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ALMOST-PERIPHERAL GRAPHS

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Abstract. The center $C(G)$ and the periphery $P(G)$ of a connected graph G consist of the vertices of minimum and maximum eccentricity, respectively. Almost-peripheral (AP) graphs are introduced as graphs G with $|P(G)| = |V(G)|-1$ (and $|C(G)| = 1$). AP graph of radius r is called an r-AP graph. Several constructions of AP graph are given, in particular implying that for any $r \geq 1$, any graph can be embedded as an induced subgraph into some r-AP graph. A decomposition of AP-graphs that contain cut-vertices is presented. The r-embedding index $\Phi_r(G)$ of a graph G is introduced as the minimum number of vertices which have to be added to G such that the obtained graph is an r-AP graph. It is proved that $\Phi_2(G) \leq 5$ holds for any non-trivial graphs and that equality holds if and only if G is a complete graph.

1. INTRODUCTION

Graphs considered in this paper are finite, simple, and, unless stated otherwise, also connected. If G is a graph, then the *distance* $d_G(u, v)$ between vertices u and v is the usual shortest-path distance. The *eccentricity* $e_G(u)$ of the vertex u is $\max\{d_G(u, v) : v \in V(G)\}$. The *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ are the minimum eccentricity and the maximum eccentricity, respectively. The *center* $C(G)$ and the *periphery* $P(G)$ consist of the vertices of minimum and maximum eccentricity, respectively. Vertices within $C(G)$ and $P(G)$ are called *central* and *peripheral*, respectively.

The above centrality notions are utmost important in location theory because it is frequently required that a network has the property that the maximum eccentricity of any vertex is as small as possible in order to efficiently locate facilities (at central locations). In the case when $C(G) = V(G)$ holds, the graph G is called *self-centered* or *eccentric*. These graphs were extensively studied by now, see the survey [4] on the early investigations and a selection of more recent papers [5, 8, 12].

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If a graph is not self-centered, then it contains at least two vertices that do not belong to its center. Therefore *almost self-centered graphs* were recently introduced in [9] as the graphs with exactly two non-central vertices. The paper [9] brings constructions of almost self-centered graphs and also investigates embeddings of graphs into smallest almost self-centered graphs. The study of almost self-centered graphs was continued in [2], where in particular almost self-centered graphs are characterized among median graphs and among chordal graphs. For instance, it is proved that a graph is an almost self-centered chordal graph if and only if it is an edge-removed complete graph or belongs to a relatively rich family of graphs that in particular includes joins of a complete graph and a totally disconnected graph, to which two simplicial vertices are added whose neighborhoods are disjoint subcliques of the complete graph.

The other extreme is when almost none of the vertices lies in the center. Such networks could be of interest when it is required that most of the resources do not lie in the center. We hence say that a graph G is *almost-peripheral*, AP for short, if all but one of its vertices lie in the periphery, that is, if $|P(G)| = |V(G)| - 1$ holds. If G is an AP graph and if the eccentricity of the unique vertex from $C(G)$ is r, we will more precisely say that G is an *r-almost-peripheral graph*, or *r-AP graph* for short.

The paper is organized as follows. In the rest of this section some additional definitions are given. Then, in Section 2, we consider general properties of AP-graphs. Several constructions of such graphs are given and it is shown that these graphs are universal in the sense that for any $r \geq 1$, any graph can be embedded as an induced subgraph into some r-AP graph. We also prove that if an AP-graph contains a cut-vertex then it admits a natural decomposition. In Section 3 the r-embedding index $\Phi_r(G)$ of a graph G is introduced as the minimum number of vertices to be added to G such that the obtained graph is an r-AP graph and it is proved that $\Phi_2(G) \leq 5$ holds for any non-trivial graph. Interestingly, the equality holds if and only if G is a complete graph which can be understood as a non-trivial characterization of complete graphs.

If u is a vertex of a graph G, then $N_G(u)$ is the open neighborhood of u and $N_G[u]$ its closed neighborhood. If H is a subgraph of G, then $N_H(u)$ (resp. $N_H[u]$) is $N_G(u) \cap V(H)$ (resp. $N_G[u] \cap V(H)$. A (connected) subgraph H of a (connected) graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$.

2. General Properties

Clearly, a graph G is a 1-AP graph if and only if it has exactly one universal vertex, that is, a vertex of degree $|V(G)| - 1$. Examples of 2-, 3-, and 4-AP graphs are shown in Fig. 1, where in each case the unique central vertex is drawn with a filled circle.

Recall that the d-cube Q_d, $d \geq 1$, is the graph whose vertices are all binary vectors of length d, two vertices being adjacent if they differ in precisely one position. With Q_d^- we denote the graph obtained from Q_d by removing one of its vertices. (Note that by the symmetry of Q_d we get the same graph no matter which vertex is removed.) The first example from Fig. 1 is actually Q_3^- which leads to:

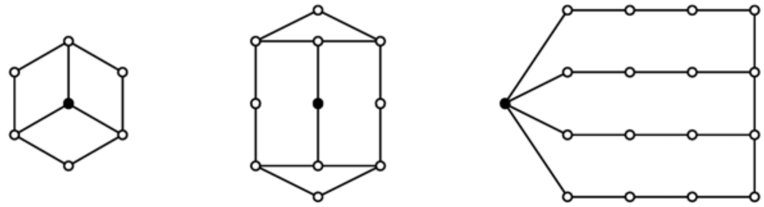

Fig. 1. 2-, 3-, and 4-AP graphs.

Proposition 2.1. *If $d \geq 3$, then Q_d^- is a $(d-1)$-AP graph.*

Proof. We may assume without loss of generality that Q_d^- is obtained from Q_d by removing the vertex $00\ldots0$. It is well-known that Q_d^- embeds isometrically into Q_d (cf. [10, p. 1211]), which implies that the distance between two vertices of Q_d^- is the number of position in which they differ. It then follows easily that the $e_{Q_d^-}(11\ldots1) = d-1$ while for any other vertex u, $e_{Q_d^-}(u) = d$. ∎

The number of vertices in Q_d^- grows exponentially with respect to d. We next show that AP graphs with their order being a linear function of their eccentricity can be constructed. The idea comes from the third example of Fig. 1.

Proposition 2.2. *For any integer $r \geq 2$ there exists an r-AP graph of order $4r+1$.*

Proof. Let G_r be the graph as shown in Fig. 2.

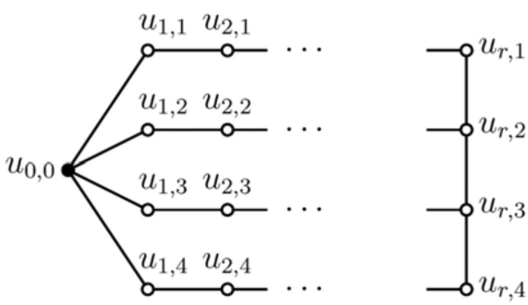

Fig. 2. r-AP graph.

Clearly, $|V(G_r)| = 4r+1$ and $e_{G_r}(u_{00}) = r$. Consider now the cycles

$$C_1 : u_{00}, u_{11}, \ldots, u_{r1}, u_{r2}, u_{r3}, u_{r-1,3}, \ldots, u_{13}, u_{00}$$

and

$$C_2 : u_{00}, u_{12}, \ldots, u_{r2}, u_{r3}, u_{r4}, u_{r-1,4}, \ldots, u_{14}, u_{00}\,.$$

Note that both C_1 and C_2 are of length $2r+2$ and that the distances along C_1 are the same as in G_r except that $d_{G_r}(u_{00}, u_{r2}) = r$ while $d_{C_1}(u_{00}, u_{r2}) = r+1$. Similarly, the only non-isometricity on C_2 is due to $d_{G_r}(u_{00}, u_{r3}) = r$ and $d_{C_2}(u_{00}, u_{r3}) = r+1$. It now readily follows that any vertex of G_r different from u_{00} has eccentricity $r+1$. ■

The 2- and 3-AP graphs from Fig. 1 are smaller than the graphs constructed in Proposition 2.2. We have no such examples for $k \geq 4$ and hence pose the following question: do there exist r-AP graphs of order $n < 4r+1$ for $r \geq 4$?

In order to construct additional AP graphs, we introduce the following operation. If G and H are arbitrary graphs and $u \in V(G)$, then let

$$G \oplus_u H$$

be the graph obtained from the disjoint union of G and H by joining u to every vertex of H.

Theorem 2.3. *If G is an r-AP graph, $r \geq 1$, and u is the center vertex of G, then $G \oplus_u H$ is an r-AP graph for any graph H.*

Proof. Set $X = G \oplus_u H$. Clearly, $e_X(u) = e_G(u) = r$. Let $x \neq u$ be a vertex of $V(X) \cap V(G)$. Since G is r-AP graph and $x \notin C(G)$, there exists $y \in V(X) \cap V(G)$ such that $d_G(x, y) = d_X(x, y) = r+1$. Moreover, if $z \in V(X) \cap V(H)$, then $d_X(x, z) = d_X(x, u) + d_X(u, z) = d_X(x, u) + 1 \leq r+1$. We conclude that $e_X(x) = r+1$. We similarly infer that $e_X(z) = r+1$ holds for any vertex $z \in V(X) \cap V(H)$. ■

Corollary 2.4. *Let $r \geq 1$. Then any graph G can be embedded as an induced subgraph into some r-AP graph.*

Proof. Combine Theorem 2.3 with the existence of r-AP graphs for any $r \geq 1$. ■

For related embeddings of arbitrary graphs into host graphs such that the embeddings have required properties see [1, 3, 7, 11].

Each AP graph constructed in Corollary 2.4 contains a cut-vertex. In fact, AP-graphs that contain a cut-vertex can be described using the $\oplus_u$ operation as follows:

Theorem 2.5. *If G is an AP-graph with a cut-vertex u, then there exists graphs G' and G'', where G'' need not be connected, such that*

$$G = G' \oplus_u G''\,.$$

Proof. Since u is a cut-vertex and G is an AP-graph, we infer that $C(G) = \{u\}$. (This can also be deduced from [6, Corollary 5.4].) Hence $e(u) = r$. Let v be a vertex of G with $d_G(u, v) = r$ and let G' be the block of G containing v. Then, because

u is the unique cut-vertex, we also have $u \in G'$. Let $x \in V(G) - V(G')$. Then $d(v,x) \geq d(v,u) + d(u,x) \geq r+1$. Since $e(v) = r+1$ we infer that $d(u,x) = 1$, that is, x is adjacent to u. Setting G'' to be the subgraph of G induced by the vertices $V(G) - V(G')$ we conclude that $G = G' \oplus_u G''$. ∎

3. Embedding Index

Recall from Corollary 2.4 that if r is an arbitrary positive integer, then a graph G can be embedded as an induced subgraph into some r-AP graph. From optimization point of view it is desirable that the host graph is as small as possible. Hence, if G is a graph and r a positive integer, let

$$\Phi_r(G) = \min\{|V(H)| - |V(G)| : \ H \text{ is } r\text{−AP graph}, G \text{ is induced in } H\}\,.$$

We call $\Phi_r(G)$ the *r-embedding index of G.*

Clearly, $\Phi_r(G) = 0$ if and only if G is an r-AP graph. Note also that for any graph G, $\Phi_1(G) \leq 1$. Indeed, if G does not contain a (unique) universal vertex (equivalently $\Phi_1(G) > 0$), then let H be the graph obtained from G by adding a new vertex and joining it to all vertices of G. Then H is a 1-AP graph.

If $r \geq 2$, then combining Proposition 2.2 with Theorem 2.3 we get $\Phi_r \leq 4r+1$. For $r = 2$ we can improve this bound as follows:

Theorem 3.1. *If G is an arbitrary graph on at least two vertices, then $\Phi_2(G) \leq 5$. Moreover, equality holds if and only if G is a complete graph.*

Proof. Suppose first that G is not complete. Then G contain an induced path on three vertices $v_1v_2v_3$. Define the graph H as follows. Let $V(H) = V(G) \cup \{u_1,u_2,u_3,u_4\}$ and $E(H) = E(G) \cup \{u_1u_2, u_2u_3, u_3u_4, u_1u_4, v_1u_1, v_2u_4, v_3u_3\} \cup \{xu_4 : \ x \in V(G), x \neq v_1, v_2, v_3\}$, see Fig. 3.

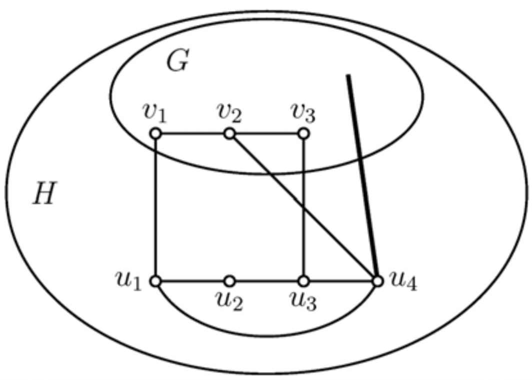

Fig. 3. Construction from the proof.

It is straightforward to verify that $e_H(u_4) = 2$ and that $e_H(x) = 3$ for any vertex $x \neq u_4$. While verifying the distances note that it is essential that $v_1v_3 \notin E(G)$, for otherwise the eccentricity of u_3 in H would be 2. We therefore conclude that $\Phi_2(G) \leq 4$ holds for any non-complete graph G.

We are left with the problem to determine $\Phi_2(K_n)$ for $n \geq 2$. That $\Phi_2(K_2) \leq 5$ holds, follows from the left graph of Fig. 1. Consider next K_n, $n \geq 3$. Let x and y be arbitrary vertices of K_n and construct a graph H with the vertex set $V(H) = V(G) \cup \{u_i : 1 \leq i \leq 5\}$ and the edge set as shown in Fig. 4.

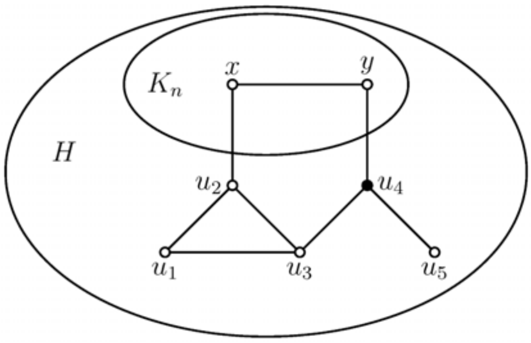

Fig. 4. K_n, $n \geq 3$, embedded into H.

Since $n \geq 3$, K_n contains at least one vertex different from x and y. It is then straightforward to check that $e_H(u_4) = 2$ and $e_H(w) = 3$ for any vertex $w \neq u_4$. Hence $\Phi_2(K_n) \leq 5$ holds for any $n \geq 3$.

Clearly, $\Phi_2(K_n) > 0$. Likewise, $\Phi_2(K_n)$ cannot be 1, because any (connected) graph of order $n + 1$ which contains K_n has diameter at most 2. Suppose next that $\Phi_2(K_n) = 2$ and let H be a 2-AP graph that contains K_n. Let u and v be vertices of $V(H) - V(K_n)$. If both u and v have neighbors in K_n, then $e_H(x) \leq 2$ holds for any vertex x of K_n, but this is clearly not possible because H is a 2-AP graph and hence contains a unique central vertex x, that is, a unique vertex x with $e_H(x) = 2$. Therefore assume without loss of generality that u has no neighbor in K_n, so that then u is necessarily adjacent to v which has in turn some neighbors in K_n. But now $e_H(v) \leq 2$ as well as $e_H(x) = 2$ for any neighbor $x \neq u$ of v which is the same contradiction as above.

We have shown by now that $\Phi_2(K_n) \geq 3$. Assume that $\Phi_2(K_n) = 3$ and let H be a 2-AP graph that contains K_n, where $V(H) - V(K_n) = \{u, v, w\}$. Again, if each of u, v, and w has at least one neighbor in K_n, the eccentricity of any vertex of K_n is at most 2, a contradiction. Assume therefore that u has no neighbor in K_n. If $d(u, K_n) = 3$, let $u - v - w - x$ be a path from u to a vertex $x \in K_n$. Since $d(u, K_n) = 3$, v has no neighbor in K_n. Hence w is adjacent to all vertices of K_n, for otherwise $e_H(u) = 4$. But then $e_H(v) = e_H(w) = 2$. As this is not possible we conclude that $d(u, K_n) = 2$.

Let $u - v - x$ be a path of length 2, where $x \in V(K_n)$. Consider now the vertex w. If w is adjacent to v, then $v, x \in C(H)$, a contradiction. If w is adjacent to a neighbor x' of v in K_n (where it is possible that $x' = x$), then $v, x' \in C(H)$. Therefore $N_{K_n}(w) \cap N_{K_n}(v) = \emptyset$. If also $wu \notin E(H)$, that is, if $N_H(w) \cap N_H(v) = \emptyset$, then $d_H(u, w) = 4$, a contradiction. So w must necessarily be adjacent to u. Then $N_{K_n}(w) \neq \emptyset$, for otherwise $d_H(w, K_n) = 3$, which is a possibility we have already ruled out. Hence let $x' \in N_{K_n}(w)$. Then $e_H(x') = 2$ and because also $e_H(x) = 2$ and $x \neq x'$ we have the final contradiction for the assumption $\Phi_2(K_n) = 3$. We conclude that $\Phi_2(K_n) > 3$.

The last part of the proof is to exclude the possibility $\Phi_2(K_n) = 4$. For this sake assume on the contrary that H is a 2-AP graph that contains K_n, where $Y = \{u, v, w, z\}$ is the set of vertices of H not in K_n. As above we first infer that not every vertex from Y can have a neighbor in K_n. Clearly, a vertex from Y is at distance at most 3 from K_n. Suppose $d_H(u, K_n) = 3$ and let $u - v - w - x$ be a shortest path with $x \in K_n$. Then $N_{K_n}(w) = V(K_n)$. But then in any of the possibilities for the adjacencies of z we have that $e_H(v) = 2 = e_H(w)$, a contradiction. It follows that $d_H(y, K_n) \leq 2$ for any $y \in Y$ and hence there is at least one vertex from Y, say u, with $d_H(u, K_n) = 2$. Let $u - v - x$ be an induced path with $x \in V(K_n)$. We now distinguish the following two cases.

Case 1. $d_H(w, K_n) = 1 = d_H(z, K_n)$.
Note first that x is the unique neighbor of v in K_n, since any other of its neighbors in K_n would also be in the center of H. Suppose next that $wx \in E(H)$. Then $zx \notin E(H)$ for otherwise $x, v \in C(H)$. Now, the only possibility that $d_H(z, u) = 4$ does not happen is that there exists a z, u-shortest path that passes v or w. But in the first case $v \in C(H)$ and in the other case $w \in C(H)$ none of which is possible. We have thus proved that $wx \notin E(H)$. Analogously, $zx \notin E(H)$. Since $d_H(w, u) \leq 3$ and $d_H(z, u) \leq 3$, it follows that no w, u-shortest path uses a vertex of K_n, and also no shortest z, u-path uses such a vertex. Now we have the following cases. If $zw, wv \in E(H)$, then $w \in C(H)$. Similarly, if $zw, wu \in E(H)$, then $w \in C(H)$. And if $zw, zu \in E(H)$, then $z \in C(H)$. Hence we conclude that $zw \notin E(H)$. If $zu, wu \in E(H)$, then $w, z \in C(H)$. If $zv, wv \in E(H)$, then $v \in C(H)$. Hence it must be that $wv, zu \in E(H)$ (or vice versa). But then $v \in C(H)$, the final contradiction.

Case 2. $d_H(w, K_n) = 1$, $d_H(z, K_n) = 2$.
Note that in this case z must be adjacent to at least one of the vertices v and w. This observation now leads to several subcases that can be considered analogously as the analysis was done in Case 1. Not to repeat tedious analysis we leave out the details. In any case, however, a contradiction is reached. ■

We conclude the paper with three additional examples of 2-AP graphs presented in Fig. 5. They respectively contain K_3, K_4, and K_5 and are of orders 8, 9, and 10. Hence these examples are optimal with respect to the 2-embedding index of the corresponding complete graphs. Note that none of these embeddings is the one from the proof of Theorem 3.1 (in which a pendant vertex is present, see Fig. 4). Hence these examples show that a minimum embedding is not unique.

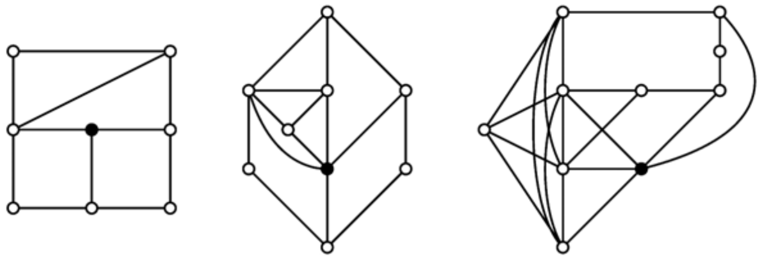

Fig. 5. Three 2-AP graphs.

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