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Computing fault-tolerant metric dimension of graphs using their primary subgraphs

S. Prabhu[a,*], Sandi Klavžar[b,c,d], K. Bharani Dharan[e], S. Radha[e]

[a] Department of Mathematics, Rajalakshmi Engineering College, Chennai 602105, India

[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[c] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[e] School of Advanced Sciences, Vellore Institute of Technology, Chennai 600127, India

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Abstract

The metric dimension of a graph is the cardinality of a minimum resolving set, which is the set of vertices such that the distance representations of every vertex with respect to that set are unique. A fault-tolerant metric basis is a resolving set with a minimum cardinality that continues to resolve the graph even after the removal of any one of its vertices. The fault-tolerant metric dimension is the cardinality of such a fault-tolerant metric basis. In this article, we investigate the fault-tolerant metric dimension of graphs formed through the point-attaching process of primary subgraphs. This process involves connecting smaller subgraphs to specific vertices of a base graph, resulting in a more complex structure. By analyzing the distance properties and connectivity patterns, we establish explicit formulae for the fault-tolerant resolving sets of these composite graphs. Furthermore, we extend our results to specific graph products, such as rooted products. For these products, we determine the fault-tolerant metric dimension in terms of the fault-tolerant metric dimension of the primary subgraphs. Our findings demonstrate how the fault-tolerant dimension is influenced by the structural characteristics of the primary subgraphs and the attaching vertices. These results have potential applications in network design, error correction, and distributed systems, where robustness against vertex failures is crucial.

*Corresponding author: drsavariprabhu@gmail.com

1 Introduction

The metric dimension of a graph is a fundamental metric in the study of networks, identifying the smallest set of landmark vertices needed to uniquely locate all other vertices using distance measures [1, 2]. This concept underpins numerous applications including network verification, autonomous navigation, and strategic resource placement. However, real-world networks often face disruptions due to system faults, attacks, or environmental changes [3]. To enhance robustness, the notion of the fault-tolerant metric dimension (FTMD) was developed [4], ensuring the uniqueness of vertex identification persists even with the failure of any landmark node. This property enhances the resilience [5], dependability [6], and robustness of networks [7], which is crucial for domains like defense technology, sensor-based infrastructures, distributed systems, and intelligent transport networks. Calculating FTMD is instrumental for constructing networks capable of withstanding failures while preserving their operational integrity, facilitating continuous monitoring and swift system adaptation during breakdowns [7]. The concept of FTMD was first explored in tree structures where its connection to the traditional metric dimension was examined [4]. Later works delved into fault-tolerant resolving sets [8], while optimization methods using integer linear programming were applied to tackle the FTMD problem [9]. Investigations also identified graphs with extremal FTMD values with respect to the order [10]. FTMD studies extended to graphs such as molecules like bismuth tri-iodide, lead chloride, and quartz [11], convex polytopes [12], prism-based graphs [13], and specific compound graphs like $P(n, 2) \odot K_1$ [14]. Exact FTMD values were established for grid graphs [15], and interconnection models like honeycomb and hexagonal networks were analyzed [16]. For cographs, linear-time algorithms were proposed for weighted FTMD [17]. Various graph families including gear and anti-web structures were examined for closed-form FTMD values, with some cases exhibiting constant values [18]. Circulant graphs with degrees 4, 6, and 8 were thoroughly studied [19, 20]. FTMD has also been applied to convex polytopes [21] and extended honeycomb-based silicate networks [22].

Further analysis of well-known networks such as butterfly and Beneš topologies led to refined FTMD results [22]. Algebraic graph models including zero-divisor graphs and their line graphs were also studied [23]. FTMD has been determined for grids [24] and hexagonal ladders [25]. More recent efforts have broadened the application of FTMD to optical interconnects [26], fractal-based topologies [27], and specific nanotube structures with constant FTMD [28]. Further work includes FTMD analysis on arithmetic graphs [29], barycentric subdivisions [30], and generalized fat tree networks [31]. In algebraic contexts, annihilator graphs over various ring structures have also been investigated [32]. The utility of primary subgraphs, which were introduced for the first time in [33], has been demonstrated across various graph parameters. Topological indices like the Hosoya polynomial [33], atom-bond connectivity [34], Graovac-Ghorbani index [34], and elliptic Sombor index [35], have been efficiently computed via this approach. Additionally, parameters such as the total domination polynomial [36], distinguishing number and index [37] and strong domination number [38] have all seen computational benefits from subgraph decomposition. Metric-based parameters in particular have shown promising results through this strategy. Both the metric dimension [39] and local metric dimension [40] have been successfully determined using primary subgraph frameworks, reducing computational demands.

2 Preliminaries

In a simple, connected graph G, the metric $d_G : V(G) \times V(G) \to \mathbb{N}_0$ is defined to assign to each pair of vertices x and y the length of a shortest path connecting them. Define $r(v|X) = (d_G(v, x_1), d_G(v, x_2), \ldots, d_G(v, x_l))$ as the representation of a vertex $v \in G$ concerning the ordered subset $X = \{x_1, x_2, \ldots, x_l\}$. The subset X is termed as a *resolving set* if, for any vertices $x, y \in V(G)$, the condition $r(x|X) \neq r(y|X)$ holds true. Let the set X be classified as a *fault-tolerant resolving set* if, for any element $s \in X$, the set $X \setminus \{s\}$ remains a resolving set. A fault-tolerant resolving set X is considered minimal if there is no other fault-tolerant resolving set X' such that $X' \subset X$. A minimal fault-tolerant resolving set that possesses the least number of elements is referred to as a *fault-tolerant basis*. The cardinality of a fault-tolerant basis is referred to as the *fault-tolerant metric dimension* of the graph G, denoted by $\text{fdim}(G)$. A minimal fault-tolerant resolving set with maximum cardinality is referred to as an upper fault-tolerant basis. The cardinality of this upper fault-tolerant basis is termed the upper fault-tolerant metric dimension of G, denoted by $\text{fdim}^+(G)$. For instance,

- for path graphs P_n, $n \geq 2$, we have $\text{fdim}(P_n) = 2 < \text{fdim}^+(P_n) = 3$;
- for cycle graph C_n, $n \geq 5$, we have $\text{fdim}(C_n) = \text{fdim}^+(C_n) = 3$;
- for star graphs $K_{1,t}$, $t \geq 3$, we have $\text{fdim}(K_{1,t}) = \text{fdim}^+(K_{1,t}) = t$;
- for complete graphs of order n, we have $\text{fdim}(K_n) = \text{fdim}^+(K_n) = n$.

Let G be a connected graph formed by merging a collection of pairwise disjoint connected graphs $G_1, \ldots, G_k$, $k \geq 2$, through the following process. First, select a vertex from G_1 and a vertex from G_2, and identify them. Next, continue this process inductively: assume that the graphs $G_1, \ldots, G_i$ have already been included in the construction, where $2 \leq i \leq k-1$. Select a vertex from the current graph (possibly one of the previously identified vertices) and a vertex from G_{i+1}, then identify them. The resulting graph G has a tree-like structure, with the graphs G_i serving as its building blocks (see Figure 1). We refer to this construction as *point-attaching* of $G_1, \ldots, G_k$, where the graphs G_i are called the *primary subgraphs* of G [33]. A special case of this construction is the decomposition of a connected graph into its blocks.

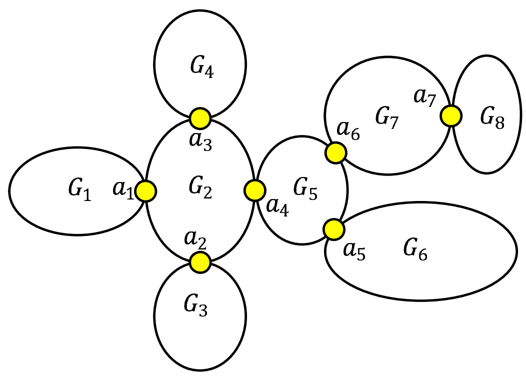

Figure 1: Graph G with attaching vertices

G is obtained by point-attaching the pairwise disjoint connected graphs $G_1, \ldots, G_k$, where each G_i serves as a primary subgraph of G. The vertices of G formed by identifying two vertices from different primary subgraphs are called the *attachment vertices* of G. The set of all attachment vertices of G is denoted by $\mathrm{At}(G)$, and the attachment vertices within each primary subgraph G_i are represented by $\mathrm{At}(G_i) = \mathrm{At}(G) \cap V(G_i)$. Furthermore, for any two vertices $x, y \in V(G_i)$, the distance between them in G is the same as their distance in G_i, i.e., $d_G(x, y) = d_{G_i}(x, y)$. Note that, $V(G_i) \cap V(G_j) = \mathrm{At}(G_i) \cap \mathrm{At}(G_j)$ and $E(G_i) \cap E(G_j) = \emptyset$, for any $i \neq j$ and $1 \leq i, j \leq k$.

Cactus graphs, rooted products of graphs, circuits of graphs, block graphs, bouquets of graphs, generalized corona products of graphs, etc., are some examples of graphs formed by point-attaching.

A primary subgraph G_i is referred to as a *primary end-subgraph* if it contains exactly one attachment vertex, i.e., $|\mathrm{At}(G_i)| = 1$. It is called a *primary internal subgraph* if it contains two or more attachment vertices, i.e., $|\mathrm{At}(G_i)| \geq 2$. Clearly, any graph formed by point-attaching contains at least two primary end-subgraphs.

In this paper, we present closed-form expressions for the FTMD of graphs formed through point-attaching. We apply the main result to specific graph constructions, including the rooted product, the corona product, block graphs, and chain graphs. To introduce the necessary terminology, let G be a simple graph. The neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and its eccentricity by $\epsilon_G(v)$. The diameter of G is represented by $\mathrm{diam}(G)$. For a subset $S \subseteq V(G)$, the subgraph induced by S is written as $\langle S \rangle$. A graph G is said to be even graph [41] if, for every vertex $x \in V(G)$, there is exactly one vertex $y \in V(G)$ such that $d_G(x, y) = \mathrm{diam}(G)$. Examples of even graphs include even cycles and hypercubes. Throughout the paper, definitions and concepts are introduced as they become relevant.

3 Main results

To establish a foundation for our discussion, we first derive a lower bound for the FTMD of graphs formed from primary subgraphs in a general setting. In this case, no specific rule governs the construction process through point-attaching, making the analysis more intricate. Since these constructions rely on the attachment vertices of the primary subgraphs, they present additional challenges in determining their FTMD. To address this complexity, we introduce an additional parameter that directly pertains to the FTMD of graphs derived from primary subgraphs. The precise definition of this parameter is given below.

Let G be a connected graph formed by point-attaching the primary subgraphs $G_1, \dots, G_k$. An *attaching fault-tolerant resolving set* for a primary subgraph G_i is a subset $\mathcal{F}_i \subset V(G_i) \setminus \mathrm{At}(G_i)$ such that $\{\mathcal{F}_i \cup \mathrm{At}(G_i)\} \setminus \{x\}$ is a resolving set of G_i for every $x \in \mathcal{F}_i$. An *attaching fault-tolerant basis* of G_i is an attaching fault-tolerant resolving set of minimum cardinality, and its size is called the *attaching fault-tolerant metric dimension* of G_i, denoted by $\mathrm{fdim}^*(G_i, \mathrm{At}(G_i))$. For a given primary subgraph G_i, we will assume that its set $\mathrm{At}(G_i)$ is fixed, so in the following we may simplify the notation $\mathrm{fdim}^*(G_i, \mathrm{At}(G_i))$ to $\mathrm{fdim}^*(G_i)$.

For example, if $|\mathrm{At}(G_i)| = 1$, then $\mathrm{fdim}^*(G_i) \in \{\mathrm{fdim}(G_i), \mathrm{fdim}(G_i) - 1\}$. Consider next the case $G_i = P_n$. Then $\mathrm{fdim}^*(P_n) = 2$, if $\mathrm{At}(P_n)$ consists of a single vertex of degree 2, while in all the other cases $\mathrm{fdim}^*(P_n) = 0$. For the cycle graph C_n, the value of $\mathrm{fdim}^*(C_n)$ is 2 when $\mathrm{At}(C_n)$ contains exactly one vertex, or when it consists of exactly two antipodal vertices and the cycle has even length. In all the other situations, $\mathrm{fdim}^*(C_n) = 0$. Furthermore, for a complete graph we have that $\mathrm{fdim}^*(K_n) = n - |\mathrm{At}(K_n)|$. For a complete graph K_n, the attaching fault-tolerant metric dimension is given by $\mathrm{fdim}^*(K_n) = n - |\mathrm{At}(K_n)|$, if $|\mathrm{At}(K_n)| < n - 1$, and $\mathrm{fdim}^*(K_n) = 0$ otherwise.

Proposition 1. *If G is a graph formed by point-attaching $G_1, \ldots, G_k$, $k \geq 1$, then*

$$\mathrm{fdim}(G) \geq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i).$$

Proof. Let $\mathcal{F}$ be a fault-tolerant basis of G, and for $i \in \mathbb{N}_k$, set

$$\mathcal{F}_i = \mathcal{F} \cap (V(G_i) \setminus \mathrm{At}(G_i)).$$

We claim that $\mathcal{F}_i$ is an attaching fault-tolerant resolving set of G_i. That is, we need to prove $\mathcal{F}_i \cup \mathrm{At}(G_i) \setminus \{x\}$ is a resolving set for every $x \in \mathcal{F}_i$.

Fix $x \in \mathcal{F}_i$, and let u and v be arbitrary two vertices of G_i. Since $\mathcal{F}$ is the fault-tolerant basis of G, the vertices u and v are resolved in G by some vertex y from $\mathcal{F} \setminus \{x\}$. Assume first that $y \in V(G_i)$. Then having in mind that G_i is an isometric subgraph of G, vertices u and v are also resolved in G_i. Assume second that $y \in V(G_j)$ for some $j \neq i$. Then there exists a unique vertex $a \in \mathrm{At}(G_i)$ such that $d_G(u, y) = d_G(u, a) + d_G(a, y)$ and $d_G(v, y) = d_G(v, a) + d_G(a, y)$. Since $d_G(u, y) \neq d_G(v, y)$, it follows that $d_G(u, a) \neq d_G(x, a)$. This in turn gives $d_{G_i}(u, a) \neq d_{G_i}(x, a)$, hence also in this case vertices u and v are also resolved in G_i with a vertex from $\mathcal{F}_i \cup \mathrm{At}(G_i) \setminus \{x\}$. We have thus proved that $\mathcal{F}_i$ is an attaching fault-tolerant resolving set of G_i and therefore $|\mathcal{F}_i| \geq \mathrm{fdim}^*(G_i)$. This implies

$$\mathrm{fdim}(G) = |\mathcal{F}| \geq \sum_{i=1}^{k} |\mathcal{F}_i| \geq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i)$$

and we are done. □

For a graph G constructed through point-attaching, we define the following properties for each primary subgraph G_i.

Condition 1 ($\mathcal{C}1$)**:** For any $a_1 \in \mathrm{At}(G_i)$ and $v \in V(G_i) \setminus \mathrm{At}(G_i)$, there exists $a_2 \in \mathrm{At}(G_i)$ such that $d_{G_i}(a_1, a_2) \geq d_{G_i}(v, a_2)$.

Condition 2 ($\mathcal{C}2$)**:** $\mathrm{At}(G_i) = \{a\}$ and either G_i is a path and a is not a leaf, or G_i is not a path.

Note that condition $\mathcal{C}1$ is satisfied by a large class of connected graphs. For example, this holds when a primary internal subgraph G_i meets one of the following conditions.

1. $V(G_i) = \mathrm{At}(G_i)$.
2. $\mathrm{At}(G_i)$ is any independent set for G_i and $\mathrm{diam}(G_i) = 2$.
3. $\mathcal{E}_{G_i}(u_1) = \mathcal{E}_{G_i}(u_2) = d_{G_i}(u_1, u_2)$ for any pair of distinct vertices $u_1, u_2 \in \mathrm{At}(G_i)$. In particular, this includes all non-trivial complete graphs.

4. G_i is an even graph and if $u \in \mathrm{At}(G_i)$, then the vertex antipodal to u also belongs to $\mathrm{At}(G_i)$.

A graph G has fault-tolerant metric dimension 2 if and only if it is a path graph. Moreover, a set $\{v_1, v_2\}$ forms a fault-tolerant metric basis for a path if and only if both v_1 and v_2 are leaves of the path. Therefore, if a graph G_i satisfies condition $\mathcal{C}2$, then its fault-tolerant metric dimension must be at least 1, that is, $\mathrm{fdim}^*(G_i) \geq 1$.

To demonstrate that the bound of Proposition 1 is tight, we impose restrictions on primary subgraphs using conditions $\mathcal{C}1$ and $\mathcal{C}2$ as follows.

Theorem 2. *If G is a graph formed by point-attaching $G_1, \ldots, G_k$, $k \geq 3$, such that each primary internal subgraph satisfies $\mathcal{C}1$, every primary end-subgraph satisfies $\mathcal{C}2$, and $\mathrm{At}(G_i) \cap \mathrm{At}(G_j) = \emptyset$ for any pair of primary end-subgraphs G_i and G_j, then*

$$\mathrm{fdim}(G) = \sum_{i=1}^{k} \mathrm{fdim}^*(G_i).$$

Proof. By Proposition 1, we have $\mathrm{fdim}(G) \geq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i)$, hence it remains to show that $\mathrm{fdim}(G) \leq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i)$. Let $\mathcal{F}_i$, $i \in [k]$, be an attaching fault-tolerant basis of G_i. We will demonstrate that the set $\mathcal{F} = \cup_{i=1}^{k} \mathcal{F}_i$ forms a fault-tolerant resolving set for G. To do this, we consider the following cases for any two distinct vertices x_1 and x_2 of G.

Case 1: $x_1, x_2 \in V(G_i)$, $x_1 \neq x_2$.

Subcase 1.1: $\mathcal{F}_i \neq \emptyset$.

Since $\mathcal{F}_i$ is non-empty and is at the same an attaching fault-tolerant basis for G_i, there exists $u_1, u_2 \in \mathcal{F}_i$ such that $d_G(x_1, u_1) \neq d_G(x_2, u_1)$ and $d_G(x_1, u_2) \neq d_G(x_2, u_2)$. If $u_1, u_2 \in \mathcal{F}_i$, then we are done. Now if $u_1 \in \mathrm{At}(G_i)$, then there exists a primary end-subgraph G_j, $j \neq i$, such that for any $w \in \mathcal{F}_j$ we have $d_G(u_1, w) = \min\limits_{v \in V(G_i)} \{d_G(v, w)\}$. Notice that since G_j satisfies $\mathcal{C}2$, we get $\mathcal{F}_j \geq 2$. Hence

$$d_G(x_1, w) = d_G(x_1, u_1) + d_G(u_1, w) \neq d_G(x_2, u_1) + d_G(u_1, w) = d_G(x_2, w).$$

Subcase: 1.2: $\mathcal{F}_i = \emptyset$.

Since $\mathrm{At}(G_i)$ is a resolving set for G_i, for any two vertices of $x_1, x_2 \in V(G_i)$ there exist $a \in \mathrm{At}(G_i)$, such that $d_G(x_1, a) \neq d_G(x_2, a)$. This implies that there exists a primary end-subgraph G_j such that for any $w \in \mathcal{F}_j$ we have $d_G(a, w) = \min\limits_{v \in V(G_i)} \{d_G(v, w)\}$. Since G_j satisfies $\mathcal{C}2$, $|\mathcal{F}_j| \geq 2$. Hence, for any $w \in \mathcal{F}_j$,

$$d_G(x_1, w) = d_G(x_1, a) + d_G(a, w_1) \neq d_G(x_2, a) + d_G(a, w) = d_G(x_2, w).$$

Case 2: $x_1 \in V(G_i)$, $x_2 \in V(G_j)$, where $i \neq j$.

Let $a_1 \in V(G_i)$ and $a_2 \in V(G_j)$ be the attachment vertices such that $d_G(x_1, x_2) = d_G(x_1, a_1) + d_G(a_1, a_2) + d_G(a_2, x_2)$. Note that if G_i and G_j have a common attachment vertex, then $a_1 = a_2$. If $x_2 = a_2 = a_1$ or $x_1 = a_1 = a_2$, then we proceed as Case 1. Hence, in the rest we may assume that x_1 and x_2 do not belong to the same primary subgraph, that is, $x_2 \neq a_1$ and $x_1 \neq a_2$.

Subcase 2.1: $|\operatorname{At}(G_i)| \geq 2$ or $|\operatorname{At}(G_j)| \geq 2$.

Assume without loss of generality that $|\operatorname{At}(G_i)| \geq 2$. Since G_i satisfies $\mathcal{C}1$, there exists $c \in \operatorname{At}(G_i) \setminus \{a_1\}$, such that $d_G(a_1, c) \geq d_G(x_1, c)$. Now, let G_ℓ, $\ell \neq i$, be a primary end-subgraph such that for any $t \in \mathcal{F}_\ell$, $d_G(c, t) = \min_{v \in V(G_i)}\{d_G(v, t)\}$, ($\mathcal{F}_\ell \geq 2$, as G_ℓ satisfies $\mathcal{C}2$). Then for any $t_1, t_2 \in \mathcal{F}_\ell$,

$$\begin{aligned} d_G(x_1, t_1) &= d_G(x_1, c) + d_G(c, t_1) \leq d_G(a_1, c) + d_G(c, t_1) \\ &< d_G(x_2, a_1) + d_G(a_1, c) + d_G(c, t_1) = d_G(x_2, t_1), \quad \text{and} \\ d_G(x_1, t_2) &= d_G(x_1, c) + d_G(c, t_2) \leq d_G(a_1, c) + d_G(c, t_2) \\ &< d_G(x_2, a_1) + d_G(a_1, c) + d_G(c, t_2) = d_G(x_2, t_2). \end{aligned}$$

Subcase 2.2: $|\operatorname{At}(G_i)| = |\operatorname{At}(G_j)| = 1$.

Clearly G_i and G_j are primary end-subgraphs and since they satisfy $\mathcal{C}2$, it follows that $|\mathcal{F}_i| \geq 2$ and $|\mathcal{F}_j| \geq 2$. Hence, let $p_1, p_2 \in \mathcal{F}_i$ and $q_1, q_2 \in \mathcal{F}_j$. If there exist two vertices in $\{p_1, p_2, q_1, q_2\}$ that distinguish x_1 and x_2, then we are done. On the contrary, suppose that there do not exist at least two vertices in $\{p_1, p_2, q_1, q_2\}$, such that they individually distinguishes the vertices x_1 and x_2. We may suppose without loss of generality that in this case we have:

$$d_G(x_1, p_1) = d_G(x_2, p_1) = d_G(x_2, a_2) + d_G(a_2, a_1) + d_G(a_1, p_1), \tag{1}$$

$$d_G(x_1, p_2) = d_G(x_2, p_2) = d_G(x_2, a_2) + d_G(a_2, a_1) + d_G(a_1, p_2), \tag{2}$$

$$d_G(x_2, q_1) = d_G(x_1, q_1) = d_G(x_1, a_1) + d_G(a_1, a_2) + d_G(a_2, q_1). \tag{3}$$

Observe that since $|\operatorname{At}(G_i) \cap \operatorname{At}(G_j)| = \emptyset$, we have $a_1 \neq a_2$. Moreover,

$$d_G(x_1, p_1) \leq d_G(x_1, a_1) + d_G(a_1, p_1), \tag{4}$$

$$d_G(x_1, p_2) \leq d_G(x_1, a_1) + d_G(a_1, p_2), \tag{5}$$

$$d_G(x_2, q_1) \leq d_G(x_2, a_2) + d_G(a_2, q_1). \tag{6}$$

From (1), (2), (4) and (5) we obtain

$$d_G(x_2, a_2) + d_G(a_2, a_1) \leq d_G(x_1, a_1). \tag{7}$$

From (3) and (6)

$$d_G(x_1, a_1) + d_G(a_1, a_2) \leq d_G(x_2, a_2). \tag{8}$$

Adding (7) and (8), we get

$$2 \cdot d_G(a_1, a_2) \leq 0,$$

which is a contradiction. Hence, $\mathrm{fdim}(G) \leq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i)$. □

The following sections focus on deriving several consequences of Theorem 2. Specifically, we present closed-form expressions for the FTMD of certain families of graphs in terms of parameters associated with their primary subgraphs. This is done in cases where the point-attaching process can be described as a graph composition scheme or when the primary subgraphs satisfy specific properties.

4 An extremal case

As before, let G be a graph formed by point-attaching the primary subgraphs $G_1, \ldots, G_k$. In this section, we examine the case where every minimal fault-tolerant basis of a primary subgraph is also of minimum cardinality, that is, when $\mathrm{fdim}(G_i) = \mathrm{fdim}^+(G_i)$ for each G_i. Let $\mathcal{F}(G_i)$ denote the collection of all fault-tolerant metric bases of G_i and let

$$\theta_i = \begin{cases} \max\limits_{F \in \mathcal{F}(G_i)} \{|F \cap \mathrm{At}(G_i)|\}; & \mathrm{At}(G_i) \text{ is not a resolving set of } G_i, \\ \mathrm{fdim}(G_i); & \text{otherwise.} \end{cases}$$

In other words, θ_i represents the maximum number of attachment vertices of G that are also part of a metric basis of G_i.

Corollary 1. *Let G be a graph formed by point-attaching $G_1, \ldots, G_k$, $k \geq 3$, which satisfy the conditions of Theorem 2. If also $\mathrm{At}(G_i) \neq V(G_i)$ and $\mathrm{fdim}(G_i) = \mathrm{fdim}^+(G_i)$, $i \in [k]$, then*

$$\mathrm{fdim}(G) = \sum_{i=1}^{k} (\mathrm{fdim}(G_i) - \theta_i).$$

Proof. For any primary subgraph G_i of G such that $\mathrm{fdim}(G_i) = \mathrm{fdim}^+(G_i)$, we have $\mathrm{fdim}^*(G_i) = \mathrm{fdim}(G_i) - \theta_i$. Hence, the result follows from Theorem 2. □

Consider the graph with five primary subgraphs from Figure 2, where $\mathrm{At}(G_1) = \{a_1\}$, $\mathrm{At}(G_2) = \{a_1, a_2, a_3\}$, $\mathrm{At}(G_3) = \{a_2\}$, $\mathrm{At}(G_4) = \{a_3, a_4\}$, and $\mathrm{At}(G_5) = \{a_4\}$. Using Corollary 1 we get

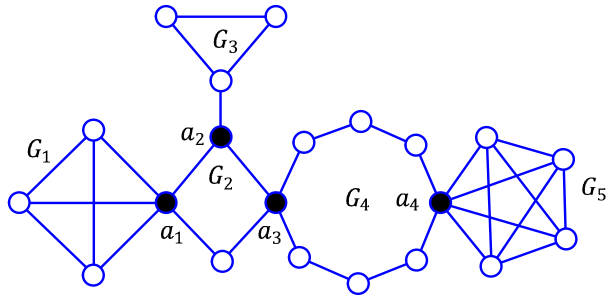

Figure 2: A graph G formed by point-attaching from $G_1 \cong K_4$, $G_2 \cong K_3$, G_3 being the paw graph, $G_4 \cong C_8$, and $G_5 \cong K_5$.

$\text{fdim}(G) = (\text{fdim}(G_1) - 1) + (\text{fdim}(G_2) - 4) + (\text{fdim}(G_3) - 1) + (\text{fdim}(G_4) - 1) + (\text{fdim}(G_5) - 1) = 3 + 0 + 2 + 2 + 4 = 11.$

A block graph is a graph in which each biconnected component (block) forms a clique. Observe that any block graph can be constructed by point-attaching a family of connected graphs. Recalling that $\text{fdim}(K_n) = n = \text{fdim}^+(K_n)$, we get the following remark as a special case of Corollary 1.

Remark. *Let G be a block graph formed by point-attaching $\{K_{r_1}, K_{r_2}, \ldots, K_{r_k}\}$, where $k \geq 3$ and $r_i \geq 3$, $i \in [k]$. If $\text{At}(K_{r_i}) \cap \text{At}(K_{r_j}) = \emptyset$ for any primary end-subgraphs K_{r_i} and K_{r_j}, then,*

$$\text{fdim}(G) = \sum_{\substack{i \in [k] \\ |\text{At}(G_i)| < r_i - 1}} (r_i - |\text{At}(K_{r_i})|).$$

5 Rooted products

In this section, we explore a notable special case of graphs constructed through point-attaching: the rooted product of graphs.

A rooted graph is a graph with a designated vertex that is uniquely labeled to distinguish it from the others. This distinguished vertex is referred to as the root of the graph. Let G be a labeled graph with n vertices, and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a collection of rooted graphs. The rooted product graph $G[\mathcal{H}]$ is formed by identifying the root of each graph H_i with the i^{th} vertex of G. It is evident that any rooted product graph $G[\mathcal{H}]$ can be viewed as a graph obtained by point-attaching. Here, the primary internal subgraph is G with $\text{At}(G) = V(G)$, while the family $\mathcal{H}$ consists of primary end-subgraphs, where each attachment vertex corresponds to the root of the respective graph. Using Theorem 2, we derive the following result.

Corollary 2. *Let G be a connected graph of order $n \geq 2$, and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of rooted graphs, each satisfying $\mathcal{C}2$, with roots $v_1, \dots, v_n$, respectively. Then*

$$\mathrm{fdim}(G[\mathcal{H}]) = \sum_{H_i \in \mathcal{H}_1} \mathrm{fdim}(H_i) + \sum_{H_j \in \mathcal{H}_2} (\mathrm{fdim}(H_j) - 1),$$

where $H_i \in \mathcal{H}_1$ if v_i is not part of any fault-tolerant basis of H_i, and $H_j \in \mathcal{H}_2$ otherwise.

Next, we examine the case where the family $\mathcal{H}$ consists of vertex-transitive graphs. Let $\mathrm{Aut}(H)$ denote the automorphism group of a graph H. For any two vertices $x_1, x_2 \in V(H)$ and any automorphism $f \in \mathrm{Aut}(H)$, the distance between the vertices is preserved, i.e., $d(x_1, x_2) = d(f(x_1), f(x_2))$. Consequently, if $\mathcal{F}$ is a fault-tolerant basis of a connected graph H and $f \in \mathrm{Aut}(H)$, then the image of the basis under the automorphism, $f(\mathcal{F})$, is also a fault-tolerant basis of H. Thus, each vertex of H must belongs to some fault-tolerant basis. Applying Corollary 1, we derive the following remark.

Remark. *Let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a collection of vertex-transitive graphs with orders greater than two. Then for any connected graph G of order $n \geq 2$, we have*

$$\mathrm{fdim}(G[\mathcal{H}]) = \sum_{i=1}^{n} \big(\mathrm{fdim}(H_i) - 1\big).$$

In particular, if $\mathcal{H} = \{K_{r_1}, \dots, K_{r_n}\}$, then

$$\mathrm{fdim}(G[\mathcal{H}]) = \sum_{i=1}^{n} (r_i - 1),$$

and if $\mathcal{H} = \{C_{r_1}, \dots, C_{r_n}\}$, then

$$\mathrm{fdim}(G[\mathcal{H}]) = 2 \cdot n.$$

A special case of rooted product graphs arises when the family $\mathcal{H}$ consists of n isomorphic rooted graphs. Formally, let $V(G) = \{u_1, \dots, u_n\}$, and let v be the root vertex of a graph H. The rooted product graph $G \circ_v H$ is defined with the vertex set $V(G \circ_v H) = V(G) \times V(H)$ and the edge set

$$E(G \circ_v H) = \bigcup_{i=1}^{n} \{(u_i, b)(u_i, y) : by \in E(H)\} \cup \{(u_i, v)(u_j, v) : u_i u_j \in E(G)\}.$$

For this case, Corollary 1 reduces to the following.

Proposition 3. *If H is a connected graph not isomorphic to a path and $v \in V(H)$, then the following hold.*

(i) If v is not a part of any fault-tolerant basis of H, then for any connected graph G of order n,

$$\mathrm{fdim}(G \circ_v H) = n \cdot \mathrm{fdim}(H).$$

(ii) If v belongs to a fault-tolerant basis of H, then for any connected graph G of order $n \geq 2$,

$$\mathrm{fdim}(G \circ_v H) = n \cdot (\mathrm{fdim}(H) - 1).$$

Proposition 3 raises the question of identifying the necessary and/or sufficient conditions for a vertex $v \in V(H)$ to be part of a fault-tolerant basis of H. For example, it is straightforward to verify that a vertex v in a path graph P belongs to fault-tolerant basis of P if and only if v is one of its leaf vertices. Building on this observation, Proposition 3 (i) yields:

Corollary 3. *Let H be a connected graph, $v \in V(H)$ a vertex that is not part of any fault-tolerant basis of H, and let G be a connected graph of order n. Then $\mathrm{fdim}(G \circ_v H) = 2n$ if and only if H is a path graph and v is not a leaf.*

In view of Proposition 3 and Corollary 3, the remaining case to be considered is when the second factor of a rooted product graph is a path with the root as a leaf. For this specific case, the following bounds are established.

Proposition 4. *If G is a connected graph of order $n \geq 2$, and v is a leaf of a non-trivial path P, then*

$$\mathrm{fdim}(G) \leq \mathrm{fdim}(G \circ_v P) \leq n.$$

Proof. G appears as an induced subgraph of $G \circ_v P$. Since any fault-tolerant resolving set of $G \circ_v P$ must in particular resolve G, the lower bound follows.

To establish the upper bound we claim that $V(G) \times \{v'\}$ is a fault-tolerant resolving set for $G \circ_v P$, where v' denotes the leaf of P distinct from v. Let (x, y) and (x', y') be any two vertices of $G \circ_v P$. If $x = x'$, then for any $u_1, u_2 \in V(G)$,

$$d\big((u_1, v'), (x, y)\big) \neq d\big((u_1, v'), (x, y')\big)\,.$$

And if $x \neq x'$, then

$$d\big((x, v'), (x, y)\big) < d\big((x, v'), (x', y')\big) \text{ and } d\big((x', v'), (x, y)\big) > d\big((x', v'), (x', y')\big)\,.$$

Hence $V(G) \times \{v'\}$ is indeed a fault-tolerant resolving set for $G \circ_v P$, hence $\mathrm{fdim}(G \circ_v P) \leq n$. □

To see that the upper bound of Proposition 4 is sharp, consider the example from Figure 3.

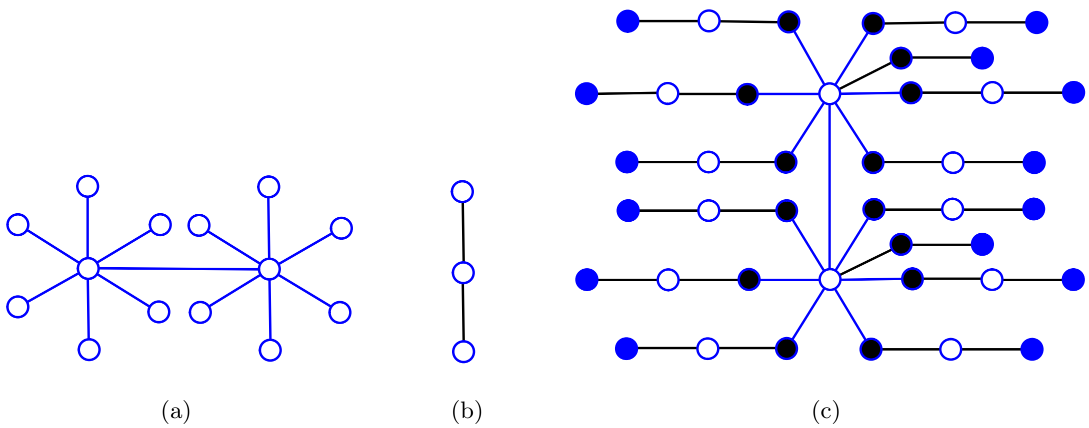

Figure 3: (a) Graph G (b) Graph $H = P_3$ (c) $G \circ_v P_3$, where the blue vertices form a fault-tolerant basis, and the black vertices are the attaching vertices.

6 Conclusion

In this paper, we have introduced an effective methodology for computing the fault-tolerant metric dimension of graphs constructed through point-attaching techniques involving primary subgraphs. By systematically analyzing the role of distance relations and connectivity among the attached substructures, we established explicit formulas for determining fault-tolerant resolving sets. This framework not only simplifies the computation for complex graphs but also enhances our understanding of how the structural properties of the primary subgraphs influence the overall fault tolerance.

Furthermore, we extended our approach to specific graph products, including the rooted product, by expressing their fault-tolerant metric dimensions in terms of the fault-tolerant metric dimensions of their component subgraphs. This generalization highlights the versatility and applicability of our method across various graph constructions.

The results presented contribute significantly to the study of fault-tolerant graph invariants and offer a modular approach for analyzing large-scale networks. Such insights are especially relevant for applications in network design, fault detection, and resilient communication systems, where maintaining unique identifiability despite node failures is essential.

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