq 2d+2$. Let $T' = T - T_{v_{d+2}}$. Since $\{v_{d+3}, \ldots, v_{2d+3}\} \subseteq V(T')$, we obtain

$$d+1 \leq n' = |V(T')| \leq n - 2d - 2.$$

Let D' be a $\gamma_d^1(T')$-set. Then $D = D' \cup (N(v_{d+2}) \setminus \{v_{d+3}\})$ is a d-distance independent dominating set of T. Let $p = \deg(v_{d+2})$. Observe that $p \geq 2$ and $n' \leq n - (2d+1)(p-1) - 1$. By Corollary 2.2, we get the following contradiction:

$$\begin{aligned}
\gamma_d^1(T) &\leq |D| = \gamma_d^1(T') + p - 1 \\
&\leq \frac{n'}{d+1} + p - 1 \\
&\leq \frac{n - (2d+1)(p-1) - 1}{d+1} + p - 1 \\
&= \frac{n - d(p-1) - 1}{d+1} \\
&< \frac{n}{d+1}.
\end{aligned}$$

Case 2. There is a vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ with $\deg(v) \leq 2$.

If $\deg(v) = 2$ and T_v contains a vertex u with $\deg(u) \geq 3$, then Lemma 3.3 (iii) implies the existence of a leaf $w \in V(T_u)$ with $d(w, u) = d$. It follows then that $d(w, v_{d+2}) \geq d + 2$ and $d(w, v_{s+1}) \geq s + 1 = \text{diam}(T) + 1$, a contradiction. Therefore, $\deg(v) \leq 2$ implies that T_v is a path and v_{d+2} is a P_i-support vertex for some $i \geq 1$. By our assumption, $i \leq d$. Further, by Claim A, we have the following properties.

- If v_{d+2} is a P_i-support vertex of T for some $i \in [d-1]$, then there is only one pendant path attached to v_{d+2}, and it is clearly of order i.
- If v_{d+2} is a P_d-support vertex of T, then v_{d+2} is not a P_i-support vertex of T for any $i \in [d-1]$, and there is at least one copy of P_d attached to v_{d+2}.

Case 2.1. $L(v_{d+2}) \neq \emptyset$.

In this case v_{d+2} is a P_1-support vertex of T. Let $x \in L(v_{d+2})$ and $T' = T - x$. Now for each vertex $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$, the subtree T'_v is isomorphic to the P_d-subdivision of a star K_{1,t_v} for $t_v \geq 2$. Clearly, $n' = |V(T')| = n - 1 \geq 2d + 2$.

Let D' be a $\gamma_d^1(T')$-set. If $v_{d+2} \in D'$, then D' is also a d-distance independent dominating set of T. If $v_{d+2} \notin D'$, then since $|D' \cap V(T'_{v_{d+1}})| \geq 1$, we may assume that $v_{d+1} \in D'$. The set D' is also a d-distance independent dominating set of T. For any subcase, $\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'}{d+1} = \frac{n-1}{d+1} < \frac{n}{d+1}$ by Corollary 2.2.

Case 2.2. $L(v_{d+2}) = \emptyset$.

In this case, v_{d+2} is a P_i-support vertex of T for some $i \in [d] \setminus \{1\}$ (where if $i = d$, then there could be multiple copies of P_d attached to v_{d+2}). Let $P' := x_1 x_2 \ldots x_i$ be the (selected) copy of P_i attached to v_{d+2}, where $x_i v_{d+2} \in E(T)$. Then $\deg(x_k) = 2$ for all $k \in [i] \setminus \{1\}$ and $\deg(x_1) = 1$. Consider $T' = T - T_{v_d} - T_{x_i}$. Then $n' = |V(T')| = n - d - i \leq n - d - 2$ and $n' \geq d + 3$ since $v_{d+1}, v_{d+2}, \ldots, v_{2d+3} \in V(T')$.

Let D' be a $\gamma_d^1(T')$-set. Then $|D' \cap \{v_{d+1}, v_{d+2}\}| \leq 1$. If $v_{d+1} \in D'$ and $v_{d+2} \notin D'$, then let $D = D' \cup \{x_i\}$. If $v_{d+1} \notin D'$ and $v_{d+2} \in D'$, then let $D = D' \cup \{v_1\}$. If $v_{d+1}, v_{d+2} \notin D'$, then since v_{d+1} is attached to at least two copies of P_d, we have $D' \cap (V(T_{v_{d+1}}) \setminus V(T_{v_d})) \neq \emptyset$. Let $D = D' \cup \{v_{d+1}, x_i\} \setminus (V(T_{v_{d+1}}) \setminus V(T_{v_d}))$. For any subcase, D is a d-distance independent dominating set of T, and $\gamma_d^1(T) \leq |D'| + 1 = \gamma_d^1(T') + 1 \leq \frac{n'}{d+1} + 1 \leq \frac{n-d-2}{d+1} + 1 < \frac{n}{d+1}$ by Corollary 2.2.

This completes the proof of Theorem 3.4. □

4 Upper bounds on γ_d and γ_d^1 of trees in terms of the order and the number of leaves

For any tree T of order n and with ℓ leaves, the set of non-leaves is a dominating set of T. Hence, $\gamma_1(T) \leq n - \ell$. Note that the equality holds if and only if each vertex of T is either a leaf or a support vertex. If there exists a vertex $u \in V(T)$ that is neither a leaf nor a support vertex, then

$V(T) \setminus (\{u\} \cup L(T))$ is a dominating set of T, implying that $\gamma_1(T) < n - \ell$. On the other hand, the upper bound $\gamma_1^1(T) \le n - \ell$ is not true for every tree T. For example, let $T' = T^* \circ P_1 \in \mathcal{T}_1$ for some tree T^*, and let T be the tree obtained from T' by adding $r \ge 2$ leaves to each vertex of T'. It can be checked that

$$\gamma_1^1(T) = |V(T^*)| + r|V(T^*)| > 2|V(T^*)| = 2(r+1)|V(T^*)| - 2r|V(T^*)| = n - \ell.$$

Set now

$$\mathcal{F}_2 = \{T : \ T - L(T) \in \zeta_1\} \ ,$$

and if $d \ge 3$, then set

$$\mathcal{F}_d = \{T : \ T - L(T) \text{ is a tree of order } d \text{ or belongs to } \mathcal{T}_{d-1}\} \ .$$

Note that each graph from $\mathcal{F}_d$, $d \ge 2$, is a tree, and the following property is equivalent to the definition of $\mathcal{F}_d$.

($\star$) If $d \ge 3$, a tree T belongs to $\mathcal{F}_d$ if and only if it can be obtained from some tree T', which satisfies $|V(T')| = d$ or $T' \in \mathcal{T}_{d-1}$, by adding at least one pendant vertex to each leaf of T', and some number (possibly zero) to other vertices of T'. For $d = 2$, a tree T belongs to $\mathcal{F}_2$ if and only if it can be obtained similarly from a tree $T' \in \zeta_1$.

For $d \ge 2$, we prove the following result.

Theorem 4.1. *Let $d \ge 2$ be an integer and T be a tree of order n and with ℓ leaves. If $n - \ell \ge d$, then $\gamma_d^1(T) \le \frac{n-\ell}{d}$ with equality if and only if $T \in \mathcal{F}_d$.*

Proof. Consider the tree $T' = T - L(T)$. Let $n' = |V(T')| = n - \ell \ge d$. Let D' be a $\gamma_{d-1}^1(T')$-set. By Corollary 2.2, $|D'| = \gamma_{d-1}^1(T') \le \frac{n'}{d}$. Moreover, D' is also a d-distance independent dominating set of T, implying that

$$\gamma_d^1(T) \le |D'| = \gamma_{d-1}^1(T') \le \frac{n'}{d} = \frac{n-\ell}{d}. \tag{1}$$

Assume that $\gamma_d^1(T) = \frac{n-\ell}{d}$ holds for a tree T. Inequalities in (1) therefore imply $\gamma_d^1(T) = \gamma_{d-1}^1(T') = \frac{n'}{d}$. By Theorems 3.2 and 3.4, we know that $T' \in \zeta_1$ when $d = 2$, and T' is a tree of order d or $T' \in \mathcal{T}_{d-1}$ when $d \ge 3$. Since $T' = T - L(T)$, we conclude $T \in \mathcal{F}_d$.

It remains to prove that $\gamma_d^1(T) \ge \frac{n-\ell}{d}$ holds for every $T \in \mathcal{F}_d$. Consider first a tree T from $\mathcal{F}_2$ and let $T' = T - L(T)$. Hence $T' \in \zeta_1$. We will prove the inequality by induction on T' according to the recursive definition of ζ_1. If $T' \cong K_2$, then T is a double star and $\gamma_2^1(T) = 1 = \frac{n-\ell}{2}$. If $T' \cong D_{r,r}$, for $r \ge 1$, then any $\gamma_1^1(T')$-set is a smallest 2-distance independent dominating set of T, implying that

$$\gamma_2^1(T) = \gamma_1^1(T') = r + 1 = \frac{2r+2}{2} = \frac{n'}{2} = \frac{n-\ell}{2}.$$

Assume next that $T' = T - L(T)$ is a tree from ζ_1 which is neither K_2 nor a double star. Let $T_2' = T'$ and let T_1' be the tree from ζ_1 such that T_2' is obtained from T_1' by the recursive construction of ζ_1, that is, T_2' can be obtained by joining the center u of a new copy of $K_{1,t}$ ($t \ge 1$) to a support vertex v of T'_1, and adding $t-1$ leaves at v. For $i \in [2]$, let T_i be a tree from $\mathcal{F}_2$, which is obtained from T'_i according to ($\star$). Moreover, let $n'_i = |V(T'_i)|$, $n_i = |V(T_i)|$, and $\ell_i = |L(T_i)|$, $i \in [2]$.

Assume that $\gamma_2^1(T_1) = \frac{n_1-\ell_1}{2}$. We are going to prove that $\gamma_2^1(T_2) \geq \frac{n_2-\ell_2}{2}$. Note that $n'_i = n_i - \ell_i$ and $n'_2 = n'_1 + 2t$. Let D_2 be a $\gamma_2^1(T_2)$-set that contains as few leaves from T_2 as possible and let $D_1 = D_2 \cap V(T_1)$. If $v \in D_2$, then $u \notin D_2$ and, by the minimality of $|D_2 \cap L(T_2)|$, we have $L_{T'_2}(u) \subset D_2$. Now $D_1 = D_2 \setminus L_{T'_2}(u)$ is a 2-distance independent dominating set of T_1, implying that $\gamma_2^1(T_1) \leq |D_1| = |D_2| - t$. If $v \notin D_2$, then we may assume that $u \in D_2$. Also, $L_{T'_2}(v) \setminus L_{T'_1}(v) \subset D_2$ holds by the minimality of $|D_2 \cap L(T_2)|$. Further, $\emptyset \neq L_{T'_1}(v) \subset D_1$, and v and the leaves added to $L_{T'_1}(v)$ in T_1 will be independently dominated by $L_{T'_1}(v)$. Hence, $D_1 = D_2 \setminus \{u\} \setminus (L_{T'_2}(v) \setminus L_{T'_1}(v))$ is a 2-distance independent dominating set of T_1, implying that $\gamma_2^1(T_1) \leq |D_1| = |D_2| - t$. Hence no matter whether v belongs to D_2 or not, we have

$$\gamma_2^1(T_2) \geq \gamma_2^1(T_1) + t = \frac{n_1 - \ell_1}{2} + \frac{n'_2 - n'_1}{2} = \frac{n'_2}{2} = \frac{n_2 - \ell_2}{2}.$$

Assume now that $T \in \mathcal{F}_d$ and $d \geq 3$. For $T' = T - L(T)$, let $n' = |V(T')| = n - \ell \geq d$. If $n' = d$, then $\gamma_d^1(T) \geq 1 = \frac{n-\ell}{d}$. If $T' \in \mathcal{T}_{d-1}$, then let $T' = T^* \circ P_{d-1}$ for some non-trivial tree T^*. For each $u \in V(T^*)$, let T'_u be the subtree of T' induced by u and the vertices of the copy of P_{d-1} attached to u, and let T_u be the subtree of T induced by $V(T'_u)$ and the leaves added to $V(T'_u)$ in T. If D is a $\gamma_d^1(T)$-set, then $|D \cap V(T_u)| \geq 1$ for every $u \in V(T^*)$. Thus, we have

$$\gamma_d^1(T) = |D| \geq |V(T^*)| = \frac{n'}{d} = \frac{n-\ell}{d}.$$

This completes the proof of Theorem 4.1. □

We note that the condition of $n \geq d + \ell$ is necessary in Theorem 4.1. Let T' be a tree of order at most $d-1$. Consider the tree T obtained from T' by adding at least one pendant vertex to each leaf of T' and some number to other vertices of T'. Then $n' = |V(T')| = n - \ell \leq d - 1$ and we may infer $\gamma_d^1(T) \geq 1 > \frac{d-1}{d} \geq \frac{n-\ell}{d}$.

Favaron [6] proved that if T is a tree of order $n \geq 2$ and with ℓ leaves, then $\gamma_1^1(T) \leq \frac{n+\ell}{3}$, and gave the full list of extremal trees for this bound. Our next theorem extends Favaron's result to all $d \geq 2$.

Theorem 4.2. *Let $d \geq 2$ be an integer and T a tree of order n and with ℓ leaves. If $n \geq d$, then $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ with equality if and only if $T \in \{P_d\} \cup \mathcal{T}_d$.*

Proof. If $T \cong P_d$, then $\gamma_d^1(T) = 1 = \frac{d+2}{d+2} = \frac{n+\ell}{d+2}$. If $T \in \mathcal{T}_d$, then by Proposition 3.1, $\gamma_d^1(T) = \frac{n}{d+1} = \frac{n+\frac{n}{d+1}}{d+2} = \frac{n+\ell}{d+2}$. To prove the upper bound and that the equality implies $T \in \{P_d\} \cup \mathcal{T}_d$, we proceed by induction on n. If $\mathrm{diam}(T) \leq 2d$, then $\gamma_d^1(T) = 1 = \frac{d+2}{d+2} \leq \frac{n+\ell}{d+2}$. The equality holds if and only if $n = d$ and $\ell = 2$, implying that $T \cong P_d$. So, we may assume that $\mathrm{diam}(T) \geq 2d + 1$ and $n \geq 2d + 2$. Note that if $\ell > \frac{n}{d+1}$, then by Corollary 2.2, $\gamma_d^1(T) \leq \frac{n}{d+1} < \frac{n+\ell}{d+2}$.

Claim C. *Let $i \in [d-1]$ and $j \in [d]$ with $i \leq j$. If T has a vertex v that is a (P_i, P_j)-support vertex, then $\gamma_d^1(T) < \frac{n+\ell}{d+2}$.*

Proof. Let $P' := x_1x_2\dots x_i$ and $P'' := y_1y_2\dots y_j$ be a copy of P_i and P_j, respectively, attached to v in T, where $x_iv, y_jv \in E(T)$. Since $n \geq 2d+2$ and $|V(P')| \leq |V(P'')| \leq d$, we have $\deg(v) \geq 3$. Consider $T' = T - V(P')$. Then $\ell' = |L(T')| = \ell - 1$ and $n' = |V(T')| = n - i \geq d+3$. As in the proof of Claim A it can be proved that there exists a $\gamma_d^1(T')$-set D' that is a d-distance independent dominating set of T. Using the induction hypothesis, we have

$$\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'+\ell'}{d+2} = \frac{n-i+\ell-1}{d+2} < \frac{n+\ell}{d+2}.$$

This proves Claim C. (□)

Claim D. *If T has a P_{d+1}-support vertex v, then $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ and if equality holds, then $T \in \mathcal{T}_d$.*

Proof. Let $P' := x_1x_2\dots x_{d+1}$ be a copy of P_{d+1} attached to v, where $x_{d+1}v \in E(T)$. Then $\deg_T(x_k) = 2$ for all $k \in [d+1] \setminus \{1\}$ and $\deg_T(x_1) = 1$. Consider $T' = T - V(P')$. Since $n \geq 2d+2$, we have $\deg_T(v) \geq 2$. Then $\ell' = |L(T')| = \ell$ if $\deg_T(v) = 2$ and $\ell' = \ell - 1$ if $\deg_T(v) \geq 3$. We observe that $n' = |V(T')| = n - (d+1) \geq d+1$ and consider two cases according to the degree of v.

Case D1. $\deg_T(v) \geq 3$.

Let D' be a $\gamma_d^1(T')$-set. The set $D' \cup \{x_1\}$ is a d-distance independent dominating set of T. By the induction hypothesis,

$$\gamma_d^1(T) \leq |D' \cup \{x_1\}| = \gamma_d^1(T') + 1 \leq \frac{n'+\ell'}{d+2} + 1 = \frac{n-(d+1)+\ell-1}{d+2} + 1 = \frac{n+\ell}{d+2},$$

and the equality holds if and only if $\gamma_d^1(T) = \gamma_d^1(T')+1$ and $\gamma_d^1(T') = \frac{n'+\ell'}{d+2}$. Note that $n' \geq d+1$, so $T' \not\cong P_d$ and $T' \in \mathcal{T}_d$.

Let $T' = T'_* \circ P_d$ for some non-trivial tree T'_*. Then $\ell' = \frac{n'}{d+1}$. Since $\deg(v) \geq 3$, we infer that $v \notin L(T')$. If $v \in V(T'_*)$, then $T = T_* \circ P_d \in \mathcal{T}_d$, where T_* is the tree obtained from T'_* by adding a new vertex x_{d+1} to it such that $x_{d+1}v \in E(T_*)$. If $v \notin V(T'_*) \cup L(T')$, then let u be the P_d-support vertex of T'_* attached to the copy of P_d containing v, and u' be the leaf of T' corresponding to u. Note that $v \neq u'$ and $D = (L(T') \setminus \{u'\}) \cup \{x_{d+1}\}$ is a d-distance independent dominating set of T. Thus,

$$\gamma_d^1(T) \leq |D| = |L(T')| = \frac{n'}{d+1} = \frac{n'+\ell'}{d+2} = \frac{n-(d+1)+\ell-1}{d+2} < \frac{n+\ell}{d+2}.$$

Case D2. $\deg_T(v) = 2$.

Let $P'' := x_1x_2\dots x_{d+1}v$ be a copy of P_{d+2} attached to v', where $vv' \in E(T)$. Consider $T'' = T - V(P'') = T' - v$. Then $n'' = |V(T'')| = n - (d+2) \geq d$, and $\ell'' = |L(T'')| \leq \ell$ with equality if and only if $\deg_T(v') = 2$. Let D'' be a $\gamma_d^1(T'')$-set. The set $D'' \cup \{x_2\}$ is a d-distance independent dominating set of T. By the induction hypothesis,

$$\gamma_d^1(T) \leq |D'' \cup \{x_2\}| = |D''| + 1 = \gamma_d^1(T'') + 1 \leq \frac{n''+\ell''}{d+2} + 1 \leq \frac{n-(d+2)+\ell}{d+2} + 1 = \frac{n+\ell}{d+2},$$

and the equality holds if and only if $\gamma_d^1(T) = \gamma_d^1(T'') + 1$, $\ell'' = \ell$, and $\gamma_d^1(T'') = \frac{n''+\ell''}{d+1}$, i.e., $T'' \in \{P_d\} \cup \mathcal{T}_d$.

Note that $\deg_T(v') = 2$ and $\deg_{T''}(v') = 1$. If $T'' \cong P_d$, then $T \cong P_{2d+2} \in \mathcal{T}_d$. Suppose that $T'' \in \mathcal{T}_d$. Let $T'' = T''_* \circ P_d$ for some non-trivial tree T''_*. Then $\ell'' = \frac{n''}{d+1}$. Clearly, $v' \in L(T'')$. Let $u'_1 \in V(T''_*)$ be the P_d-support vertex in T'', which is attached to the copy of P_d containing v'. Since $|V(T''_*)| \geq 2$, there exists a neighbor $u'_2 \in V(T''_*)$ of u'_1. It is clear that v' is the leaf of T' corresponding to u'_1. Let u''_2 be the leaf of T' corresponding to u'_2. Since $d \geq 2$, the set $D = (L(T'') \setminus \{v', u''_2\}) \cup \{u'_2, x_{d+1}\}$ is a d-distance independent dominating set of T. Thus, we have

$$\gamma_d^1(T) \leq |D| = |L(T'')| = \frac{n''}{d+1} = \frac{n'' + \ell''}{d+2} = \frac{n - (d+2) + \ell}{d+2} < \frac{n+\ell}{d+2}.$$

This completes the proof of Claim D. (□)

In the continuation, we may suppose that there is no P_{d+1}-support vertex in T and also that if v is a (P_i, P_j)-support vertex, then $i = j = d$. Let $s = \mathrm{diam}(T) \geq 2d + 1$ and let $P := v_1 v_2 \dots v_{s+1}$ be a diametrical path in T. Root T at v_{s+1}. Hence, by Lemma 3.3, $\deg(v_k) \leq 2$ for each $k \in [d] \cup ([s+1] \setminus [s-d+1])$, and $\deg(v_k) \geq 3$ for each $k \in \{d+1, s-d+1\}$. It also follows that the subtree $T_{v_{d+1}}$ is isomorphic to the $(d-1)$-subdivision of a star $K_{1,t}$ for some $t \geq 2$.

If $s = \mathrm{diam}(T) = 2d + 1$, then by Lemma 3.3 (v), T is obtained from the $(d-1)$-subdivision of a star K_{1,t_1} and the $(d-1)$-subdivision of a star K_{1,t_2} by joining the centers v_{d+1} and v_{d+2}. We may assume that $t_1 \geq t_2 \geq 2$. Then $N(v_{d+2})$ is a d-distance independent dominating set of T. Since $d \geq 2$, we have

$$\begin{aligned}
\gamma_d^1(T) \leq |N(v_{d+2})| = \deg(v_{d+2}) = t_2 + 1 &= \frac{(d+1)t_2 + d + t_2 + 2}{d+2} \\
&< \frac{(d+1)t_2 + dt_2 + t_2 + 2}{d+2} = \frac{2(d+1)t_2 + 2}{d+2} \\
&\leq \frac{d(t_1 + t_2) + 2 + (t_1 + t_2)}{d+2} = \frac{n+\ell}{d+2}.
\end{aligned}$$

So, we may assume that $\mathrm{diam}(T) \geq 2d + 2$ and $n \geq 2d + 3$. Regarding v_{d+2}, we divide the rest of the proof into two cases and prove that the strict inequality $\gamma_d^1(T) < \frac{n+\ell}{d+2}$ holds in each case.

Case 1. Every vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ is of degree at least 3.

For each vertex $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$ we have $\deg(v) \geq 3$, and the subtree T_v is isomorphic to the $(d-1)$-subdivision of a star K_{1,t_v} for $t_v \geq 2$. Let $T' = T - T_{v_{d+2}}$ and $p = \deg(v_{d+2})$. It holds that

$$d + 1 \leq n' = |V(T')| \leq n - 1 - (2d+1)(p-1).$$

Moreover, we have

$$\ell' = |L(T')| \leq \ell - 2(p-1) + 1 = \ell - 2p - 1,$$

with equality if and only if $\deg(v_{d+3}) = 2$, and for each $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$, $t_v = 2$.

Let D' be a $\gamma_d^1(T')$-set. Then $D = D' \cup (N(v_{d+2}) \setminus \{v_{d+3}\})$ is a d-distance independent dominating set of T. Since $d \geq 2$ and $p \geq 2$, by the induction hypothesis we get

$$\begin{aligned}\gamma_d^1(T) \leq |D| = |D'| + p - 1 = \gamma_d^1(T') + p - 1 &\leq \frac{n' + \ell'}{d+2} + p - 1 \\ &\leq \frac{n - 1 - (2d+1)(p-1) + \ell - 2p - 1}{d+2} + p - 1 \\ &= \frac{n + \ell - d(p-1) - p - 3}{d+2} < \frac{n+\ell}{d+2}.\end{aligned}$$

Case 2. There is a vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ with $\deg(v) \leq 2$.

Since v_{d+2} is not a P_{d+1}-support vertex and P is a diametrical path, Lemma 3.3 (iii) implies that T_v is a pendant path P_i for some $i \in [d]$. Moreover, we have the following.

- If v_{d+2} is a P_i-support vertex of T for some $i \in [d-1]$, then there is no other pendant path attached to v_{d+2}.
- If v_{d+2} is a P_d-support vertex of T, then v_{d+2} is not a P_i-support vertex of T for any $i \in [d-1]$, and there is at least one copy of P_d attached to v_{d+2}.

Case 2.1. $L(v_{d+2}) \neq \emptyset$.

Let $x \in L(v_{d+2})$ and $T' = T - x$. Clearly, $\deg_T(v_{d+2}) \geq 3$ and $\deg_{T'}(v_{d+2}) \geq 2$. Then $\ell' = |L(T')| = \ell - 1$ and $n' = |V(T')| = n - 1 \geq 2d + 2$. Let D' be a $\gamma_d^1(T')$-set. By considering whether v_{d+2} is in D' or not, we observe that D' can be chosen such that it is also a d-distance (independent) dominating set of T. By the induction hypothesis, we have $\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'+\ell'}{d+2} = \frac{n-1+\ell-1}{d+2} < \frac{n+\ell}{d+2}$.

Case 2.2. $L(v_{d+2}) = \emptyset$.

Let $P' := x_1 x_2 \dots x_i$ be a copy of P_i attached to v_{d+2}, where $x_i v_{d+2} \in E(T)$. Then $i \in [d] \setminus \{1\}$, and $\deg(x_k) = 2$ for all $k \in [i] \setminus \{1\}$, while $\deg(x_1) = 1$. Consider $T' = T - T_{v_d} - T_{x_i}$. By Lemma 3.3 (ii), $\deg(v_{d+1}) \geq 3$ and, by our condition, $\deg(v_{d+2}) \geq 3$. Therefore, $\ell' = |L(T')| = \ell - 2$. We also know that $n' = |V(T')| = n - d - i \leq n - d - 2$ and $n' \geq d + 3$.

Let D' be a $\gamma_d^1(T')$-set. Then $|D' \cap \{v_{d+1}, v_{d+2}\}| \leq 1$. As in Case 2.2 of Theorem 3.4, let

$$D = \begin{cases} D' \cup \{x_i\}, & \text{if } v_{d+1} \in D' \text{ and } v_{d+2} \notin D', \\ D' \cup \{v_1\}, & \text{if } v_{d+1} \notin D' \text{ and } v_{d+2} \in D', \\ D' \cup \{v_{d+1}, x_i\} \setminus (V(T_{v_{d+1}}) \setminus V(T_{v_d})), & \text{if } v_{d+1}, v_{d+2} \notin D'. \end{cases}$$

For any subcase, D is a d-distance independent dominating set of T. By the induction hypothesis, we have $\gamma_d^1(T) \leq |D| \leq |D'| + 1 = \gamma_d^1(T') + 1 \leq \frac{n'+\ell'}{d+2} + 1 \leq \frac{n-d-2+\ell-2}{d+2} + 1 < \frac{n+\ell}{d+2}$.

This completes the proof of Theorem 4.2. □

Now we set

$$\mathcal{F}_2' = \{T : T - L(T) \in \{K_2\} \cup \mathcal{T}_1\},$$

and if $d \geq 3$, then set

$$\mathcal{F}'_d = \mathcal{F}_d.$$

By Theorems 4.1 and 4.2, we have the following two corollaries, respectively.

Corollary 4.3. *Let $d \geq 2$ be an integer and T be a tree of order n and with ℓ leaves. If $n - \ell \geq d$, then $\gamma_d(T) \leq \frac{n-\ell}{d}$ with equality if and only if $T \in \mathcal{F}'_d$.*

Corollary 4.4. *Let $d \geq 2$ be an integer and T be a tree of order n and with ℓ leaves. If $n \geq d$, then $\gamma_d(T) \leq \frac{n+\ell}{d+2}$ with equality if and only if $T \in \{P_d\} \cup \mathcal{T}_d$.*

Combining the above results with Corollary 2.2, we obtain

Corollary 4.5. *If $d \geq 2$, and T is a tree with ℓ leaves and of order $n \geq d + \ell$, then*

$$\gamma_d(T) \leq \gamma_d^1(T) \leq \begin{cases} \frac{n-\ell}{d}, & \text{if } n < (d+1)\ell, \\ \frac{n}{d+1}, & \text{if } n = (d+1)\ell, \\ \frac{n+\ell}{d+2}, & \text{if } n > (d+1)\ell. \end{cases}$$

Moreover, these bounds are best possible.

5 A conjecture

Recall that Ma and Chen [9] described equivalently bipartite graphs G of order n with $\gamma_1^1(G) = \frac{n}{2}$. For $d \geq 2$ we pose:

Conjecture 5.1. *If $d \geq 2$ and G is a connected bipartite graph G of order n, then $\gamma_d^1(G) = \frac{n}{d+1}$ if and only if $G \in \{C_{2d+2}\} \cup \mathcal{B}_d$ or $n = d + 1$.*

Since $\gamma_1^1(K_{r,r}) = r = \frac{r+r}{2} = \frac{n}{2}$, the condition of $d \geq 2$ of the conjecture above is necessary. If Conjecture 5.1 holds true, then it generalizes Theorem 3.4. Moreover, the result [13, Theorem 3] due to Topp and Volkmann, restricted to bipartite graphs, gives exactly the same characterization for graphs G with $\gamma_d(G) = \frac{n}{d+1}$ as we pose in Conjecture 5.1 for the d-distance independent domination.

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