Computing fault-tolerant metric dimension of graphs using their primary subgraphs

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Abstract

The metric dimension of a graph is the cardinality of a minimum resolving set, which is the set of vertices such that the distance representations of every vertex with respect to that set are unique. A fault-tolerant metric basis is a resolving set with a minimum cardinality that continues to resolve the graph even after the removal of any one of its vertices. The fault-tolerant metric dimension is the cardinality of such a fault-tolerant metric basis. In this article, we investigate the fault-tolerant metric dimension of graphs formed through the point-attaching process of primary subgraphs. This process involves connecting smaller subgraphs to specific vertices of a base graph, resulting in a more complex structure. By analyzing the distance properties and connectivity patterns, we establish explicit formulae for the fault-tolerant resolving sets of these composite graphs. Furthermore, we extend our results to specific graph products, such as rooted products. For these products, we determine the fault-tolerant metric dimension in terms of the fault-tolerant metric dimension of the primary subgraphs. Our findings demonstrate how the fault-tolerant dimension is influenced by the structural characteristics of the primary subgraphs and the attaching vertices. These results have potential applications in network design, error correction, and distributed systems, where robustness against vertex failures is crucial.

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1 Introduction

The metric dimension of a graph is a fundamental metric in the study of networks, identifying the smallest set of landmark vertices needed to uniquely locate all other vertices using distance measures [1, 2]. This concept underpins numerous applications including network verification, autonomous navigation, and strategic resource placement. However, real-world networks often face disruptions due to system faults, attacks, or environmental changes [3]. To enhance robustness, the notion of the fault-tolerant metric dimension (**FTMD**) was developed [4], ensuring the uniqueness of vertex identification persists even with the failure of any landmark node. This property enhances the resilience [5], dependability [6], and robustness of networks [7], which is crucial for domains like defense technology, sensor-based infrastructures, distributed systems, and intelligent transport networks. Calculating **FTMD** is instrumental for constructing networks capable of withstanding failures while preserving their operational integrity, facilitating continuous monitoring and swift system adaptation during breakdowns [7]. The concept of **FTMD** was first explored in tree structures where its connection to the traditional metric dimension was examined [4]. Later works delved into fault-tolerant resolving sets [8]. Further investigations on characterization of graphs with respect to maximum **FTMD** were erroneously done in [9], later it was corrected by Prabhu et al. in [10]. This studies extended to convex polytopes [11], grid graphs [12], and interconnection models like honeycomb and hexagonal networks were analyzed [13]. Circulant graphs with degrees 4, 6, and 8 were thoroughly studied [14, 15]. The parameter **FTMD** has also been investigated for convex polytopes [16], butterfly, Beneš, and extended honeycomb-based silicate networks [10].

Further analysis of well-known networks such as generalized fat-tree [17], biswapped network [18], and fractal cubic network [19]. Algebraic graph models including zero-divisor graphs and their line graphs were also studied [20], and annihilator graphs over various ring structures have also been investigated [21]. More recent efforts have broadened the application of **FTMD** in specific nanotube structures with constant **FTMD** [22]. Further work includes **FTMD** analysis on arithmetic graphs [23], and barycentric subdivisions [24]. The other fault-tolerant variants reported in [25–30] are also interesting to investigate.

2 Preliminaries

Throughout this paper we denote the simple, undirected, connected graph as G, and the metric $d_G : V(G) \times V(G) \to \mathbb{N}_0$ ($d : V(G) \times V(G) \to \mathbb{N}_0$ if G is understood) is defined to assign to each pair (x, y), the minimum number of edges required to connect them. Here, we denote $\{0, 1, 2, \ldots\}$ as $\mathbb{N}_0$. Define $r(v|X) = (d(v, x_1), d(v, x_2), \ldots, d(v, x_l))$ as the representation of a vertex $v \in G$ concerning the ordered subset $X = \{x_1, x_2, \ldots, x_l\}$. The subset X is called a *resolving set* if, for any vertices $x, y \in V(G)$, the condition $r(x|X) \neq r(y|X)$ holds true. Let the set X be classified as a fault-tolerant resolving set (**FTRS**) if, for any element $s \in X$, the set $X \setminus \{s\}$ remains a resolving set. A **FTRS** X is considered minimal if there is no other fault-tolerant resolving set X' such that $X' \subset X$. A minimal **FTRS** that possesses the least number of elements is referred to as a fault-tolerant basis (**FTB**). The cardinality of a fault-tolerant basis is referred to as the **FTMD** of the graph G, coined by $\text{fdim}(G)$. A minimal **FTRS** with maximum cardinality is referred to as an upper fault-tolerant basis. The cardinality of this upper **FTB** is termed the upper **FTMD** of G, denoted as $\text{fdim}^+(G)$. The upper **FTMD** of some standard graphs are as follows.

- for $n \geq 2$, $\text{fdim}(P_n) = 2 < \text{fdim}^+(P_n) = 3$;
- for $n \geq 5$, $\text{fdim}(C_n) = \text{fdim}^+(C_n) = 3$;
- for $t \geq 3$, $\text{fdim}(K_{1,t}) = \text{fdim}^+(K_{1,t}) = t$;
- for $n \geq 3$, $\text{fdim}(K_n) = \text{fdim}^+(K_n) = n$.

Let G be a resulting graph formed by merging a collection of pairwise disjoint graphs $G_1, \ldots, G_k$, $k \geq 2$, through the following process. First, select two vertices one from G_1 and the other from G_2, and identify them. Next, continue this process inductively: assume that the graphs $G_1, \ldots, G_i$ have already been included with $i \in \{2, 3, \ldots k-1\}$. Choose a vertex from the current graph (possibly one of the previously identified vertices) and a vertex from G_{i+1}, then identify them. The resulting graph G has a tree-like structure, with the graphs G_i serving as its building blocks (see Figure 1). We refer to this construction as *point-attaching* of $G_1, \ldots, G_k$, where the graphs G_i are called the *primary subgraphs* of G [31]. Throughout this paper, the notation $[k]$ denotes the index set $\{1, 2, \ldots, k\}$. Here we would like to emphasize that if each of the graphs G_i, $i \in [k]$, is 2-connected, then the graphs G_i are precisely the blocks of the graph constructed in the described way, see Figure 1 again.

A vertex $a \in V(G)$ is said to be an attachment vertex of G, if $V(G_i) \cap V(G_j) = \{a\}$, for some i and j. The collection of all such vertices of G is denoted by $\text{At}(G)$, $\text{At}(G_i) = \text{At}(G) \cap V(G_i)$. Note that, $V(G_i) \cap V(G_j) = \text{At}(G_i) \cap \text{At}(G_j)$ and $E(G_i) \cap E(G_j) = \emptyset$, for any $i \neq j$ and $1 \leq i, j \leq k$.

The utility of primary subgraphs, which were introduced for the first time in [31], has been demonstrated across various graph parameters. Topological indices like the Hosoya polynomial [31],

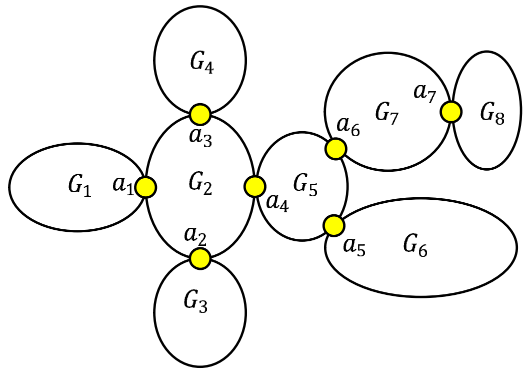

Figure 1: Graph G with attaching vertices

atom-bond connectivity [32], Graovac-Ghorbani index [32], and elliptic Sombor index [33], have been efficiently computed via this approach. Additionally, parameters such as the total domination polynomial [34], distinguishing number and distinguishing index [35], and strong domination number [36] have all seen computational benefits from subgraph decomposition. Metric-based parameters in particular have shown promising results through this strategy. Certain metric dimension parameters have been successfully determined using primary subgraph frameworks [37,38], reducing computational demands.

It is interesting to note that the Cactus graphs [39], rooted products of graphs [40], block graphs [41], corona products of graphs [42] are some few prominent examples of graphs formed by point-attaching process.

We now give a couple of basic properties of graphs formed by point-attaching. For this sake recall that a subgraph H of a graph G is *isometric*, if for every two vertices $u, v \in V(H)$ we have $d_H(u,v) = d_G(u,v)$, and that H is *convex* if every shortest u,v-path from G lies completely in H. Clearly, every convex subgraph is isometric (but not necessarily the other way around).

Lemma 1. *If G is a graph formed by point-attaching $G_1, \ldots, G_k$, $k \geq 1$, then the following properties hold.*

(i) G_i, $i \in [k]$, is a convex subgraph of G.

(ii) If $u \in V(G_i)$ and $v \in V(G_j)$, where $i \neq j$, then there exist attachment vertices $x_i \in V(G_i)$ and $x_j \in V(G_j)$, such that every shortest u,v-path contains x_i and x_j.

Proof. (i) Let u and v be arbitrary vertices from some primary subgraph G_i, where $i \in [k]$. Consider an arbitrary shortest u,v-path P in G. If P contains no attachment vertex, then P clearly lies completely in G_i. Assume next that P contains some attachment vertex x and suppose that x has a neighbor $y \notin V(G_i)$ on P, so that P contain vertices $u, \ldots, x, y, \ldots, v$ in that order. Then by the point-attaching procedure, P cannot re-enter G_i, that is, P is not a shortest $u.v$-path. This contradiction proves that all the vertices of P belong to $V(G_i)$ and we can conclude that G_i is a convex subgraph of G.

(ii) Let $u \in V(G_i)$ and $v \in V(G_j)$, where $i \neq j$, and let P be an arbitrary shortest u, v-path. Then by the (tree-like) point-attaching procedure, there exist a unique sequence of primary subgraphs $G_{k_1}, \ldots, G_{k_r}$, $r \geq 1$, such that P sequentially contains vertices from $G_i, G_{k_1}, \ldots, G_{k_r}, G_j$. Then the claimed attachment vertex $x_i \in V(G_i)$ is the unique common vertex of G_i and G_{k_1}, and the claimed attachment vertex $x_j \in V(G_j)$ is the unique common vertex of G_{k_r} and G_j. □

We add that in Lemma 1(ii), it is possible that $x_i = u$, or that $x_j = v$, or that $x_i = x_j$.

A *primary end-subgraph* is a primary subgraph G_i with exactly one attachment vertex, i.e., $|\operatorname{At}(G_i)| = 1$. It is called a *primary internal subgraph* if it contains two or more attachment vertices, i.e., $|\operatorname{At}(G_i)| \geq 2$.

In this paper, we present exact expressions for the **FTMD** of graphs formed through point-attaching. We apply the main result to specific graph constructions, including the rooted product and block graphs. An open neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and its eccentricity by $\epsilon_G(v)$. A graph G is said to be even graph [43] if, for every $x \in V(G)$, there exist an unique $y \in V(G)$, such that they are diametrically opposite. The hypercubes and even cycles are few examples of even graphs. Throughout the paper, definitions and concepts are introduced as they become relevant.

3 Main results

To establish a foundation for our discussion, we first derive a some bound for the FTMD of G formed by the point-attaching process. In this case, no specific rule governs the construction process through point-attaching, making the analysis more intricate. Since these constructions rely on the attachment process, they present additional challenges in determining their FTMD. To address this complexity, we introduce an additional parameter that directly pertains to the FTMD of graphs derived from primary subgraphs.

An attaching **FTRS** of G_i is a subset $\mathcal{F}_i \subset V(G_i) \setminus \operatorname{At}(G_i)$ such that $\{\mathcal{F}_i \cup \operatorname{At}(G_i)\} \setminus \{x\}$ is a resolving set of G_i for every $x \in \mathcal{F}_i$. An attaching **FTB** of G_i is an attaching **FTRS** of minimum cardinality, and its size is called the attaching **FTMD** of G_i, denoted by $\mathrm{fdim}^*(G_i, \mathrm{At}(G_i))$. For a given primary subgraph G_i, we will assume that its set $\mathrm{At}(G_i)$ is fixed, so in the following we may simplify the notation $\mathrm{fdim}^*(G_i, \mathrm{At}(G_i))$ to $\mathrm{fdim}^*(G_i)$.

For example, if $|\mathrm{At}(G_i)| = 1$, then $\mathrm{fdim}^*(G_i) \in \{\mathrm{fdim}(G_i), \mathrm{fdim}(G_i)-1\}$. Consider next the case $G_i = P_n$. Then $\mathrm{fdim}^*(P_n) = 2$, if $\mathrm{At}(P_n)$ consists of a single vertex of degree 2, while in all the other cases $\mathrm{fdim}^*(P_n) = 0$. For the cycle graph C_n, the value of $\mathrm{fdim}^*(C_n)$ is 2 when $\mathrm{At}(C_n)$ contains exactly one vertex, or when it consists of exactly two antipodal vertices and the cycle has even length. In all the other situations, $\mathrm{fdim}^*(C_n) = 0$. Furthermore, $\mathrm{fdim}^*(K_n) = n - |\mathrm{At}(K_n)|$. For a complete graph K_n, the attaching fault-tolerant metric dimension is given by $\mathrm{fdim}^*(K_n) = n - |\mathrm{At}(K_n)|$, if $|\mathrm{At}(K_n)| < n - 1$, and $\mathrm{fdim}^*(K_n) = 0$ otherwise.

Proposition 2. *If G is formed by the point-attaching process over $\{G_i : i \in [k]\}$, $k \geq 1$, then*

$$\mathrm{fdim}(G) \geq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i).$$

Proof. For $i \in \mathbb{N}_k$, let $\mathcal{F}_i = \mathcal{F} \cap (V(G_i) \setminus \mathrm{At}(G_i))$, where $\mathcal{F}$ is a **FTB** of G. We claim that $\mathcal{F}_i$ is an attaching **FTRS** of G_i. That is, we need to prove $\mathcal{F}_i \cup \mathrm{At}(G_i) \setminus \{x\}$ is a resolving set for each $x \in \mathcal{F}_i$.

Fix $x \in \mathcal{F}_i$, and $u, v \in V(G_i)$. Since $\mathcal{F}$ is the **FTB** of G, the vertices u and v are resolved in G by some vertex y from $\mathcal{F} \setminus \{x\}$. Assume first that $y \in V(G_i)$. Then having in mind that G_i is an isometric subgraph of G, vertices u and v are also resolved in G_i. Assume second that $y \in V(G_j)$ for some $j \neq i$. Then there exists a unique vertex $a \in \mathrm{At}(G_i)$ such that $d(y,u) = d(y,a) + d(a,u)$ and $d(y,v) = d_G(y,a) + d(a,v)$. Since $d(y,u) \neq d(y,v)$, it follows that $d(a,u) \neq d(a,x)$. This in turn gives $d_{G_i}(u,a) \neq d_{G_i}(x,a)$, hence u and v are also resolved in G_i with a vertex from $\mathcal{F}_i \cup \mathrm{At}(G_i) \setminus \{x\}$. We have thus proved that $\mathcal{F}_i$ is an attaching **FTRS** of G_i and therefore $|\mathcal{F}_i| \geq \mathrm{fdim}^*(G_i)$. This implies

$$\mathrm{fdim}(G) = |\mathcal{F}| \geq \sum_{i=1}^{k} |\mathcal{F}_i| \geq \sum_{i=1}^{k} \mathrm{fdim}^*(G_i)$$

and we are done. □

By Proposition 2, for the graph illustrated in Figure 2, we obtain a lower bound of 4 for the fault-tolerant metric dimension. However, the exact fault-tolerant metric dimension of this graph is 6, showing that the bound is not tight in general. To address this gap, we introduce Condition 1 ($\mathcal{C}_1$). Similarly, for the graph illustrated in Figure 3, we obtain a lower bound of 2 for the fault-tolerant metric dimension, whereas the exact value is 4. This example highlights the necessity that each primary end subgraph must contain at least two fault-tolerant resolvers; therefore, we introduce Condition 2 ($\mathcal{C}_2$).

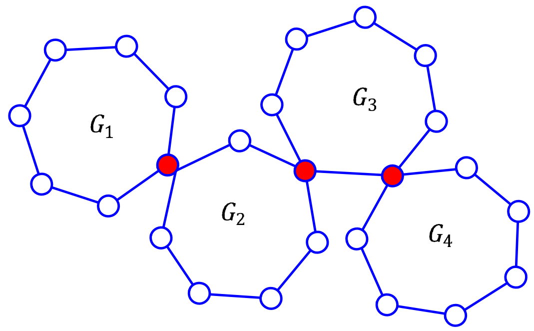

Figure 2: A graph obtained by point attachment of G_i, $i \in [4]$, where $G_i \cong C_8$, $i \in [4]$.

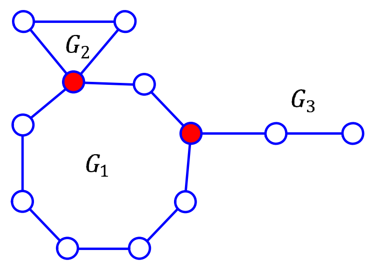

Figure 3: A graph obtained by point attachment of $G_1 \cong C_8$, $G_2 \cong K_3$ and $G_3 \cong P_3$.

Condition 1 ($\mathcal{C}1$)**:** For each $a_1 \in \text{At}(G_i)$ and $v \in V(G_i) \setminus \text{At}(G_i)$, there exists $a_2 \in \text{At}(G_i)$ such that $d_{G_i}(a_1, a_2) \geq d_{G_i}(v, a_2)$.

Condition 2 ($\mathcal{C}2$)**:** $\text{At}(G_i) = \{a\}$ and either $G_i \cong P_n$ and a is a vertex which is not a leaf, or G_i is not a path.

Note that there are large class of connected graphs satifying condition $\mathcal{C}1$. For example, this holds when G_i meets one of the following.

1. $V(G_i) = \text{At}(G_i)$.
2. Any two vertices in $\text{At}(G_i)$ is not adjacent in G_i and $\text{diam}(G_i) = 2$.
3. $\mathcal{E}_{G_i}(u_1) = \mathcal{E}_{G_i}(u_2) = d_{G_i}(u_1, u_2)$ for any pair of distinct vertices $u_1, u_2 \in \text{At}(G_i)$. In particular, this includes all non-trivial complete graphs.
4. G_i is an even graph and if $u \in \text{At}(G_i)$, then the vertex antipodal to u also belongs to $\text{At}(G_i)$.

A graph G has fault-tolerant metric dimension 2 if and only if it is a path graph [4]. Moreover, a set $\{v_1, v_2\}$ forms a fault-tolerant metric basis for a path if and only if both v_1 and v_2 are leaves of the path. Therefore, if a graph G_i satisfies condition $\mathcal{C}2$, then its fault-tolerant metric dimension must be at least 2, that is, $\text{fdim}^*(G_i) \geq 2$.

To demonstrate that the bound of Proposition 2 is tight, we impose restrictions on primary subgraphs using conditions $\mathcal{C}1$ and $\mathcal{C}2$ as follows.

Theorem 3. *If G is formed by the point-attaching process over $\{G_i : i \in [k]\}$, $k \geq 3$, such that each internal subgraph and end-subgraph respectively satisfies $\mathcal{C}1$ and $\mathcal{C}2$, and no two primary end-subgraphs share a vertex, then*

$$\text{fdim}(G) = \sum_{i=1}^{k} \text{fdim}^*(G_i).$$

Proof. By Proposition 2, we have $\text{fdim}(G) \geq \sum_{i=1}^{k} \text{fdim}^*(G_i)$, hence it remains to show that $\text{fdim}(G) \leq \sum_{i=1}^{k} \text{fdim}^*(G_i)$. Let $\mathcal{F}_i$, $i \in [k]$, be an attaching **FTB** of G_i. We will demonstrate that the set $\mathcal{F} = \cup_{i=1}^{k} \mathcal{F}_i$ forms a **FTRS** for G. To do this, we have the following possibilities.

Case 1: $x_1, x_2 \in V(G_i)$, $x_1 \neq x_2$.

Subcase 1.1: $\mathcal{F}_i \neq \emptyset$.

Since $\mathcal{F}_i$ is non-empty and an attaching fault-tolerant basis for G_i, there exists $u_1, u_2 \in \mathcal{F}_i$ such that $d(u_1, x_1) \neq d(u_1, x_2)$ and $d(u_2, x_1) \neq d(u_2, x_2)$. If $u_1, u_2 \in \mathcal{F}_i$, then the proof is clear. If $u_1 \in \text{At}(G_i)$, then $\exists\ G_j \neq G_i$, such that for each $w \in \mathcal{F}_j$, $d(w, u_1) = \min\limits_{v \in V(G_i)} \{d(w, v)\}$. Since, G_j obeys $\mathcal{C}2$, we get $\mathcal{F}_j \geq 2$. Therefore,

$$d(w, x_1) = d(w, u_1) + d(u_1, x_1) \neq d(x_2, u_1) + d(u_1, w) = d(x_2, w).$$

Subcase: 1.2: $\mathcal{F}_i = \emptyset$.

Since $\text{At}(G_i)$ is a resolving set for G_i, for any two vertices of $x_1, x_2 \in V(G_i)$ there exist $a \in \text{At}(G_i)$, such that $d(x_1, a) \neq d(x_2, a)$. This implies that there exists a G_j such that for each $w \in \mathcal{F}_j$ we have $d(w, a) = \min\limits_{v \in V(G_i)} \{d(w, v)\}$. Now, $|\mathcal{F}_j| \geq 2$ due to G_j satisfying $\mathcal{C}2$. Hence, for any $w \in \mathcal{F}_j$,

$$d(x_1, w) = d(x_1, a) + d(a, w_1) \neq d(x_2, a) + d(a, w) = d(x_2, w).$$

Case 2: $x_1 \in V(G_i)$, $x_2 \in V(G_j)$, $G_i \neq G_j$.

Let $a_1 \in \text{At}(G_i)$ and $a_2 \in \text{At}(G_j)$, such that $d(x_1, x_2) = d(x_1, a_1) + d(a_1, a_2) + d(a_2, x_2)$. Note that $a_1 = a_2$, whenever $\text{At}(G_i) \cap \text{At}(G_j) \neq \emptyset$. If $x_2 = a_2 = a_1$ or $x_1 = a_1 = a_2$, then the discussion is similar to Case 1. Hence, in the rest we may assume that $x_2 \neq a_1$ and $x_1 \neq a_2$. **Subcase 2.1**:

$|\,\text{At}(G_i)| \geq 2$ or $|\,\text{At}(G_j)| \geq 2$.

Let $|\operatorname{At}(G_i)| > 1$. As G_i obeys $\mathcal{C}1$, $\exists\ c \in \operatorname{At}(G_i) \setminus \{a_1\}$, such that $d_G(a_1, c) \geq d_G(x_1, c)$. Assume $G_{\ell \neq i}$, with $|\operatorname{At}(G_{\ell \neq i})| = 1$, s.t. for every $t \in \mathcal{F}_\ell$, $d(t, c) = \min_{v \in V(G_i)}\{d(t, v)\}$, ($\mathcal{F}_\ell \geq 2$, as G_ℓ obeys $\mathcal{C}2$). Then for every $t_1, t_2 \in \mathcal{F}_\ell$,

$$\begin{aligned} d(x_1, t_1) &= d(c, x_1) + d(c, t_1) \leq d(c, a_1) + d(c, t_1) \\ &< d(x_2, a_1) + d(c, a_1) + d(c, t_1) = d(x_2, t_1), \quad \text{and} \\ d(x_1, t_2) &= d(c, x_1) + d(c, t_2) \leq d(c, a_1) + d(c, t_2) \\ &< d(x_2, a_1) + d(c, a_1) + d(c, t_2) = d(x_2, t_2). \end{aligned}$$

Subcase 2.2: $|\operatorname{At}(G_i)| = 1 = |\operatorname{At}(G_j)|$.

Since, G_i and G_j obeys $\mathcal{C}2$, $|\mathcal{F}_i| \geq 2$ and $|\mathcal{F}_j| \geq 2$. Hence, let $p_1, p_2 \in \mathcal{F}_i$ and $q_1, q_2 \in \mathcal{F}_j$. If there exist two vertices in $\{p_1, q_1, p_2, q_2\}$ that distinguish x_1 and x_2, then we are done. Suppose that there do not exist at least two vertices in $\{p_1, q_1, p_2, q_2\}$, s.t. they individually distinguishes the vertices x_1 and x_2. We may suppose w.l.o.g. that in this case we have:

$$d(x_1, p_1) = d(x_2, a_2) + d(a_2, a_1) + d(a_1, p_1) = d(x_2, p_1), \tag{1}$$

$$d(x_1, p_2) = d(x_2, a_2) + d(a_2, a_1) + d(a_1, p_2) = d(x_2, p_2), \tag{2}$$

$$d(x_2, q_1) = d(x_1, a_1) + d(a_1, a_2) + d(a_2, q_1) = d(x_1, q_1). \tag{3}$$

Observe that since $\operatorname{At}(G_i) \cap \operatorname{At}(G_j) = \emptyset$, we have $a_1 \neq a_2$. Moreover,

$$d(x_1, p_1) \leq d(p_1, a_1) + d(a_1, x_1), \tag{4}$$

$$d(x_1, p_2) \leq d(p_2, a_1) + d(a_1, x_1), \tag{5}$$

$$d(x_2, q_1) \leq d(q_1, a_2) + d(a_2, x_2). \tag{6}$$

From (1), (2), (4) and (5) we obtain

$$d(a_1, a_2) + d(a_2, x_2) \leq d(x_1, a_1). \tag{7}$$

From (3) & (6)

$$d(x_1, a_1) + d(a_2, a_1) \leq d(x_2, a_2). \tag{8}$$

Adding (7) & (8), we get

$$2 \cdot d(a_1, a_2) \leq 0,$$

which is a contradiction. Hence, $\operatorname{fdim}(G) \leq \sum_{i=1}^{k} \operatorname{fdim}^*(G_i)$. $\square$

Let us now illustrate Theorem 3 with two examples, the first of which is a block graph.

Example 4. *The graph G illustrated in Figure 4 satisfies conditions* $(\mathcal{C}_1)$ *and* $(\mathcal{C}_2)$. *Hence, Theorem 3 can be applied to compute the fault-tolerant metric dimension of G using the attaching metric dimensions of its primary subgraphs as follows:*

$$\begin{aligned}\operatorname{fdim}(G) &= \operatorname{fdim}^*(G_1) + \operatorname{fdim}^*(G_2) + \operatorname{fdim}^*(G_3) + \operatorname{fdim}^*(G_4) + \operatorname{fdim}^*(G_5)\\ &= 2 + 4 + 3 + 2 + 4\\ &= 15.\end{aligned}$$

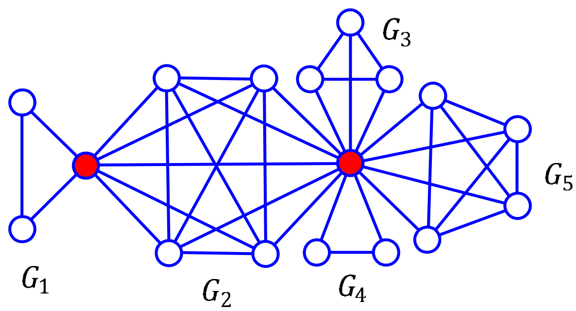

Figure 4: A block graph G constructed by point attaching $G_1 \cong K_3$, $G_2 \cong K_6$, $G_3 \cong K_4$, $G_4 \cong K_3$, and $G_5 \cong K_5$.

Example 5. *The graph G illustrated in Figure 5 satisfies conditions* $(\mathcal{C}_1)$ *and* $(\mathcal{C}_2)$. *Hence, Theorem 3 can be applied to compute the fault-tolerant metric dimension of G using the attaching metric dimensions of its primary subgraphs as follows:*

$$\begin{aligned}\operatorname{fdim}(G) &= \operatorname{fdim}^*(G_1) + \operatorname{fdim}^*(G_2) + \operatorname{fdim}^*(G_3) + \operatorname{fdim}^*(G_4) + \operatorname{fdim}^*(G_5)\\ &= 2 + 4 + 0 + 2 + 2\\ &= 10.\end{aligned}$$

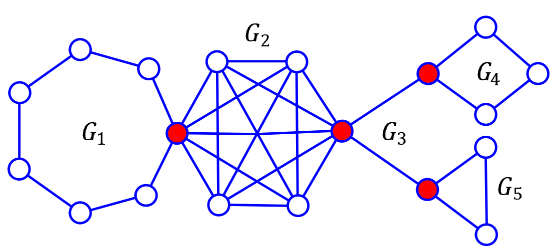

Figure 5: A graph G constructed by point attaching $G_1 \cong C_7$, $G_2 \cong K_6$, $G_3 \cong P_3$, $G_4 \cong C_4$, and $G_5 \cong K_3$.

The following sections focus on deriving several consequences of Theorem 3. Specifically, we present closed-form expressions for the **FTMD** of certain collection of graphs which are derived by point-attaching process.

4 An extremal case

The investigation of the case where each minimal fault-tolerant basis of a primary subgraph is also of minimum cardinality, that is, when $\mathrm{fdim}(G_i) = \mathrm{fdim}^+(G_i)$ for each G_i were done in this section. Let $\mathcal{F}(G_i)$ denote the collection of all **FTB** of G_i and let

$$\theta_i = \begin{cases} \max\limits_{F\in\mathcal{F}(G_i)} \{|F \cap \mathrm{At}(G_i)|\}; & \mathrm{At}(G_i) \text{ is not a resolving set of } G_i, \\ \mathrm{fdim}(G_i); & \text{otherwise.} \end{cases}$$

Corollary 1. *Let G is formed by the point-attaching process over $\{G_i : i \in [k]\}$, $k \geq 3$, which satisfy the conditions of Theorem 3. If also $\mathrm{At}(G_i) \neq V(G_i)$ and $\mathrm{fdim}(G_i) = \mathrm{fdim}^+(G_i)$, $i \in [k]$, then*

$$\mathrm{fdim}(G) = \sum_{i=1}^{k} (\mathrm{fdim}(G_i) - \theta_i).$$

Proof. By the assumption, for any G_i we have $\mathrm{fdim}(G_i) = \mathrm{fdim}^+(G_i)$. Note now that the identity $\mathrm{fdim}^*(G_i) = \mathrm{fdim}(G_i) - \theta_i$ follows by the definition $\theta_i = \max_{F\in\mathcal{F}(G_i)} |F \cap \mathrm{At}(G_i)|$, because for any attaching fault-tolerant resolving set of G_i, the remaining vertices must necessarily be chosen from $V(G_i) \setminus \mathrm{At}(G_i)$, and their number is therefore $\mathrm{fdim}(G_i) - \theta_i$. Hence, the result follows from Theorem 3. □

Consider the graph with five primary subgraphs from Figure 6, where $\mathrm{At}(G_1) = \{a_1\}$, $\mathrm{At}(G_2) = \{a_1, a_2, a_3\}$, $\mathrm{At}(G_3) = \{a_2\}$, $\mathrm{At}(G_4) = \{a_3, a_4\}$, and $\mathrm{At}(G_5) = \{a_4\}$. Using Corollary 1 we get $\mathrm{fdim}(G) = (\mathrm{fdim}(G_1) - 1) + (\mathrm{fdim}(G_2) - 4) + (\mathrm{fdim}(G_3) - 1) + (\mathrm{fdim}(G_4) - 1) + (\mathrm{fdim}(G_5) - 1) = 3 + 0 + 2 + 2 + 4 = 11$.

It is interesting to see that block graph can be derived by point-attaching process on a collection of complete graphs. Recalling that $\mathrm{fdim}(K_n) = n = \mathrm{fdim}^+(K_n)$, we get the remark as a special case of Corollary 1, which is as follows.

Remark. *If G is a block graph formed by the point-attaching process over $\{K_{r_i} : i \in [k]\}$, where $k \geq 3$ and $r_i \geq 3$. If $\mathrm{At}(K_{r_i}) \cap \mathrm{At}(K_{r_j}) = \emptyset$ for any primary end-subgraphs K_{r_i} and K_{r_j}, then,*

$$\mathrm{fdim}(G) = \sum_{\substack{i\in[k] \\ |\mathrm{At}(G_i)|<r_i-1}} (r_i - |\mathrm{At}(K_{r_i})|).$$

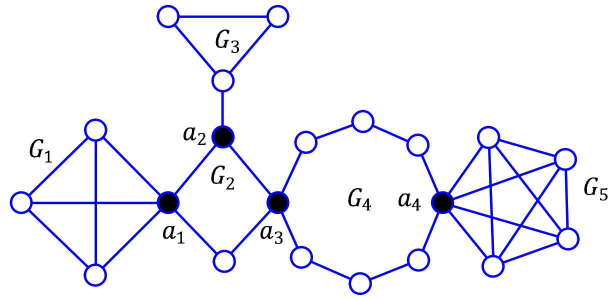

Figure 6: A graph G and its primary subgraphs $G_1 \cong K_4$, $G_2 \cong K_3$, G_3 being the paw graph, $G_4 \cong C_8$, and $G_5 \cong K_5$.

5 Rooted products

In this section, we explore a rooted product of graphs constructed through point-attaching.

A rooted graph is the one with a designated vertex that is uniquely vertex labeled to distinguish it from the others. This distinguished vertex is referred to as the root of the graph. Let G be a vertex labeled graph of order n, and let $\mathcal{H} = \{H_i : i \in [n]\}$ be a collection of rooted graphs. The rooted product graph denoted by $G[\mathcal{H}]$ is formed by identifying the root of each H_i with the i^{th} vertex of G as defined in [44]. It is evident that any rooted product graph $G[\mathcal{H}]$ can be viewed as a graph derived by point-attaching process. Using Theorem 3, we derive the following result.

Corollary 2. *Let G of order $n \geq 2$, and let $\mathcal{H} = \{H_i : i \in [n]\}$ be a collection of rooted graphs, each satisfying $\mathcal{C}2$, with roots $v_1, \dots, v_n$, respectively. Then*

$$\operatorname{fdim}(G[\mathcal{H}]) = \sum_{H_i \in \mathcal{H}_1} \operatorname{fdim}(H_i) + \sum_{H_j \in \mathcal{H}_2} (\operatorname{fdim}(H_j) - 1),$$

*where $H_i \in \mathcal{H}_1$ if v_i is not part of any **FTB** of H_i, and $H_j \in \mathcal{H}_2$ otherwise.*

Next, we examine the case where the family $\mathcal{H}$ consists of graphs that are vertex-transitive. Let $\operatorname{Aut}(H)$ denote the automorphism group of a graph H. For any two vertices $x_1, x_2 \in V(H)$ and any automorphism $f \in \operatorname{Aut}(H)$, the distance between the vertices is preserved. Consequently, if $\mathcal{F}$ is a **FTB** of H and $f \in \operatorname{Aut}(H)$, then the image of the basis under the automorphism, $f(\mathcal{F})$, is also a **FTB** of H. Thus, each vertex of H must belong to some **FTB**. The next remark is the direct consequence of Corollary 1.

Remark. *Let $\mathcal{H} = \{H_i : i \in [n]\}$ be a collection of vertex-transitive graphs of order at least 3, and G of order at least 2, then*

$$\operatorname{fdim}(G[\mathcal{H}]) = \sum_{i=1}^{n} \big(\operatorname{fdim}(H_i) - 1\big).$$

Further, if $\mathcal{H} = \{K_{r_i} : i \in [n]\}$, we have

$$\mathrm{fdim}(G[\mathcal{H}]) = \sum_{i=1}^{n}(r_i - 1),$$

and if $\mathcal{H} = \{C_{r_i} : i \in [n]\}$, then

$$\mathrm{fdim}(G[\mathcal{H}]) = 2 \cdot n.$$

A special case of rooted product graphs arises when the family $\mathcal{H}$ consists of n isomorphic rooted graphs. Let $V(G) = \{g_i : i \in [n]\}$, and let $V(H) = \{h_i : i \in [n']\}$. Declare the vertex $h = h_1$ to be the root of H. The *rooted product* $G \circ_h H$ has the vertex set

$$V(G \circ_h H) = V(G) \times V(H) = \left\{(g_i, h_j) :\ i \in [n], j \in [n']\right\},$$

and the edge set

$$E(G \circ_h H) = \left\{(g_i, h)(g_{i'}, h) :\ g_i g_{i'} \in E(G)\right\} \cup \bigcup_{i=1}^{n} \left\{(g_i, h_j)(g_i, h_{j'}) :\ h_j h_{j'} \in E(H)\right\}.$$

The following result is a special case of Corollary 1.

Proposition 6. *If H is not isomorphic to a path and $v \in V(H)$, then the following hold.*

*(i) If v is not a part of any **FTB** of H, then for any G of order n,*

$$\mathrm{fdim}(G \circ_v H) = n \cdot \mathrm{fdim}(H).$$

*(ii) If v belongs to a **FTB** of H, then for any G of order at least 2,*

$$\mathrm{fdim}(G \circ_v H) = n \cdot (\mathrm{fdim}(H) - 1).$$

Proposition 6 raises the question of identifying the conditions of necessity and sufficiency for $v \in V(H)$ to be part of a **FTB** of H. It is straightforward to verify that a v is in **FTB** of P iff v is one of its leaf vertices. Building on this observation, Proposition 6 (i) yields:

Corollary 3. *Let $v \in V(H)$ is not part of any **FTB** of H, and let G be a graph of order n. Then* $\mathrm{fdim}(G \circ_v H) = 2n$ *iff $H \cong P$ and v is not a leaf.*

In view of Proposition 6 and Corollary 3, the remaining case to be considered is when the second factor of a rooted product graph is a path with the root as a leaf. For this specific case, the following bounds are established.

Proposition 7. *If G is of order $n \geq 2$, and v is a leaf of a non-trivial path P, then*

$$\text{fdim}(G) \leq \text{fdim}(G \circ_v P) \leq n.$$

Proof. G appears as an induced subgraph of $G \circ_v P$. Since any **FTRS** of $G \circ_v P$ must in particular resolve G, the lower bound follows.

To establish the upper bound we claim that $V(G) \times \{v'\}$ is a **FTRS** of $G \circ_v P$, $v' \neq v$ denotes the leaf of P. Let $(x', y'), (x, y) \in V(G \circ_v P)$. If $x = x'$, then for any $u_1, u_2 \in V(G)$,

$$d\big((u_1, v'), (x, y)\big) \neq d\big((u_1, v'), (x, y')\big).$$

And if $x \neq x'$, then

$$d\big((x, v'), (x, y)\big) < d\big((x, v'), (x', y')\big) \text{ and } d\big((x', v'), (x, y)\big) > d\big((x', v'), (x', y')\big).$$

Hence $V(G) \times \{v'\}$ is indeed a **FTRS** for $G \circ_v P$, hence $\text{fdim}(G \circ_v P) \leq n$. $\square$

To see that the upper bound of Proposition 7 is sharp, consider the example from Figure 7.

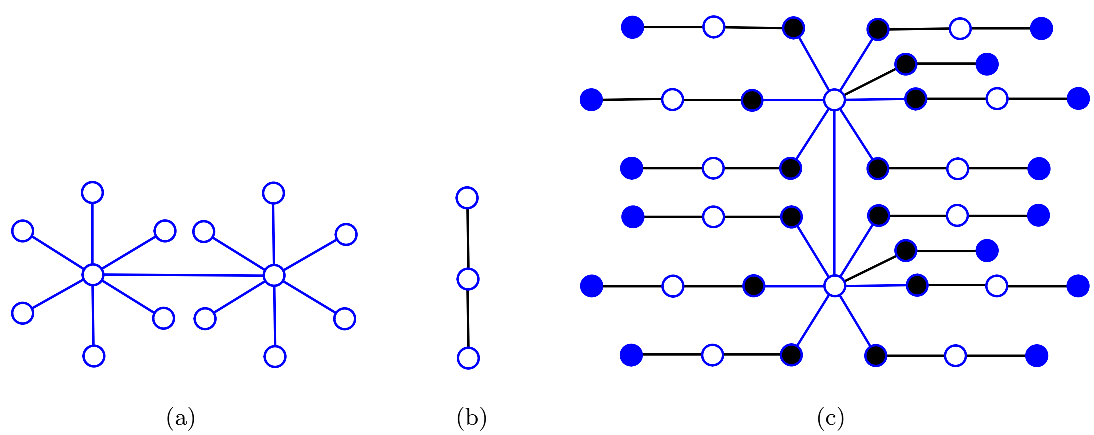

Figure 7: (a) Graph G (b) Graph $H = P_3$ (c) $G \circ_v P_3$, where the blue vertices form a fault-tolerant basis, and the black vertices are the attaching vertices.

6 Conclusion

In this paper, we have introduced an effective methodology for computing the fault-tolerant metric dimension of graphs constructed through point-attaching techniques involving primary subgraphs. By systematically analyzing the role of distance relations and connectivity among the attached substructures, we established explicit formulas for determining **FTMD** of graphs having subgraphs that satisfying both the conditions ($\mathcal{C}_1$) and ($\mathcal{C}_2$). This framework not only simplifies the computation for complex graphs but also enhances our understanding of how the structural properties of the primary subgraphs influence the overall fault tolerance. Explicit formulas for determining the **FTMD** of graphs whose subgraphs do not satisfy conditions ($\mathcal{C}_1$) or ($\mathcal{C}_2$) are under investigations.

Furthermore, we extended our approach to specific graph products, including the rooted product, by expressing their fault-tolerant metric dimensions in terms of the fault-tolerant metric dimensions of their component subgraphs. This generalization highlights the versatility and applicability of our method across various graph constructions.

The results presented contribute significantly to the study of fault-tolerant graph invariants and offer a modular approach for analyzing large-scale networks. Such insights are especially relevant for applications in network design, fault detection, and resilient communication systems, where maintaining unique identifiability despite node failures is essential.

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