SIAM J. DISCRETE MATH.
Vol. 15, No. 2, pp. 235–251

PARTIAL CUBES AND CROSSING GRAPHS*

SANDI KLAVŽAR† AND HENRY MARTYN MULDER‡

Abstract. Partial cubes are defined as isometric subgraphs of hypercubes. For a partial cube G, its crossing graph $G^{\#}$ is introduced as the graph whose vertices are the equivalence classes of the Djoković–Winkler relation Θ, two vertices being adjacent if they cross on a common cycle. It is shown that every graph is the crossing graph of some median graph and that a partial cube G is 2-connected if and only if $G^{\#}$ is connected. A partial cube G has a triangle-free crossing graph if and only if G is a cube-free median graph. This result is used to characterize the partial cubes having a tree or a forest as its crossing graph. An expansion theorem is given for the partial cubes with complete crossing graphs. Cartesian products are also considered. In particular, it is proved that $G^{\#}$ is a complete bipartite graph if and only if G is the Cartesian product of two trees.

Key words. isometric subgraph, hypercube, partial cube, crossing graph, median graph, triangle-free graph, Cartesian product of graphs

AMS subject classifications. 05C75, 05C12

PII. S0895480101383202

1. Introduction.

A *partial cube* is a connected graph that admits an isometric embedding into a hypercube. Partial cubes have first been investigated in the 1970s by Graham and Pollak [13], who used them as a model for a communication network. By now, the structure of partial cubes is relatively well understood. Djoković [10] characterized these graphs via convexity of certain vertex partitions. He also introduced the relation Θ on the edge set of a graph. This relation was later used by Winkler [28] to characterize the partial cubes as those bipartite graphs for which Θ is transitive. Chepoi [8] followed with an expansion theorem for partial cubes. Another characterization of partial cubes was obtained by Avis [4]; cf. also [27]. Partial cubes have found several applications. See, for instance, [11] for connections with oriented matroids and [9, 21] for recent applications to chemical graph theory. An important subclass of the class of partial cubes is that of the median graphs (see [24, 25]); cf. [22]. Among the median graphs the cube-free median graphs stand out; see, for instance, [5, 20, 23].

The fastest known recognition algorithm for partial cubes is of complexity $O(mn)$, where n and m are the number of vertices and edges of a given graph. Since for partial cubes $m \leq (n \log n)/2$ (cf. [2, 3, 12, 19]), this complexity reduces to $O(n^2 \log n)$. The first such algorithm is due to Aurenhammer and Hagauer [2, 3]. Another more general algorithm for recognizing partial Hamming graphs (isometric subgraphs of Cartesian products of complete graphs) of complexity $O(mn)$ is given in [1]. Applying the canonical isometric embedding theory of Graham and Winkler [14], a simple algorithm for recognizing partial Hamming graphs of the same complexity can be obtained; see [17, 19]. However, only a trivial lower bound $O(m)$ for recognizing partial cubes

*Received by the editors January 5, 2001; accepted for publication (in revised form) February 8, 2002; published electronically April 3, 2002.
http://www.siam.org/journals/sidma/15-2/38320.html

†Department of Mathematics, PEF, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia (sandi.klavzar@uni-lj.si). This author's research was supported by the Ministry of Education, Science, and Sport of Slovenia under grant 0101-P-504.

‡Econometrisch Instituut, Erasmus Universiteit, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands (hmmulder@few.eur.nl). This author's research was supported by the Ministry of Education, Science, and Sport of Slovenia and the University of Maribor, Slovenia.

is known. This contrasts with the recognition problem for median graphs, where the connection between median graphs and triangle-free graphs [20] provides strong evidence that the fastest known recognition algorithms for median graphs [15, 19] are close to being optimal.

In this paper we are interested in the structure of the Θ-classes of a partial cube G. An important feature is whether two Θ-classes cross or not. We say that two Θ-classes F_1 and F_2 cross in G if edges of F_2 occur in both the components of $G - F_1$. The *crossing graph* $G^\#$ of a partial cube G has the Θ-classes of G as its vertices, where two vertices of $G^\#$ are joined by an edge whenever they cross as Θ-classes in G.

In the next section we recall concepts needed later and collect basic properties of the relation Θ. Our results are presented in sections 3–6. We start with a theorem asserting that every connected graph is the crossing graph of some partial cube, even the crossing graph of some median graph. Thus at first sight the notion of a crossing graph may not seem very interesting. However, appearances are deceptive. There is a nontrivial relationship between the structure of a partial cube and that of its crossing graph. For instance, we prove the following results for partial cubes G: "G is 2-connected if and only if its crossing graph is connected," "the crossing graph of G is triangle-free if and only if G is a cube-free median graph," "the crossing graph of G is a tree if and only if G is a 2-connected cube-free median graph with some forbidden subgraphs," and "the crossing graph of G is a complete bipartite graph if and only if G is the Cartesian product of two trees." Along the way some other types of graphs, such as C_4-trees and C_4-cactoids, are considered. Moreover, we characterize the partial cubes with a complete graph as crossing graph. We conclude this paper with a number of open problems.

2. Preliminaries. For $u, v \in V(G)$, let $d_G(u, v)$ denote the length of a shortest path (also called *geodesic*) in G from u to v. A subgraph H of a graph G is an *isometric* subgraph if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. The *interval* $I(u, v)$ between two vertices u and v in G is the set of all vertices on shortest paths between u and v. A subgraph H of G is *convex* if we have $I(u, v) \subseteq V(H)$ for any u, $v \in V(H)$.

The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever either $ab \in E(G)$ and $x = y$ or $a = b$ and $xy \in E(H)$. The n-cube Q_n is the Cartesian product of n copies of the complete graph on two vertices K_2.

For a graph $G = (V, E)$ and $X \subseteq V$, let $\langle X \rangle$ denote the subgraph induced by X. For two vertices u and v on a path P, we denote the subpath of P between u and v by $u \to \cdots P \cdots \to v$.

The Djoković–Winkler relation Θ [10, 28] is defined on the edge set of a graph in the following way. Edges $e = xy$ and $f = uv$ of a graph G are in relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)\,.$$

Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if

$$d(x, u) = d(y, v) \text{ and } d(x, v) = d(y, u)\,.$$

Among bipartite graphs, Θ is transitive precisely for partial cubes (i.e., isometric subgraphs of hypercubes), as has been proved by Winkler in [28].

Let $G = (V, E)$ be a connected, bipartite graph. For any edge ab of G we write

$W_{ab} = \{w \in V \mid d_G(a, w) < d_G(b, w)\}$,
$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}$,
$F_{ab} = \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}$,
$G_{ab} = \langle W_{ab} \rangle$.

Note that if G is bipartite, then we have $V = W_{ab} \cup W_{ba}$. For a bipartite graph G, the sets F_{ab} are called *colors* and the subgraphs G_{ab}, G_{ba} form the *split* of the color F_{ab}. The subgraph $\langle U_{ab} \rangle$ is the *side* of color F_{ab} in G_{ab}, and $\langle U_{ba} \rangle$ is the *opposide* of $\langle U_{ab} \rangle$. Djoković [10] characterized the partial cubes as the connected bipartite graphs in which all subgraphs G_{ab} are convex.

We now state three well-known facts about the relation Θ; cf. [18].

LEMMA 2.1. *Let G be a connected, bipartite graph, and let ab be any edge of G. Then F_{ab} is the set of all edges in relation Θ with ab.*

Note that for partial cubes Lemma 2.1 asserts that Θ-classes coincide with the sets F_{ab}, a fact that will be used implicitly in what follows.

We say that a color *occurs* in a subgraph H if there is an edge of that color in H.

LEMMA 2.2. *Let C be an isometric cycle of a partial cube G, and let F_{ab} be a color which occurs in C. Then F_{ab} occurs in C exactly twice (in two antipodal edges).*

LEMMA 2.3. *Suppose P is a path connecting the endpoints of an edge e. Then P contains an edge f with $e\Theta f$.*

The conclusion of Lemma 2.3 holds also if P is a walk, as every walk containing the endpoints of e contains a path between the endpoints of e; cf. Lemma 2.4 of [19].

For our purposes it is convenient to have statements available that are slightly stronger than the above lemmas. These may also be part of folklore, but to make the paper self-contained we provide them with proofs.

LEMMA 2.4. *Let G be a partial cube. Then a path P in G is a geodesic if and only if no color occurs twice on P.*

Proof. If P is a geodesic, then, by the definition of Θ, all colors on P must be distinct.

Conversely, let $P = v_0 \to v_1 \to \cdots \to v_n$ be a path on which all colors are distinct. Assume that P is not a geodesic with n as small as possible. Note that $n \geq 3$. Then we have $d(v_0, v_{n-1}) = d(v_1, v_n) = n - 1$, $d(v_0, v_n) = n - 2$. Let Q be a v_0, v_n-geodesic. By minimality, P and Q are internally disjoint. By Lemma 2.3, the edge v_0v_1 is in relation Θ with an edge of the cycle composed of P and Q. Moreover, $v_1 \to v_0 \to \cdots Q \cdots \to v_n$ is a path of length $n-1$ and thus a geodesic; hence v_0v_1 is not in relation Θ with any edge of Q. So P contains two edges of the same color, a contradiction. ∎

Let C be an even cycle of length $2k$. We call two edges on C *antipodal* if their endpoints are joined by two paths of length $k-1$ on C.

LEMMA 2.5. *Let G be a partial cube. Then C is an isometric cycle in G if and only if every color on C occurs only on antipodal edges.*

Proof. Let C be an isometric cycle of length $2k$ in G. Then every path of length at most k on C is a geodesic in G, so that, by Lemma 2.4, all colors on such paths occur exactly once. Since each color on C occurs at least twice, it follows that each color on C occurs precisely on antipodal edges.

Conversely, let C be a cycle of length $2k$ in G such that each color on C occurs precisely on antipodal edges. Then, by Lemma 2.4, each path of length at most k must be a geodesic in G. Hence C is isometric in G. ∎

LEMMA 2.6. *Let G be a partial cube, and let F_{ab} be a color of G. If uv, xy are edges of F_{ab}, with u, x on one side and v, y on the opposide, and if P is any u, x-*

geodesic and Q is any v,y-geodesic, then P and Q contain the same colors, and each color occurs at most once on P and at most once on Q.

Proof. By definition of a color, a geodesic contains every color at most once. Take any color on P, say, edge e is of that color. Then $u \to \cdots P \cdots \to x \to y \to \cdots Q \cdots \to v \to u$ is a cycle through e, whence constitutes a path between the ends of e. Hence, by Lemma 2.3, it contains an edge f of the same color. Since f cannot be on P, it is on Q. Conversely, every color on Q occurs on P. ∎

A *median graph* is, by definition, a connected graph such that, for every triple of its vertices, there is a unique vertex lying on a geodesic (i.e., shortest path) between each pair of the triple. It follows immediately from the definition that median graphs are bipartite. By now, the class of median graphs has been well investigated and a rich structure theory is available; see the recent survey [22]. Median graphs are partial cubes (see [24, 25]), whence relation Θ is transitive on median graphs. They may be characterized as the connected bipartite graphs in which all subgraphs $\langle U_{ab} \rangle$ are convex; cf. [6]. Another relevant feature of median graphs is that any isometric cycle of length $2n$ is contained in an induced Q_n. (This again can be deduced directly from the definition.)

3. Crossing graphs. Let G be a partial cube. We say that two colors *cross* if their splits G_1, G_2 and H_1, H_2 satisfy $G_i \cap H_j \neq \emptyset$ for $1 \leq i, j \leq 2$; see [23]. The *crossing graph* $G^{\#}$ of a partial cube G has the colors of G as its vertices, and two vertices are adjacent if they cross as colors.

At first sight it is not clear which graphs are crossing graphs. However, using the following concept, the answer is clear. For a graph G, the *simplex graph* $S(G)$ of G is the graph whose vertices are the complete subgraphs of G (including the empty graph), two vertices being adjacent if, as complete subgraphs of G, they differ in exactly one vertex; see [7]. It is easily seen that a simplex graph is a median graph, and hence a partial cube, by checking that it satisfies the definition of a median graph.

THEOREM 3.1. *Every graph is a crossing graph of some median graph. More precisely, for any graph G we have $G = S(G)^{\#}$.*

Proof. Let $V(G) = \{1, 2, \ldots, n\}$. Since vertex $\emptyset$ of $S(G)$ is of degree n, we infer that $S(G)^{\#}$ has at least n vertices. Let uv be an arbitrary edge of $S(G)$. Without loss of generality we may assume that $u = \{1, 2, \ldots, k\}$ and $v = \{1, 2, \ldots, k+1\}$. It is now straightforward to check that the edge $(\emptyset, \{k+1\})$ is in relation Θ with uv. It follows that $S(G)^{\#}$ has exactly n vertices and that vertex i of G corresponds to the color of edge $(\emptyset, \{i\})$ in $S(G)$. Assume that vertices 1 and 2 are adjacent in G. Then $\emptyset$, $\{1\}$, $\{2\}$, and $\{1, 2\}$ induce C_4 in $S(G)$, and so the corresponding colors cross. Finally, if 1 is not adjacent to 2, then they are not in the same complete subgraph of G, which implies that the corresponding colors do not cross in $S(G)$. ∎

There is another (simplified) construction showing that every graph is the crossing graph of some partial cube. For a graph G, let $\widetilde{G}$ be the graph obtained from G by subdividing all edges of G and adding a new vertex z joined to all the original vertices of G; see [20]. Then we can argue similarly as above that for any graph G we have $G = \widetilde{G}^{\#}$.

To prove relations between properties of a partial cube and properties of its crossing graph, we need some simple criteria for colors to determine whether they cross.

LEMMA 3.2. *Let G be a partial cube. Then any cycle of G contains two crossing colors.*

Proof. Let C be any cycle of G. Note that each color occurs an even number of times on C. Choose two edges uv and xy of the same color F on C such that on the subpath $P = u \to v \to \cdots \to x \to y$ of C every color occurs at most once between u and x. Since uv and xy cannot be adjacent, there is at least one other color on P. By Lemma 2.3, color F crosses with each color on the subpath $v \to \cdots \to x$ of P. □

We say that two colors *alternate on a cycle* C if they both occur in C and we encounter them alternately while walking along C. Note that Lemma 2.5 in particular implies that any two colors on an isometric cycle of a partial cube alternate.

LEMMA 3.3. *Let G be a partial cube G, and let F_{ab} and F_{uv} be two different colors of G. Then the following statements are equivalent:*

(i) *F_{ab} and F_{uv} cross.*
(ii) *F_{ab} and F_{uv} alternate on an isometric cycle of G.*
(iii) *F_{ab} and F_{uv} occur on an isometric cycle of G.*
(iv) *Each of the colors F_{ab} and F_{uv} appear exactly twice on a cycle of G and they alternate.*

Proof. (i) ⇒ (ii) Suppose that the colors F_{ab} and F_{uv} cross. We may assume that uv lies in G_{ab}. As the colors cross, there is an edge $u'v'$ in G_{ba} of color F_{uv} with u and u' on the one side of F_{uv} and v and v' on the opposide, that is, $d_G(u,u') = d_G(v,v') = d_G(u,v') - 1$. We choose the edges uv and $u'v'$ such that $d_G(u,u')$ is as small as possible. Let $l(S)$ denote the length of the walk S in G.

Let P be a shortest u,u'-path, and let Q be a shortest v,v'-path, so that P lies in G_{uv} and Q lies in G_{vu}. The paths P and Q are disjoint and each of them contains exactly one edge of F_{ab}.

We claim that $C = u \to \cdots P \cdots \to u' \to v' \to \cdots Q \cdots \to v \to u$ is an isometric cycle. Assume the contrary, and let x and y be vertices of C such that $d_G(x,y) < d_C(x,y)$. Let R be a shortest x,y-path. We may select x and y such that R is internally disjoint from C and that x lies on P and y on Q. Then $C' = u \to \cdots P \cdots \to x \to \cdots R \cdots \to y \to \cdots Q \cdots \to v \to u$ is a cycle of length $l(C') < l(C)$. By Lemma 2.3, there is an edge $x'y'$ of color F_{uv} on C' with u, x', and u' on the one side and v, y', and v' on the opposide of F_{uv}. Write $P' = u \to \cdots P \cdots \to x \to \cdots R \cdots \to x'$ and $Q' = v \to \cdots Q \cdots \to y \to \cdots R \cdots \to y'$. Then we have

$$2d_G(u,u') = d_G(u,u') + d_G(v,v') = l(C) - 2$$

$$> l(C') - 2 = l(P') + l(Q') \geq d_G(u,x') + d_G(v,y') = 2d_G(u,x').$$

Hence we have $d_G(u,x') < d_G(u,u')$, which contradicts the minimality of $d_G(u,u')$. Thus we conclude that C is an isometric cycle. By Lemma 2.5 each of the two colors appears exactly twice on C and the colors alternate on C.

(ii) ⇒ (iii) This implication is trivial.

(iii) ⇒ (iv) This follows from Lemma 2.5.

(iv) ⇒ (i) Let C be a cycle of G on which the colors F_{ab} and F_{uv} appear exactly twice. Let $a'b'$ and $u'v'$ be the second edges of colors F_{ab} and F_{uv}, respectively, and let $d_G(a,a') = d_G(b,b') = d_G(a,b') - 1$. Then the cycle C can be written as $C = a \to \cdots P \cdots \to a' \to b' \to \cdots \to Q \cdots \to b \to a$, where P and Q are the corresponding paths on C connecting a with a' and b' with b. Since ab and $a'b'$ are the only edges from F_{ab} on C, we observe that P lies in G_{ab} and Q lies in G_{ba}. Moreover, as the colors alternate on C, we may assume that uv lies on P and $u'v'$ lies on Q. Without loss of generality we may assume that $a \in W_{uv}$ and $b \in W_{u'v'}$. Then we have $a \in W_{ab} \cap W_{uv}$, $a' \in W_{ab} \cap W_{vu}$, $b \in W_{ba} \cap W_{uv}$ and $b' \in W_{ba} \cap W_{vu}$. Thus the colors F_{ab} and F_{uv} cross. □

Our first result that relates properties of the crossing graph $G^{\#}$ to properties of the partial cube G involves connectivity.

THEOREM 3.4. *Let $G = (V, E)$ be a partial cube. Then G is 2-connected if and only if $G^{\#}$ is connected.*

Proof. First assume that G is not 2-connected, and let x be a cutvertex in G. Let A be a subgraph of G induced by x and one component of $G - x$, and let B be the subgraph of G induced by x and the remaining part of $G - x$. Then A and B both contain edges, and we have $A \cup B = G$ and $A \cap B = \{x\}$, so that no color in A crosses with a color in B. Hence, in $G^{\#}$, there is no path between any color in A and any color in B; that is, $G^{\#}$ is disconnected.

Conversely, let G be 2-connected and take any two incident edges uv and vw of G, with, say, uv colored red and vw colored blue. Since G is 2-connected, there exists a path between u and w in G not containing v. Let P be such a u, w-path of minimal length k. Since $P \to v \to u$ is a cycle, red and blue must occur on P. Now we walk along P from u to w. Let xy be the first red or blue edge on P, where we traverse xy from x to y.

Suppose the color of xy is blue, that is, the color of vw. Then x and v are on the same side of color blue and y and w are on the opposite. Let Q be a geodesic between x and v, and let Q' be a geodesic between y and w. Then by Lemma 2.3 red occurs on Q and, since Q is a geodesic, red occurs only once on Q.

By Lemma 2.6 red occurs exactly once also on Q'; so red and blue occur exactly twice alternately on the cycle

$$v \to \cdots Q \cdots \to x \to y \to \cdots Q' \cdots \to w \to v\,.$$

So, by Lemma 3.3 (iv), red and blue cross in G and are adjacent in $G^{\#}$.

Suppose the color of xy is red, that is, the color of uv. Now u and x are on one side of red, and v and y are on the opposite. Let P_1 be a geodesic between v and y, and let u_1 be the neighbor of v on P_1. Note that the u, x-subpath of P is a geodesic by minimality of P. Thus, using Lemma 2.3, red and the color of vu_1 cross in G, and so are adjacent in $G^{\#}$. Now,

$$u_1 \to \cdots P_1 \cdots \to y \to \cdots P \cdots \to w$$

is a walk between u_1 and w not containing v of length $k - 2$. Hence there is a path between u_1 and w not containing v of length at most $k - 2$.

Repeating the above argument, we find neighbors $u_2, \ldots, u_p$ of v such that the colors of vu_i and vu_{i+1} cross, for $i = 1, \ldots, p-1$, and also the color of vu_p and blue cross. Thus we have constructed a path in $G^{\#}$ between red and blue. Connectivity of $G^{\#}$ now follows from the connectivity of G. $\square$

Note that $P_n \Box P_n$ is 2-connected but not 3-connected, since it contains a vertex of degree 2. On the other hand, $(P_n \Box P_n)^{\#} = K_{n,n}$ is n-connected. So there does not exist an analogue of Theorem 3.4 for higher connectivities.

4. Complete crossing graphs. In this section we consider the partial cubes that have complete crossing graphs. In [23] it was proved that in a median graph G there are n pairwise crossing colors if and only if G contains an induced n-cube. We restate this result in the next proposition and its corollary.

PROPOSITION 4.1. *A median graph is a hypercube if and only if its crossing graph is complete. More precisely, if G is a median graph, then $G = Q_n$, $n \geq 1$ if and only if $G^{\#} = K_n$.*

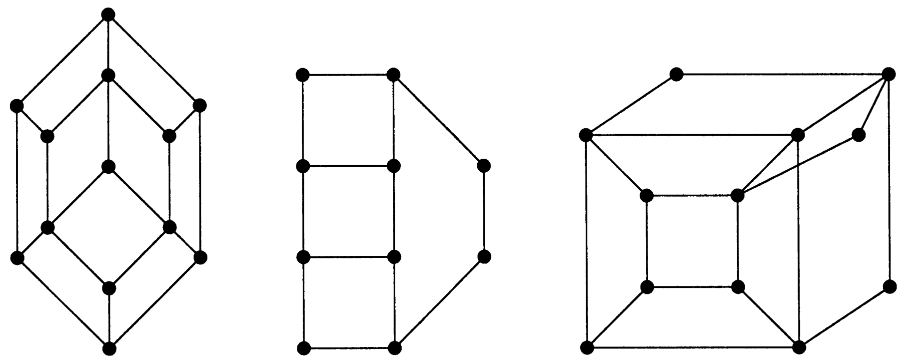

FIG. 4.1. *Partial cubes with complete crossing graphs.*

Recall that the clique number of a graph is the size of a largest complete subgraph in the graph.

COROLLARY 4.2. *Let G be a median graph. Then the clique number of $G^{\#}$ is equal to the dimension of the largest hypercube in G.*

A simple consequence of the above cited result is the following corollary; see [23].

COROLLARY 4.3. *Let G be a partial cube. Then G is a tree if and only if $G^{\#}$ is the complement of a complete graph.*

For partial cubes the variety of graphs with complete crossing graphs is much richer than for median graphs. Besides hypercubes one finds even cycles, Q_3 minus a vertex, and the graphs from Figure 4.1. Moreover, the Cartesian product preserves this property (cf. Proposition 6.1).

In order to characterize the partial cubes with complete crossing graphs, we recall the concept of expansion; see [24, 25, 26] or [19].

Let G' be a connected graph. A *proper cover* G'_1, G'_2 consists of two induced subgraphs G'_1, G'_2 of G' such that $G' = G'_1 \cup G'_2$ and $G'_0 = G'_1 \cap G'_2$ is a nonempty subgraph, called the *intersection* of the cover. The cover is *isometric* (resp., *convex*) if it consists of isometric (resp., convex) subgraphs.

Let G' be a connected graph, and let G'_1, G'_2 be a proper cover of G' with $G'_0 = G'_1 \cap G'_2$. The *expansion* of G' with respect to G'_1, G'_2 is the graph G constructed as follows. Let G_i be an isomorphic copy of G'_i, for $i = 1, 2$, and, for any vertex u' in G'_0, let u_i be the corresponding vertex in G_i for $i = 1, 2$. Then G is obtained from the disjoint union $G_1 \cup G_2$, where for each u' in G'_0 the vertices u_1 and u_2 are joined by an edge. We denote the copy of G'_0 in G_i by G_{0i} for $i = 1, 2$. Note that the set F of edges between G_{01} and G_{02} is a Θ-class, i.e., a color, with sides G_{01} and G_{02}. If the cover G'_1, G'_2 is isometric (resp., convex), then we call G an *isometric* (resp., *convex*) expansion. Finally, G is an *all-color expansion* if any of the G'_1 and G'_2 contains at least one edge of each Θ-class of G. The converse operation of expansion is *contraction*: G' is the contraction of G with respect to the split G_1, G_2, or, equivalently, with respect to the color F.

Chepoi [8] proved that a graph G is a partial cube if and only if G is obtained from the one-vertex graph K_1 by successive isometric expansions. Mulder [24, 25] proved that G is a median graph if and only if G can be obtained from K_1 by successive convex expansions.

Let G be a partial cube with a complete crossing graph, and let H be an isometric subgraph of G that meets all the Θ-classes of G. Then the expansion of G with respect to H and G is an all-color expansion, and the expanded graph has a complete crossing graph. (Note that the right-hand graph of Figure 4.1 is an expansion of Q_3 with respect to Q_3 and $K_{1,3}$.) More generally, we have the following result.

PROPOSITION 4.4. *Let G be a partial cube. Then $G^{\#}$ is a complete graph if and only if G can be obtained from K_1 by a sequence of all-color expansions.*

Proof. Assume first that $G^{\#}$ is a complete graph, and let F_{ab} be an arbitrary but fixed color of G. Let G' be the contraction of G with respect to F_{ab}, and let G'_1, G'_2 be the corresponding cover of G', so that G is the expansion of G' with respect to the cover G'_1, G'_2. Let F_{uv} be any other color of G. Then, since $G^{\#}$ is complete, we infer from Lemma 3.3 (ii) (or (iv)) that both G_1 and G_2 contain an edge from F_{uv}. Hence G'_1 and G'_2 both contain edges of this color. Induction completes the argument.

Conversely, suppose that G can be obtained from K_1 by a sequence of all-color expansions. Let G be an all-color expansion of G' with respect to the cover G'_1, G'_2, and let F_{ab} be the color of this expansion step. We need to show that F_{ab} crosses with any other color. Let F_{uv} be an arbitrary color different from F_{ab}. Since G is obtained by an all-color expansion, there is an edge xx' from F_{uv} in G_1 and an edge yy' from F_{uv} in G_2 with x and y on the one side and x' and y' on the opposide. Let P be a shortest x, y-path, and let Q be a shortest x', y'-path. Then $x' \to x \to P$ is a shortest path and thus no edge of P belongs to F_{uv}. Similarly, we see that no edge of Q is in F_{uv}. Moreover, there are exactly two edges from F_{ab} in the cycle $x' \to x \to \cdots P \cdots \to y \to y' \to \cdots Q \cdots \to x' \to x$. Therefore, by Lemma 3.3 (iv) the colors F_{ab} and F_{uv} cross. □

5. Triangle-free crossing graphs. *Cube-free median graphs* are median graphs that do not contain Q_3 as an induced subgraph. Note that a median graph is cube-free if and only if it does not contain isometric cycles of length at least 6. Moreover, each side of any color in a cube-free median graph must be a tree.

The class of cube-free median graphs may seem a rather special class of graphs. However, in [20] it was proved that there exists a one-to-one correspondence between the class of triangle-free graphs and a special subclass of cube-free median graphs. Hence, in the universe of all graphs, the density of the triangle-free graphs is as large as that of the cube-free median graphs (being triangle-free themselves). In [23] it was shown that cube-free median graphs play a special role in the theory of consensus functions on graphs. In our next result we show that the condition that the crossing graph of a partial cube G is triangle-free turns out to be a rather strong condition—it is equivalent to the fact that G is a cube-free median graph.

THEOREM 5.1. *Let G be a partial cube. Then $G^{\#}$ is triangle-free if and only if G is a cube-free median graph.*

Proof. First let G be a cube-free median graph. Then, by Theorem 11 from [23], we know that G does not contain three mutually crossing colors, so that $G^{\#}$ is triangle-free.

Conversely, let $G^{\#}$ be triangle-free; that is, G does not contain three mutually crossing colors. Take any color F in G, say between G_1 and G_2, with sides G_{01} and G_{02}, respectively. (Here and later we use the "expansion" notation introduced after Corollary 4.3.) Every color in G_{01} crosses with F. Hence, to avoid a triangle in $G^{\#}$, there are no cycles in G_{01}, by Lemma 3.2. So G_{01} is a forest.

Suppose that G_{01} consists of more than one component. Let R and S be two components of G_{01}, and choose u_R in R and u_S in S closest to each other. Since G_1 is an isometric subgraph of G, there is a geodesic P in G_1 between u_R and u_S of length at least two. Note that, by the choice of u_R and u_S, all internal vertices of P are in $G_1 - G_{01}$. Let v_R and v_S be the neighbors in G_{02} of u_R and u_S, respectively, and let Q be a geodesic between v_R and v_S. By Lemma 2.6, P and Q contain the same colors, and each color occurs at most once on P and at most once on Q. So all colors on P and Q cross color F. Let p be the vertex on P adjacent to u_R and q the vertex on Q adjacent to v_R. Since p is in $G_1 - G_{01}$, it follows that p is not adjacent to q. This implies that the edges $u_R \to p$ and $v_R \to q$ have different colors. So, by Lemma 3.3, they cross. Moreover, they both cross F, which is impossible. Hence we conclude that G_{01} is connected, so that it is a tree, as is G_{02}.

Assume that there is a color occurring twice in the tree G_{01}. Then choose edges uv and xy of the same color such that on the path $P = u \to v \to \cdots \to x \to y$ in G_{01} all colors on the subpath $u \to v \to \cdots \to x$ are distinct. Note that this subpath must be of length at least 3. Then $v \to \cdots P \cdots \to x$ is a geodesic on the one side of the color. Let $u \to \cdots Q \cdots \to y$ be a geodesic on the opposide. By Lemmas 2.6 and 3.3, the color of uv crosses with all other colors on P. Since all colors in G_{01} cross F we would get three mutually crossing colors, which is impossible. So every color in G_{01} occurs exactly once. Hence, by Lemma 2.4, subgraph G_{01} is isometric.

Next we prove that G_{01} is convex. Assume the contrary, and let u, v be vertices in G_{01}, so that there is a u, v-geodesic P, all of whose internal vertices are in $G_1 - G_{01}$. Note that P is of length at least 2. Let Q be the u, v-path in G_{01}, which is, as observed above, also a u, v-geodesic. Then $u \to \cdots P \cdots \to v \to \cdots Q \cdots \to u$ is a cycle. This implies that every color on P is also on Q, and vice versa. Since a cycle contains crossing colors, these must both occur on Q, so that they both cross F as well. Since this is impossible, G_{01} is convex in G.

Similarly, G_{02} is a convex subtree in G.

From the fact that G is bipartite and the sides of all colors are convex, we deduce that G is a median graph (cf. Theorem 1 in [6]). Finally, since three mutually crossing colors in a median graph necessarily force an induced Q_3, we conclude that G is a cube-free median graph. $\square$

Theorem 5.1 allows us to characterize several subclasses of partial cubes having nice crossing graphs. The *wheel* W_n consists of the n-cycle C_n together with an extra vertex joined to all the vertices of the cycle; cf. Figure 5.1. The cycle is called the *rim* of the wheel, the extra vertex the *center* of the wheel. The edges incident with the center are the *spokes* of the wheel.

The *cogwheel* M_n is obtained from the wheel W_n by subdividing all the edges on the rim of the wheel; cf. Figure 5.1. Note that the cogwheel M_3 is precisely the cube Q_3 minus a vertex. The *center* and the *spokes* of the cogwheel are inherited from the wheel. The cogwheel M_n is a partial cube with C_n as its crossing graph.

The next proposition is a simple corollary of Theorem 4.4, but it is also straightforward to check it directly.

PROPOSITION 5.2. *Let G be a partial cube. Then $G^{\#} = K_3$ if and only if G is Q_3, M_3, or C_6.*

THEOREM 5.3. *Let G be a partial cube. Then $G^{\#}$ is a cycle of length $n \geq 4$ if and only if $G = M_n$.*

Proof. The "if" part of the theorem is obvious. So let $G^{\#} = C_n$ with $n \geq 4$. By Theorems 3.4 and 5.1, G is a 2-connected cube-free median graph. This implies that the only isometric cycles in G are 4-cycles. Hence any two colors of G cross on some

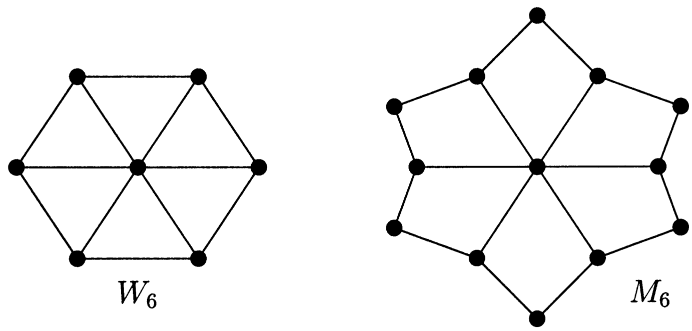

FIG. 5.1. *The wheel W_6 and the cogwheel M_6.*

4-cycle. If the side of a color would contain a P_4 or a $K_{1,3}$, then $G^{\#}$ would contain a vertex of degree at least 3. Hence each side of any color of G must be a P_2 or a P_3. (Note that P_1 is impossible, since G is 2-connected.) If some side were a P_2, say of color F, then F would be a pendant vertex in $G^{\#}$. So we conclude that all sides in G induce a P_3.

Take a vertex z of maximum degree k in G. Take any edge zu incident with z. If zu is on a 4-cycle with zw, then the colors F_{zu} and F_{zw} cross. On the other hand, color F_{zu} crosses with exactly two other colors. Moreover, F_{zu} crosses with each of these colors on a 4-cycle through z. Hence every edge incident with z is on a 4-cycle with exactly two other edges incident with z. This implies that the colors at z form a 2-regular subgraph of $G^{\#}$, so that these are all the colors of G. Moreover, the colors cross cycle-wise; that is, we may number the edges incident with z by $0, 1, \dots, n-1$, so that i is exactly on a 4-cycle with edges $i-1$ and $i+1$ modulo k. In the subgraph consisting of all these 4-cycles, all sides already induce a P_3. So this subgraph comprises all of G, and G is the cogwheel M_n. ∎

A C_4*-tree* G is recursively defined as follows: G is a 4-cycle, or G is obtained from a C_4-tree G' by gluing a 4-cycle along an edge to an edge of G'. It is straightforward to prove that G is a C_4-tree if and only if G can be obtained from two smaller C_4-trees by gluing them together along an edge (unless $G = C_4$). In [16] it was shown that the central vertices in a C_4-tree are contained in some 4-cycle or induce a P_4 such that the middle edge of the path is a common edge of two 4-cycles, whereas the first edge is on the one 4-cycle and the last edge is on the other 4-cycle.

Let G be a median graph. Let F be a color of G with split G_1, G_2. Then we call the color, or the split, *peripheral* if $G_1 = G_{01}$ or $G_2 = G_{02}$. We call the side G_i with $G_i = G_{0i}$ a *peripheral side.* In [26] peripheral colors and sides were called *extremal.* There it was proved that, for any split G_1, G_2 of a median graph, there exists a peripheral split H_1, H_2 such that $H_1 \subseteq G_1$ (and $G_2 \subseteq H_2$).

THEOREM 5.4. *Let G be a graph. Then the following statements are equivalent:*

(i) *G is a partial cube with $G^{\#}$ a tree.*
(ii) *G is obtained from K_2 by successive expansions, where the intersection of the cover is always an edge.*
(iii) *G is K_2 or a 2-connected cube-free median graph without induced cogwheels.*
(iv) *G is K_2 or a C_4-tree.*

Proof. (i) ⇒ (ii) We prove the implication by induction on the number of vertices in $G^{\#}$. If $G^{\#} = K_1$, then $G = K_2$, and we are done. So let $G^{\#}$ be a nontrivial tree T, and let the color F of G be a vertex of degree 1 in T. By Theorems 3.4 and 5.1, G is a 2-connected cube-free median graph. This implies that the sides of F cannot consist of a single vertex. If the sides contain more than one edge, then F crosses more than one other color of G. So the sides of F consist of a single edge. Let H be the contraction of G with respect to color F. Then $H^{\#} = T - F$, which is a tree with one vertex less than T. So, by induction, H is obtained by successive expansions, where the intersection of the cover is always a single edge. Hence this also holds for G.

(ii) $\Rightarrow$ (iii) If G is not K_2, then G is a 2-connected median graph. Cubes can arise only in an expansion when the intersection of the cover contains a 4-cycle. Cogwheels can arise only in an expansion when the intersection of the cover contains a P_3. So G does not contain Q_3 or any cogwheel.

(iii) $\Rightarrow$ (iv) We use induction on the number of colors in G. If $G = K_2$, then we are done. So let G contain at least two colors. Take a peripheral color F of G, and let $G_1 = G_{01}$ be a peripheral side of F. Since G is a cube-free median graph, we know that G_1 is a tree. Let u be a vertex of degree 1 in G_1 adjacent to w in G_1. Let v be the neighbor of u in G_{02} and x that of w in G_{02}, so that $C = u \to v \to x \to w \to u$ is a 4-cycle in G. Note that u is a vertex of degree 2 in G.

If G_1 consists of a single edge, then C is a "pendant" 4-cycle in G (or $G = C$), and we are done by induction. So we may assume that G_1 is a tree with more than two vertices; that is, w has other neighbors in G_1 besides u. Let z be any other neighbor of w in G_1, and let y be the neighbor of z in G_{02}.

Consider the color F_{uw}. If the sides of F_{uw} consist of a single edge, then the edges uw and vx are on C but not on any other 4-cycle. By deleting the edges uw and vx from G, we obtain two components. Let H_1 be the component containing wx, and let H_2 be the graph obtained from the other component by gluing the 4-cycle C to it along the edge uv. Now G can be obtained from H_1 and H_2 by gluing them together along the edge wx. By induction, H_1 and H_2 are two C_4-trees. Hence G is one too.

Now consider the case where the sides of F_{uw} consist of more than an edge. Since u is of degree 2, it follows that u is a pendant vertex in the tree S that constitutes the side of F_{uw} in G_{uw}. Let p be any other neighbor of v in the tree S, and let q be its neighbor in G_{vu}. If there is a path between q and y not going through x or u, then let R be a path of minimal length between q and y not going through x or u such that the sum of the distances from x to the vertices on R is as small as possible. Note that R does not use edges of the colors F_{uv} and F_{uw}. We claim that the vertices on R are alternately at distance 2 and 1 from x. If not, then there exists a subpath $r_1 \to r_2 \to r_3$ of R with $d(x, r_1) = d(x, r_2) - 1 = d(x, r_3) = k > 1$. The median of x, r_1, and r_3 is a common neighbor s of r_1 and r_3 at distance $k - 1$ from x. Thus we get another minimal path R' between q and y closer to x, which contradicts the choice of R. Now the vertices x, z, w, v, u, p together with the path R induce a cogwheel in G, which is impossible. Thus we have shown that $\{u, x\}$ is a cutset in G. Let Q_1 and Q_2 be the components of $G - \{u, x\}$, where Q_1 contains p, q, v, and Q_2 contains w, y, z. Let H_1 be the subgraph of G induced by Q_1 and x, and let H_2 be the subgraph of G induced by Q_2 and the vertices x, v, and u. By induction, H_1 and H_2 are C_4-trees. We can obtain G from H_1 and H_2 by gluing them together along the edge vx; so G is a C_4-tree as well.

(iv) $\Rightarrow$ (i) We use induction on the number of 4-cycles that are used to construct G. If G is K_2 or a 4-cycle, then we are done. So assume that more than one 4-cycle is used to construct G, and let $C = u \to v \to x \to w \to u$ be the last cycle used in the construction, where the gluing was along the edge uv. Let G' be the graph obtained from G by deleting the vertices x and w together with their incident edges. By induction, G' is a partial cube with a tree T' as its crossing graph. Then G is a partial cube as well, with one extra color. The crossing graph of G is obtained from T' by adding the vertex F_{uw} adjacent to vertex F_{uv}. This completes the proof. □

Recall that a *block* in a connected graph is a maximal 2-connected subgraph. A C_4-*cactoid* is a connected graph, each block of which is a K_2 or a C_4-tree. Loosely speaking, a C_4-cactoid can be obtained from K_2's and 4-cycles by gluing them together along vertices or edges.

THEOREM 5.5. *Let G be a graph. Then the following statements are equivalent:*

(i) *G is a partial cube with $G^{\#}$ a forest.*
(ii) *G is obtained from K_2 by successive expansions, where the intersection of the cover is always a vertex or an edge.*
(iii) *G is a cube-free median graph without induced cogwheels.*
(iv) *G is a C_4-cactoid.*

Proof. For each block of G we may apply Theorem 5.4. Just observe that for statement (ii) we can always add an edge pending at a vertex u to G by an expansion with respect to the cover $G_1 = G$, $G_2 = \langle u \rangle$. Then we can use this edge to construct a new block. □

6. Crossing graphs and Cartesian products. In this section we consider the relation between crossing graphs and Cartesian products of graphs. As it will turn out, there are several connections involving Cartesian products and joins of graphs. Recall that the *join* $G \oplus H$ of the graphs G and H is the graph obtained from the disjoint union of $G \cup H$ by joining every vertex of G with every vertex of H.

In the previous section we have observed that, if $G^{\#}$ and $H^{\#}$ are complete, then so is $(G \Box H)^{\#}$. This fact is a special case of our next result.

PROPOSITION 6.1. *Let G and H be partial cubes. Then $(G \Box H)^{\#} = G^{\#} \oplus H^{\#}$.*

Proof. It is easy to see that the Θ-classes of $G \Box H$ are in one-to-one correspondence with the union of the Θ-classes of G and the Θ-classes of H. (Instead of proving this fact directly, we refer to Lemma 4.3 of [19].) This means that $V((G \Box H)^{\#}) = V(G^{\#}) \cup V(H^{\#})$. In addition, the Θ-classes of $G \Box H$ corresponding to the Θ-classes of G induce $G^{\#}$, and, analogously, the Θ-classes of $G \Box H$ corresponding to the Θ-classes of H induce $H^{\#}$. Finally, any Θ-class from the induced $G^{\#}$ crosses with any Θ-class from the induced $H^{\#}$; in fact, by the definition of the Cartesian product, they cross on a 4-cycle. □

For instance, Proposition 6.1 implies that the crossing graph of the Cartesian product of n copies of P_3 is the n-octahedron: $(P_3^n)^{\#} = K_{2,2,\ldots,2}$.

Recall from section 3 that $S(G)$ is the simplex graph of a graph G.

PROPOSITION 6.2. *Let G and H be two disjoint graphs. Then $S(G \oplus H) = S(G) \Box S(H)$.*

Proof. First note that $S(G)$ and $S(H)$ are subgraphs of $S(G \oplus H)$ having only the vertex $\emptyset$ in common. For any complete subgraph K in $G \oplus H$, let $K_G = K \cap G$ and $K_H = K \cap H$. Note that $V(K)$ is the disjoint union of $V(K_G)$ and $V(K_H)$. Then the mapping ϕ defined by $\phi(K) = (K_G, K_H)$ is a bijection between the vertex set of $S(G \oplus H)$ and the vertex set of $S(G) \Box S(H)$. Let K, L be two complete subgraphs in $G \oplus H$. Then K and L are adjacent in $S(G \oplus H)$ if and only if $|K \bigtriangleup L| = 1$ if and only if either $|K_H \bigtriangleup L_H| = 0$ and $|K_H \bigtriangleup L_H| = 1$ or $|K_G \bigtriangleup L_G| = 1$ and $|K_H \bigtriangleup L_H| = 0$ if and only if (K_G, K_H) and (L_G, L_H) are adjacent in $S(G) \Box S(H)$.

So ϕ is an isomorphism. ∎

LEMMA 6.3. *Let G be a median graph, and let F_{uv} and F_{uw} be crossing colors. Then $v \to u \to w$ is in a 4-cycle.*

Proof. Since F_{uv} and F_{uw} are crossing, there is a vertex y in $G_{vu} \cap G_{wu}$. Then we have

$$d(y,v) = d(y,u) - 1 \quad \text{and} \quad d(y,w) = d(y,u) - 1\,.$$

Let x be the median of y, v, w; that is, x is on a geodesic between v and w, on a geodesic between y and v, and on a geodesic between y and w. Then x is a common neighbor of v and w distinct from u, so that $v \to u \to w \to x \to v$ is a 4-cycle. ∎

Let H be a subgraph of a connected graph G, and let z be a vertex of G outside H. A vertex x in H is a *gate* for z in H if, for any vertex w in H, there is a geodesic between z and w passing through x. For the proof of our next theorem we state the following well-known fact.

LEMMA 6.4. *Let H be a subgraph of a connected graph G, and let z be a vertex of $V(G) \setminus V(H)$. Then z has at most one gate in H, which must then be the unique vertex in H closest to z.*

We also recall that it is easy to check that, in a median graph G, every vertex outside a convex subgraph H has a gate in H.

THEOREM 6.5. *Let G be a partial cube. Then $G^{\#}$ is a complete bipartite graph if and only if G is the Cartesian product of two trees.*

Proof. First let $G = T_1 \Box T_2$ be the Cartesian product of two trees T_1 and T_2. Then the colors of T_i form an independent set in $G^{\#}$, for $i = 1, 2$, so $G^{\#}$ is a complete bipartite graph by Proposition 6.1.

Conversely, let $G^{\#}$ be a complete bipartite graph with bipartition X, Y. Then, by Theorem 5.1, G is a cube-free median graph, so that all sides in G are convex subtrees. In particular, G does not have three mutually crossing colors, and any color occurring in some side occurs only once in that side. Take any color F in G, say between G_1 and G_2, with sides G_{01} and G_{02}, respectively. Without loss of generality, F is in X. Since each color in Y crosses with F on some 4-cycle, each color in Y occurs in G_{01} as well as G_{02}. Since F does not cross with any other color in X, no color from X occurs in G_{01} of G_{02}.

Similarly, if Φ is any color in Y, then the sides of Φ are convex subtrees of G, in which each color from X occurs exactly once and no color from Y occurs.

Let z be any vertex of G. Note that the existence of three different neighbors of z in $I(u,z)$ would force three mutually crossing colors in G. (Just take the medians of u and any two of these neighbors of z in $I(u,z)$; these produce distinct 4-cycles through z and its three neighbors.) Hence we conclude that there are at most two neighbors of z that are closer to u for any z in G.

Now we are ready to find the appropriate subgraphs in G that will form the factors in the Cartesian product. Let F be a peripheral color with split G_1, G_2 and peripheral side $G_1 = G_{01}$. Without loss of generality, we may assume that F is in X. Let u be a vertex of degree 1 in subtree G_1 with neighbor w in G_1, and let v be the neighbor of u in G_{02}, so that $F = F_{uv}$ and $G_1 = G_{uv}$. Note that u has degree 2 in G.

First we will show that F_{uw} is a peripheral color with G_{uw} as its peripheral side. Let G'_{uw} be the side of F_{uw} in G_{uw}. Since uv is in G_{uw} and the color F of uv is in X, it follows that G'_{uw} is a tree, in which all colors of X occur exactly once but no color of Y occurs. Assume that some vertex p in G'_{uw} has a neighbor q in $G_{uw} - G'_{uw}$. Then F_{pq} is a color in Y; so it crosses with all colors in G'_{uw}. By repeatedly applying

Lemma 6.3, we proceed along a path from p to u in G'_{uw} finding an adjacent path in $G_{uw} - G'_{uw}$. Thus we find a neighbor s of u in $G_{uw} - G'_{uw}$, which must be distinct from v and w. This contradicts the fact that u has degree 2. So $G_{uw} = G'_{uw}$, and G_{uw} is a subtree containing all colors of X but no color of Y.

Let $H = G_{uv} \Box G_{uw}$. We endow the copies of G_{uv} and G_{uw} in H with the same coloring as G_{uv} and G_{uw} in G, respectively. We will prove that G is isomorphic to H, where the isomorphism preserves the coloring.

Set $|X| = m$ and $|Y| = n$. Then G_{uw} is a tree of size m and order $m+1$, and G_{uv} is a tree of size n and order $n+1$. If $G^{\#} = K_2$, then G is a 4-cycle, and we are done. So we may assume that $G^{\#} = K_{n,m}$ with $m \geq 2$. First we show that G has the right number of vertices by induction on $n+m$. Then we construct a coordinatization for the product and check adjacencies.

If we delete G_{uv} from G, then we get a partial cube with one less color and with $K_{n,m-1}$ as its crossing graph. So, by induction, we may assume that $G - G_{uv}$ is the Cartesian product of two trees of sizes n and $m-1$, respectively, so that $G - G_{uv}$ is of order $(n+1)m$. Since G_{uv} is a tree of size n, it follows that G is of order $(n+1)m + (n+1) = (n+1)(m+1)$. So G is of the right order.

Let $G_{\leq k}$ be the subgraph of G induced by the vertices of distance at most k to u, and let $H_{\leq k}$ be the subgraph of H induced by the vertices of distance at most k to (u,u). By induction on k, we will prove the following claim.

Claim. $G_{\leq k} \cong H_{\leq k}$ for $k \geq 0$.

Let z be any vertex of G, let z_w be its gate in G_{uv}, and let z_v be its gate in G_{uw}. Note that $d(z,u) = d(z,z_w) + d(z_w,u) = d(z,z_v) + d(z_v,u)$. We set $z = (z_w, z_v)$. Then $u = (u,u)$; so $G_{\leq 0} \cong H_{\leq 0}$.

Let z be any vertex of G_{uv}. Then we have $z = (z,u)$. Since G_{uv} is a convex subtree of G, there is a unique neighbor y of z closer to u, and y is the neighbor of z on the path from z to u in the tree G_{uv}. Then we have $y = (y,u)$. This implies that the subgraph G_{uv} of G is isomorphic to the subgraph $G_{uv} \Box \{u\}$ of H. Similarly, the subgraph G_{uw} of G is isomorphic to the subgraph $\{u\} \Box G_{uw}$ of H. In particular, we have shown that the claim is true for $k \leq 1$.

Now let z be a vertex of G outside $G_{uv} \cup G_{uw}$ with $d(u,z) = k$. Then we have $d(z,z_w), d(z_w,u), d(z,z_v), d(z_v,u) \geq 1$, so that $k \geq 2$. Let p be a neighbor of z on a geodesic from z to z_w. Then we have $p_w = z_w$, so that $p = (z_w, p_v)$. Moreover, we have $d(p,u) = d(p,z_w) + d(z_w,u) = d(z,u) - 1 = k-1$. By induction, we know the following facts. There is a unique geodesic P between p and z_w of length $d(p,z_w) = d(p_v,u)$, of which all the colors are in X. There is a unique geodesic Q between p and p_v of length $d(p,p_v) = d(z_w,u)$, of which all the colors are in Y.

Since $z \to P$ is a geodesic between z and its gate z_w in G_{uv} and all colors of Y occur in G_{uv}, color F_{zp} must be in X. Hence F_{zp} crosses with every color on Q. So, by repeatedly applying Lemma 6.3, we can construct a path Q' along Q from z to a neighbor r of p_v of the same length and coloring as Q. Since the last color on Q is F_{uv}, the last color on Q' is also F_{uv}, so that r is in G_{uv}. By the unicity of gates, we have $z_v = r$. Hence p_v is the unique neighbor of z_v in subtree G_{uw} closer to u. Let q be the neighbor of z on Q'. By a similar argument, we deduce that q_w is the unique neighbor of z_w in subtree G_{uv} closer to u.

Now $p = (z_w, p_v)$ and $q = (q_w, z_v)$ are two distinct neighbors of z closer to u. Hence these are all neighbors of z closer to u in G. This settles the induction step in the proof of the claim.

TABLE 7.1
Summary of the results of the paper.

$G^{\#}$	G
connected	2-connected
edgeless	tree
complete	obtained by all-color expansion
triangle-free	cube-free median
K_3	Q_3, M_3, or C_6
C_n, $n \geq 4$	M_n
tree	K_2 or C_4-tree
forest	C_4-cactoid
complete bipartite	Cartesian product of two trees

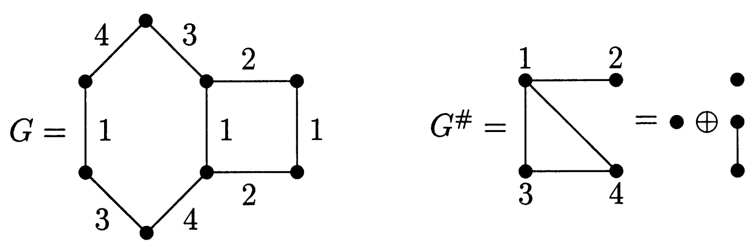

FIG. 7.1. *A partial cube and its crossing graph.*

Since G and H have the same number of vertices, we infer that

$$G = G_k \cong H_k = H\,,$$

for $k = \text{diameter}(G)$, by which the proof is complete. $\square$

7. Concluding remarks. Most of the results of this paper can be summarized in Table 7.1.

The last entry in the table, that is, Theorem 6.5, raises the following question.

PROBLEM 7.1. *What can be said about the partial cube G if its crossing graph $G^{\#}$ is the join of two other graphs that are not edgeless?*

This seems to be a tough problem as the examples in Figures 7.1 and 7.2 show. The graph in Figure 7.1 is a partial cube but not a median graph, whereas its crossing graph is still the join of two smaller graphs. The graph in Figure 7.2 is a median graph but not the Cartesian product of two smaller graphs, whereas its crossing graph is still the join of two smaller graphs.

One may define an equivalence relation $\kappa_{\#}$ on the family of all partial cubes as follows: two partial cubes are in relation $\kappa_{\#}$ to each other if they have isomorphic crossing graphs. Theorem 6.5 and Proposition 4.4 may be considered as instances of the characterization of two of the equivalence classes of this relation. A related problem is the following.

PROBLEM 7.2. *Determine all C_4-trees having the same tree as crossing graph.*

Finally, Theorem 5.3 suggests the following question for a median graph G.

PROBLEM 7.3. *Does an induced cycle C_n in $G^{\#}$ necessarily force an induced cogwheel M_n in G?*

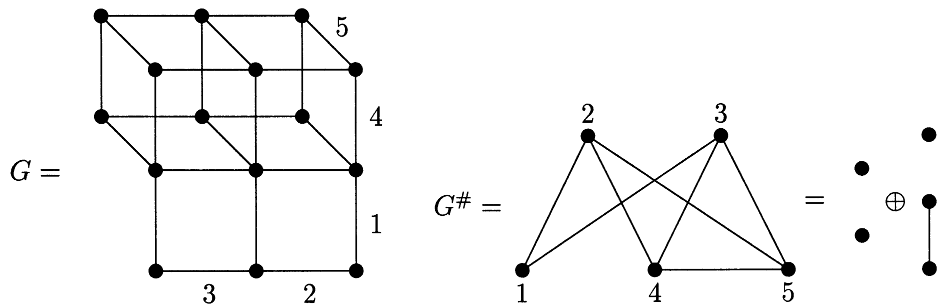

FIG. 7.2. *A median graph and its crossing graph.*

REFERENCES

[1] F. Aurenhammer, M. Formann, R. Idury, A. Schäffer and F. Wagner, *Faster isometric embeddings in products of complete graphs*, Discrete Appl. Math., 52 (1994), pp. 17–28.

[2] F. Aurenhammer and J. Hagauer, *Recognizing binary Hamming graphs in $O(n^2 \log n)$ time*, in Proceedings of the 16th International Workshop on Graph Theoretical Concepts in Computer Science, Lecture Notes in Comput. Sci. 484, Springer-Verlag, New York, 1991, pp. 90–98.

[3] F. Aurenhammer and J. Hagauer, *Recognizing binary Hamming graphs in $O(n^2 \log n)$ time,* Math. Systems Theory, 28 (1995), pp. 387–395.

[4] D. Avis, *Hypermetric spaces and the Hamming cone*, Canad. J. Math., 33 (1981), pp. 795–802.

[5] H.-J. Bandelt and J.-P. Bartélemy, *Medians in median graphs*, Discrete Appl. Math., 8 (1984), pp. 131–142.

[6] H.-J. Bandelt, H. M. Mulder, and E. Wilkeit, *Quasi-median graphs and algebras*, J. Graph Theory, 18 (1994), pp. 681–703.

[7] H.-J. Bandelt and M. van de Vel, *Superextensions and the depth of median graphs*, J. Combin. Theory Ser. A, 57 (1991), pp. 187–202.

[8] V. D. Chepoi, *d-Convexity and isometric subgraphs of Hamming graphs*, Cybernetics, 1 (1988), pp. 6–9.

[9] V. D. Chepoi and S. Klavžar, *The Wiener index and the Szeged index of benzenoid systems in linear time*, J. Chem. Inform. Comput. Sci., 37 (1997), pp. 752–755.

[10] D. Djoković, *Distance preserving subgraphs of hypercubes*, J. Combinatorial Theory Ser. B, 14 (1973), pp. 263–267.

[11] K. Fukuda and K. Handa, *Antipodal graphs and oriented matroids*, Discrete Math., 111 (1993), pp. 245–256.

[12] M. R. Garey and R. L. Graham, *On cubical graphs*, J. Combinatorial Theory Ser. B, 18 (1975), pp. 84–95.

[13] R. L. Graham and H. Pollak, *On the addressing problem for loop switching*, Bell System Tech. J., 50 (1971), pp. 2495–2519.

[14] R. L. Graham and P.M. Winkler, *On isometric embeddings of graphs*, Trans. Amer. Math. Soc., 288 (1985), pp. 527–536.

[15] J. Hagauer, W. Imrich, and S. Klavžar, *Recognizing median graphs in subquadratic time*, Theoret. Comput. Sci., 215 (1999), pp. 123–136.

[16] S. M. Hedetniemi, S. T. Hedetniemi, and P. J. Slater, *Centers and medians of $C_{(n)}$-trees*, Utilitas Math., 21 (1982), pp. 225–234.

[17] W. Imrich and S. Klavžar, *A simple $O(mn)$ algorithm for recognizing Hamming graphs*, Bull. Inst. Combin. Appl., 9 (1993), pp. 45–56.

[18] W. Imrich and S. Klavžar, *A convexity lemma and expansion procedures for bipartite graphs*, European J. Combin., 19 (1998), pp. 677–685.

[19] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, 2000.

[20] W. Imrich, S. Klavžar, and H. M. Mulder, *Median graphs and triangle-free graphs*, SIAM J. Discrete Math., 12 (1999), pp. 111–118.

[21] S. Klavžar and I. Gutman, *Wiener number of vertex-weighted graphs and a chemical appli-*

cation, Discrete Appl. Math., 80 (1997), pp. 73–81.

[22] S. Klavžar and H. M. Mulder, *Median graphs: Characterizations, location theory and related structures*, J. Combin. Math. Combin. Comput., 30 (1999), pp. 103–127.

[23] F. R. McMorris, H. M. Mulder, and F. R. Roberts, *The median procedure on median graphs*, Discrete Math., 84 (1998), pp. 165–181.

[24] H. M. Mulder, *The structure of median graphs*, Discrete Math., 24 (1978), pp. 197–204.

[25] H. M. Mulder, *The Interval Function of a Graph*, Math. Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.

[26] H. M. Mulder, *The expansion procedure for graphs*, in Contemporary Methods in Graph Theory, R. Bodendiek, ed., B.I.-Wissenschaftsverlag, Mannheim, Wien, Zürich, 1990, pp. 459–477.

[27] E. Wilkeit, *Isometric embeddings in Hamming graphs*, J. Combin. Theory Ser. B, 50 (1990), pp. 179–197.

[28] P. Winkler, *Isometric embeddings in products of complete graphs*, Discrete Appl. Math., 7 (1984), pp. 221–225.