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THE CONJECTURE ON DISTANCE-BALANCEDNESS OF GENERALIZED PETERSEN GRAPHS HOLDS WHEN INTERNAL EDGES HAVE JUMPS 3 OR 4

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Abstract. A connected graph G with $\mathrm{diam}(G) \geq \ell$ is ℓ-distance-balanced if $|W_{xy}| = |W_{yx}|$ for every $x, y \in V(G)$ with $d_G(x,y) = \ell$, where W_{xy} is the set of vertices of G that are closer to x than to y. Miklavič and Šparl (Discrete Appl. Math. 244 (2018) 143–154) conjectured that if $n > n_k$ where $n_k = 11$ if $k = 2$, $n_k = (k+1)^2$ if k is odd, and $n_k = k(k+2)$ if $k \geq 4$ is even, then the generalized Petersen graph $GP(n,k)$ is not ℓ-distance-balanced for any $1 \leq \ell < \mathrm{diam}(GP(n,k))$. In the seminal paper, the conjecture was verified for $k = 2$. In this paper we prove that the conjecture holds for $k = 3$ and for $k = 4$.

1. Introduction

Let $G = (V(G), E(G))$ be a connected graph and $u, v \in V(G)$. The set of vertices that are closer to u than to v (with respect to the standard shortest-path distance $d_G(u,v)$) is denoted by W_{uv}. When $|W_{uv}| = |W_{vu}|$ holds, the pair of vertices u and v is called *balanced*, and when every pair of adjacent vertices is balanced, G is called *distance-balanced*. Distance-balanced graphs were first considered in [11], the term "distance-balanced" was coined in [13]. For a number of reasons, both theoretical and applied, the distance-balanced graphs received a lot of attention, see [1, 3–8, 12, 15–17, 19, 21]. We should also mention in passing that distance-balanced graphs can be equivalently described as the graphs whose Mostar index (see [2]) equals 0.

More generally, let $\ell \in [\mathrm{diam}(G)] = \{1, 2, \ldots, \mathrm{diam}(G)\}$, where $\mathrm{diam}(G)$ is the diameter of G. Then G is called *ℓ-distance-balanced* [9] if each pair of vertices $u, v \in V(G)$ with $d_G(u,v) = \ell$ is balanced. For a study of 2-distance-balanced graphs see [10] and for several results on ℓ-distance-balanced graphs see [14, 20].

This paper is about the distance-balancedness of the generalized Petersen graphs. The interest in these graphs was already shown in [13] where it was conjectured that

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for any integer $k \geq 2$, there exists a positive integer n_0 such that $GP(n,k)$ is not distance-balanced for every $n \geq n_0$. The validity of the conjecture has been demonstrated in [21]. Interest in the distance-balancedeness of the generalized Petersen graphs continued in [18,20]. In [18] it was proved that $GP(n,k)$ is $\operatorname{diam}(GP(n,k))$-distance-balanced as soon as n is large relative to k, more precisely, the following theorem was proved.

Theorem 1.1. [18] *If n and k are integers, where $3 \leq k < n/2$, and*

$$n \geq \begin{cases} 8; & k = 3, \\ 10; & k = 4, \\ \frac{k(k+1)}{2}; & k \text{ is odd and } k \geq 5, \\ \frac{k^2}{2}; & k \text{ is even and } k \geq 6, \end{cases}$$

then $GP(n,k)$ is $\operatorname{diam}(GP(n,k))$-distance-balanced.

On the other hand, Miklavič and Špar posed the following:

Conjecture 1.2. [20] *Let $k \geq 2$ be an integer and let*

$$n_k = \begin{cases} 11; & k = 2, \\ (k+1)^2; & k \text{ odd}, \\ k(k+2); & k \geq 4 \text{ even}. \end{cases}$$

Then for any $n > n_k$, the graph $GP(n,k)$ is not ℓ-distance-balanced for any $1 \leq \ell < \operatorname{diam}(GP(n,k))$. Moreover, n_k is the smallest integer with this property.

In [20], Conjecture 1.2 was verified for $k = 2$. In this paper, we prove that Conjecture 1.2 holds true for $k = 3$ and for $k = 4$ by establishing the following results.

Theorem 1.3. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not ℓ-distance-balanced for any $1 \leq \ell < \operatorname{diam}(GP(n,3))$. Moreover, 16 is the smallest integer with this property.*

Theorem 1.4. *For any $n > 24$, the generalized Petersen graph $GP(n,4)$ is not ℓ-distance-balanced for any $1 \leq \ell < \operatorname{diam}(GP(n,4))$. Moreover, 24 is the smallest integer with this property.*

To prove these two theorems, it suffices to prove the first assertion of each of them. With these results in hand, the facts that 16 is the smallest integer in Theorem 1.3 and that 24 is the smallest integer in Theorem 1.4, follow by computer experiments presented in [20, Table 1].

Full proofs of Theorems 1.3 and 1.4 are very long and repetitive. We therefore present in the next two sections only selected, representative cases. The other cases of the proofs however can be found in Appendix A and Appendix B. We conclude the paper by suggesting a problem in Section 4.

To conclude the introduction recall that the *generalized Petersen graph* $GP(n,k)$, $n \geq 3$, $1 \leq k < n/2$, is defined by

$$V(GP(n,k)) = \{u_i : i \in \mathbb{Z}_n\} \cup \{v_i : i \in \mathbb{Z}_n\},$$
$$E(GP(n,k)) = \{u_i u_{i+1} : i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} : i \in \mathbb{Z}_n\} \cup \{u_i v_i : i \in \mathbb{Z}_n\}.$$

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2. Sketch proof of Theorem 1.3

As mentioned in the introduction, it suffices to prove that for any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not ℓ-distance-balanced for any $1 \leq \ell < \operatorname{diam}(GP(n,3))$. We split the argument into the cases $\ell = 1$, $\ell = 2$, and $3 \leq \ell < \operatorname{diam}(GP(n,3))$ to be respectively covered by Propositions 2.1, 2.2, and 2.3.

Proposition 2.1. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not 1-distance-balanced.*

Proof. Since $d_{GP(n,3)}(u_0,v_0) = 1$, it suffices to prove that $|W_{u_0v_0}| < |W_{v_0u_0}|$. We divide the discussion into six cases based on $n \bmod 6$, and for transparency and non-replication purposes, present only the first case in detail. Details for the other five cases are given in Appendix A.

Let $n = 6m$, where $m \geq 3$. By symmetry, it suffices to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$. Then the following holds.

- If $1 \leq t \leq m$, then $d(u_0,u_{3t}) = 2+t$ and $d(v_0,u_{3t}) = 1+t$.
- If $1 \leq t \leq m$, then $d(u_0,v_{3t}) = 1+t$ and $d(v_0,v_{3t}) = t$.
- If $1 \leq t < m$, then $d(u_0,u_{3t+1}) = 3+t$ and $d(v_0,u_{3t+1}) = 2+t$.
- If $0 \leq t < m$, then $d(u_0,v_{3t+1}) = 2+t$ and $d(v_0,v_{3t+1}) = 3+t$.
- If $1 \leq t < m$, then $d(u_0,u_{3t+2}) = 4+t$ and $d(v_0,u_{3t+2}) = 3+t$.
- If $0 \leq t < m$, then $d(u_0,v_{3t+2}) = 3+t$ and $d(v_0,v_{3t+2}) = 4+t$.
- $d(u_0,u_1) = 1$ and $d(v_0,u_1) = 2$.
- $d(u_0,u_2) = 2$ and $d(v_0,u_2) = 3$.

In the above consideration, we have $2m+2$ vertices from $W_{u_0v_0}$ and $4m-2$ vertices from $W_{v_0u_0}$. Since we have considered only the vertices u_i and v_i with $1 \leq i \leq \frac{n}{2}$, there are in total twice as many vertices, except that u_{3m} and v_{3m} are considered twice (and they lie in $W_{v_0u_0}$). Since clearly $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$, we conclude that

$$|W_{u_0v_0}| = 2(2m+2)+1 = 4m+5,$$
$$|W_{v_0u_0}| = 2(4m-2)+1-2 = 8m-5.$$

Because $m \geq 3$, we indeed have $|W_{u_0v_0}| < |W_{v_0u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 6m+1$, $m \geq 3$, then $|W_{u_0v_0}| = 4m+3$ and $|W_{v_0u_0}| = 8m-3$.
- If $n = 6m+2$, $m \geq 3$, then $|W_{u_0v_0}| = 4m+4$ and $|W_{v_0u_0}| = 8m$.
- If $n = 6m+3$, $m \geq 3$, then $|W_{u_0v_0}| = 4m+7$ and $|W_{v_0u_0}| = 8m-1$.
- If $n = 6m+4$, $m \geq 3$, then $|W_{u_0v_0}| = 4m+6$ and $|W_{v_0u_0}| = 8m+2$.
- If $n = 6m+5$, $m \geq 2$, then $|W_{u_0v_0}| = 4m+5$ and $|W_{v_0u_0}| = 8m+3$.

Note that in some cases there may be vertices which are of equal distance to u_0 and v_0. Anyhow, in each case we have $|W_{u_0v_0}| < |W_{v_0u_0}|$. □

Proposition 2.2. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not 2-distance-balanced.*

Proof. Since $d_{GP(n,3)}(u_0,v_{-3}) = 2$, it suffices to prove that $|W_{u_0v_{-3}}| < |W_{v_{-3}u_0}|$. We divide the discussion into the six cases based on n mod 6, and for transparency

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and non-replication purposes, present only the first case in detail. Details for the other five cases are given in Appendix A.

Firstly we consider vertices $v_{-1}, v_{-2}, u_{-1}, u_{-2}$.

- $d(u_0, v_{-1}) = 2$ and $d(v_{-3}, v_{-1}) = 4$.
- $d(u_0, v_{-2}) = d(v_{-3}, v_{-2}) = 3$.
- $d(u_0, u_{-1}) = 1$ and $d(v_{-3}, u_{-1}) = 3$.
- $d(u_0, u_{-2}) = d(v_{-3}, u_{-2}) = 2$.

So $u_{-1}, v_{-1} \in W_{u_0 v_{-3}}$ and no vertex of $\{v_{-1}, v_{-2}, u_{-1}, u_{-2}\}$ is in $W_{v_{-3} u_0}$.

Next we consider vertices v_i where $0 \leq i < n-3$ and u_j where $1 \leq j \leq n-3$. Let $n = 6m$ where $m \geq 3$. Note that $v_{-3} = v_{n-3}$ and $n-3 = 6m-3 = 3(2m-1)$.

- If $0 \leq t \leq m-1$, then $d(u_0, v_{3t}) = d(v_{6m-3}, v_{3t}) = 1+t$.
  If $m \leq t < 2m-1$, then $d(v_{6m-3}, v_{3t}) = 2m-1-t$ and $d(u_0, v_{3t}) > d(v_{6m-3}, v_{3t})$.
- If $0 \leq t \leq m-1$, then $d(u_0, v_{3t+1}) = 2+t$ and $d(u_0, v_{3t+1}) < d(v_{6m-3}, v_{3t+1})$.
  If $m \leq t < 2m-1$, then $d(u_0, v_{3t+1}) = d(v_{6m-3}, v_{3t+1}) = 2m-t+2$.
- If $0 \leq t \leq m-2$, then $d(u_0, v_{3t+2}) = 3+t$ and $d(u_0, v_{3t+2}) < d(v_{6m-3}, v_{3t+2})$.
  If $m-1 \leq t < 2m-1$, then $d(u_0, v_{3t+2}) = d(v_{6m-3}, v_{3t+2}) = 2m-t+1$.
- If $1 \leq t \leq m-1$, then $d(u_0, u_{3t}) = d(v_{6m-3}, u_{3t}) = 2+t$.
  If $m \leq t \leq 2m-1$, then $d(v_{6m-3}, u_{3t}) = 2m-t$ and $d(u_0, u_{3t}) > d(v_{6m-3}, u_{3t})$.
- If $1 \leq t \leq m-1$, then $d(u_0, u_{3t+1}) = d(v_{6m-3}, u_{3t+1}) = 3+t$.
  If $m \leq t < 2m-1$, then $d(v_{6m-3}, u_{3t+1}) = 2m-t+1$ and $d(u_0, u_{3t+1}) > d(v_{6m-3}, u_{3t+1})$.
- If $1 \leq t \leq m-2$, then $d(u_0, u_{3t+2}) = d(v_{6m-3}, u_{3t+2}) = 4+t$.
  If $m-1 \leq t < 2m-1$, then $d(v_{6m-3}, u_{3t+2}) = 2m-t$ and $d(u_0, u_{3t+2}) > d(v_{6m-3}, u_{3t+2})$.
- $d(u_0, u_1) = 1$, $d(v_{6m-3}, u_1) = 2m+1$, $d(u_0, u_2) = 2$, $d(v_{6m-3}, u_2) = 2m$.

Note that $u_0 \in W_{u_0 v_{6m-3}}$ and $v_{6m-3} \in W_{v_{6m-3} u_0}$. Combined with the above discussion we get $|W_{u_0 v_{6m-3}}| = 2m+4$ and $|W_{v_{6m-3} u_0}| = 4m-1$. Because $m \geq 3$, we can conclude that $|W_{u_0 v_{6m-3}}| < |W_{v_{6m-3} u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 6m+1$, $m \geq 3$, then $|W_{u_0 v_{-3}}| = 2m+4$ and $|W_{v_{-3} u_0}| = 4m+2$.
- If $n = 6m+2$, $m \geq 3$, then $|W_{u_0 v_{-3}}| = 2m+3$ and $|W_{v_{-3} u_0}| = 4m+1$.
- If $n = 6m+3$, $m \geq 3$, then $|W_{u_0 v_{-3}}| = 2m+6$ and $|W_{v_{-3} u_0}| = 4m+3$.
- If $n = 6m+4$, $m \geq 3$, then $|W_{u_0 v_{-3}}| = 2m+4$ and $|W_{v_{-3} u_0}| = 4m+2$.
- If $n = 6m+5$, $m \geq 2$, then $|W_{u_0 v_{-3}}| = 2m+5$ and $|W_{v_{-3} u_0}| = 4m+5$.

In each case we have $|W_{u_0 v_{-3}}| < |W_{v_{-3} u_0}|$. □

Proposition 2.3. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not ℓ-distance-balanced for any $3 \leq \ell < \mathrm{diam}(GP(n,3))$.*

Proof. For a given fixed n, we set $D = \mathrm{diam}(GP(n,3))$.

For any $3 \leq \ell < D$, we first show that there exists v_j such that $d(u_0, v_j) = \ell$, where $6 \leq j \leq n/2$. From [18] we recall that there exists j^* such that $d(u_0, u_{j^*}) = D$.

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If $n = 6m$ ($m \geq 3$) or $n = 6m+1$ ($m \geq 3$), then we know from [18] that $j^* = 3(m-1)+2$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{3s+2}) = s+3$, where $2 \leq s \leq m-1$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

If $n = 6m+2$ ($m \geq 3$) or $n = 6m+3$ ($m \geq 3$), then from [18] we know that $j^* = 3m+1$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{3s+1}) = s+2$, where $2 \leq s \leq m$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

If $n = 6m+4$ ($m \geq 3$), then from [18] we know that $j^* = 3m+2$ and $D = d(u_0, u_{j^*}) = m+4$. Note that $d(u_0, v_{3s+2}) = s+3$, where $2 \leq s \leq m$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

If $n = 6m+5$ ($m \geq 2$), then (again by [18]) $j^* = 3m+1$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{3s+1}) = s+2$, where $2 \leq s \leq m$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

From the above discussion, for any ℓ where $3 \leq \ell < D$, there exists a j such that $6 \leq j \leq n/2$ and $d(u_0, v_j) = \ell$. Define the following sets of vertices:

$$\begin{aligned}
V_1 &= \{u_i : \ 1 \leq i \leq j-1\} \cup \{v_i : \ 1 \leq i \leq j-1\}, \\
V_2 &= \{u_i : \ j+1 \leq i \leq n-1\} \cup \{v_i : \ j+1 \leq i \leq n-1\}, \\
W^1_{u_0 v_j} &= W_{u_0 v_j} \cap (V_1 \cup \{u_0, v_0, u_j, v_j\}), \\
W^1_{v_j u_0} &= W_{v_j u_0} \cap (V_1 \cup \{u_0, v_0, u_j, v_j\}), \\
W^2_{u_0 v_j} &= W_{u_0 v_j} \cap (V_2 \cup \{u_0, v_0, u_j, v_j\}), \\
W^2_{v_j u_0} &= W_{v_j u_0} \cap (V_2 \cup \{u_0, v_0, u_j, v_j\}).
\end{aligned}$$

Because $6 \leq j \leq n/2$, we have $|W^2_{u_0 v_j}| = |W^1_{u_0 v_{n-j}}|$ and $|W^2_{v_j u_0}| = |W^1_{v_{n-j} u_0}|$. So

$$\begin{aligned}
|W_{u_0 v_j}| &= |W^1_{u_0 v_j}| + |W^2_{u_0 v_j}| - 2 = |W^1_{u_0 v_j}| + |W^1_{u_0 v_{n-j}}| - 2 \quad \text{and} \\
|W_{v_j u_0}| &= |W^1_{v_j u_0}| + |W^2_{v_j u_0}| - 2 = |W^1_{v_j u_0}| + |W^1_{v_{n-j} u_0}| - 2.
\end{aligned}$$

In the following we will compute $|W^1_{u_0 v_j}|$ and $|W^1_{v_j u_0}|$ where $6 \leq j \leq n-6$. The computation is divided into six cases, and for transparency and non-replication purposes, we present only the first case in detail. Details for the other five cases are given in Appendix A.

The computation of $|W^1_{u_0 v_{3s}}|$ and $|W^1_{v_{3s} u_0}|$, where s is odd and $s \geq 5$, is as follows.

- If $0 \leq t < s$, then $d(u_0, v_{3t}) = 1+t$ and $d(v_{3s}, v_{3t}) = s-t$.
  If $0 \leq t < \frac{s-1}{2}$, then $d(u_0, v_{3t}) < d(v_{3s}, v_{3t})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{3t}) > d(v_{3s}, v_{3t})$.
- If $0 \leq t < s$, then $d(u_0, v_{3t+1}) = 2+t$ and $d(v_{3s}, v_{3t+1}) = s-t+3$.
  If $0 \leq t < \frac{s+1}{2}$, then $d(u_0, v_{3t+1}) < d(v_{3s}, v_{3t+1})$.
  If $\frac{s+1}{2} < t < s$, then $d(u_0, v_{3t+1}) > d(v_{3s}, v_{3t+1})$.
- If $0 \leq t < s$, then $d(u_0, v_{3t+2}) = 3+t$ and $d(v_{3s}, v_{3t+2}) = s-t+2$.
  If $0 \leq t < \frac{s-1}{2}$, then $d(u_0, v_{3t+2}) < d(v_{3s}, v_{3t+2})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{3t+2}) > d(v_{3s}, v_{3t+2})$.

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- If $1 \le t \le s$, then $d(u_0, u_{3t}) = 2 + t$ and $d(v_{3s}, u_{3t}) = s - t + 1$.
  If $1 \le t < \frac{s-1}{2}$, then $d(u_0, u_{3t}) < d(v_{3s}, u_{3t})$.
  If $\frac{s-1}{2} < t \le s$, then $d(u_0, u_{3t}) > d(v_{3s}, u_{3t})$.
- If $1 \le t < s$, then $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_{3s}, u_{3t+1}) = s - t + 2$.
  If $1 \le t < \frac{s-1}{2}$, then $d(u_0, u_{3t+1}) < d(v_{3s}, u_{3t+1})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, u_{3t+1}) > d(v_{3s}, u_{3t+1})$.
- If $1 \le t < s$, then $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_{3s}, u_{3t+2}) = s - t + 1$.
  If $1 \le t < \frac{s-3}{2}$, then $d(u_0, u_{3t+2}) < d(v_{3s}, u_{3t+2})$.
  If $\frac{s-3}{2} < t < s$, then $d(u_0, u_{3t+2}) > d(v_{3s}, u_{3t+2})$.
- $d(u_0, u_1) = 1$, $d(v_{3s}, u_1) = s + 2$, $d(u_0, u_2) = 2$, and $d(v_{3s}, u_2) = s + 1$.

Note that $u_0 \in W^1_{u_0v_{3s}}$ and $v_{3s} \in W^1_{v_{3s}u_0}$. Combined with the above discussion we obtain $|W^1_{u_0v_{3s}}| = 3s - 3$ and $|W^1_{v_{3s}u_0}| = 3s - 1$.

The conclusions in the remaining cases are as follows:

- If $s \ge 4$ and s is even, then $|W^1_{u_0v_{3s}}| = 3s$ and $|W^1_{v_{3s}u_0}| = 3s + 2$.
- If $s \ge 3$ and s is odd, then $|W^1_{u_0v_{3s+1}}| = 3s + 1$ and $|W^1_{v_{3s+1}u_0}| = 3s + 3$.
- If $s \ge 4$ and s is even, then $|W^1_{u_0v_{3s+1}}| = 3s - 2$ and $|W^1_{v_{3s+1}u_0}| = 3s$.
- If $s \ge 5$ and s is odd, then $|W^1_{u_0v_{3s+2}}| = 3s - 1$ and $|W^1_{v_{3s+2}u_0}| = 3s + 1$.
- If $s \ge 4$ and s is even, then $|W^1_{u_0v_{3s+2}}| = 3s + 2$ and $|W^1_{v_{3s+2}u_0}| = 3s + 4$.
- $|W^1_{u_0v_6}| = 7$ and $|W^1_{v_6u_0}| = 7$.
- $|W^1_{u_0v_7}| = 6$ and $|W^1_{v_7u_0}| = 6$.
- $|W^1_{u_0v_8}| = 9$ and $|W^1_{v_8u_0}| = 9$.
- $|W^1_{u_0v_9}| = 7$ and $|W^1_{v_9u_0}| = 8$.
- $|W^1_{u_0v_{11}}| = 9$ and $|W^1_{v_{11}u_0}| = 10$.

When $n \ge 17$, from the above computation of $|W^1_{u_0v_j}|$ and $|W^1_{v_ju_0}|$ where $6 \le j \le n - 6$, for any $3 \le \ell < D$, we know that there exists j where $d(u_0, v_j) = \ell$ and $6 \le j \le n/2$ such that $|W_{u_0v_j}| < |W_{v_ju_0}|$. □

3. Sketch proof of Theorem 1.4

As mentioned in the introduction, it suffices to prove that for any $n > 24$, the generalized Petersen graph $GP(n, 4)$ is not ℓ-distance-balanced for any $1 \le \ell < \mathrm{diam}(GP(n, 4))$. We split the argument into the cases $\ell = 1$, $\ell = 2$, and $3 \le \ell < \mathrm{diam}(GP(n, 4))$ to be respectively covered by Propositions 3.1, 3.2, and 3.3.

Proposition 3.1. *For any $n > 24$, the generalized Petersen graph $GP(n, 4)$ is not 1-distance-balanced.*

Proof. Since $d_{GP(n,4)}(u_0, v_0) = 1$, it suffices to prove that $|W_{u_0v_0}| < |W_{v_0u_0}|$. We divide the discussion into eight cases based on $n \bmod 8$, and for transparency and non-replication purposes, present only the first case in detail. Details for the other seven cases are given in Appendix B.

Let $n = 8m$, where $m \ge 4$. By symmetry, it suffices to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$. Then the following holds.

- If $1 \le t \le m$, then $d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$.
- If $0 \le t < m$, then $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$.

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- If $0 \leq t < m$, then $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$.
- If $0 \leq t < m$, then $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_0, v_{4t+3}) = 4 + t$.
- If $1 \leq t \leq m$, then $d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$.
- If $1 \leq t < m$, then $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$.
- If $1 \leq t < m$, then $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$.
- If $1 \leq t < m$, then $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$.
- $d(u_0, u_1) = 1$, $d(v_0, u_1) = 2$, $d(u_0, u_2) = 2$, $d(v_0, u_2) = 3$, $d(u_0, u_3) = 3$, and $d(v_0, u_3) = 3$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion we arrive at $|W_{u_0v_0}| = 2(3m+2)+1 = 6m+5$ and $|W_{v_0u_0}| = 2(5m-5)+3 = 10m-7$. Because $m \geq 4$, we can conclude that $|W_{u_0v_0}| < |W_{v_0u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 8m+1$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+3$ and $|W_{v_0u_0}| = 10m-5$.
- If $n = 8m+2$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+4$ and $|W_{v_0u_0}| = 10m-2$.
- If $n = 8m+3$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+5$ and $|W_{v_0u_0}| = 10m-1$.
- If $n = 8m+4$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+8$ and $|W_{v_0u_0}| = 10m-2$.
- If $n = 8m+5$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+7$ and $|W_{v_0u_0}| = 10m+1$.
- If $n = 8m+6$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+6$ and $|W_{v_0u_0}| = 10m+2$.
- If $n = 8m+7$, where $m \geq 3$, then $|W_{u_0v_0}| = 6m+7$ and $|W_{v_0u_0}| = 10m+3$.

In each case we have $|W_{u_0v_0}| < |W_{v_0u_0}|$. □

Proposition 3.2. *For any $n > 24$, the generalized Petersen graph $GP(n, 4)$ is not 2-distance-balanced.*

Proof. Since $d_{GP(n,4)}(u_0, v_{-4}) = 2$, it suffices to prove that $|W_{u_0v_{-4}}| < |W_{v_{-4}u_0}|$. We divide the discussion into the eight cases based on $n \bmod 8$, and for transparency and non-replication purposes, present only the first case in detail. Details for the other seven cases are given in Appendix B.

Firstly we consider vertices $v_{-1}, v_{-2}, v_{-3}, u_{-1}, u_{-2}, u_{-3}$:

- $d(u_0, v_{-1}) = 2$ and $d(v_{-4}, v_{-1}) = 4$,
- $d(u_0, v_{-2}) = 3$ and $d(v_{-4}, v_{-2}) = 4$,
- $d(u_0, v_{-3}) = d(v_{-4}, v_{-3}) = 3$,
- $d(u_0, u_{-1}) = 1$ and $d(v_{-4}, u_{-1}) = 3$,
- $d(u_0, u_{-2}) = 2$ and $d(v_{-4}, u_{-2}) = 3$,
- $d(u_0, u_{-3}) = 3$ and $d(v_{-4}, u_{-3}) = 2$.

Next we consider vertices v_i, $0 \leq i < n-4$, and u_j, $1 \leq j \leq n-4$. Let $n = 8m$, $m \geq 4$. Note that $n - 4 = 8m - 4 = 4(2m-1)$.

- If $0 \leq t \leq m-1$, then $d(u_0, v_{4t}) = d(v_{8m-4}, v_{4t}) = 1 + t$.
  If $m \leq t < 2m-1$, then $d(v_{8m-4}, v_{4t}) = 2m - t - 1$ and $d(u_0, v_{4t}) > d(v_{8m-4}, v_{4t})$.
- If $0 \leq t \leq m-1$, then $d(u_0, v_{4t+1}) = 2+t$ and $d(u_0, v_{4t+1}) < d(v_{8m-4}, v_{4t+1})$.
  If $m \leq t < 2m-1$, then $d(u_0, v_{4t+1}) = d(v_{8m-4}, v_{4t+1}) = 2m - t + 2$.
- If $0 \leq t \leq m-1$, then $d(u_0, v_{4t+2}) = 3+t$ and $d(u_0, v_{4t+2}) < d(v_{8m-4}, v_{4t+2})$.
  If $m \leq t < 2m-1$, then $d(u_0, v_{4t+2}) = d(v_{8m-4}, v_{4t+2}) = 2m - t + 2$.

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- If $0 \le t \le m-2$, then $d(u_0, v_{4t+3}) = 3+t$ and $d(u_0, v_{4t+3}) < d(v_{8m-4}, v_{4t+3})$. If $m-1 \le t < 2m-1$, then $d(u_0, v_{4t+3}) = d(v_{8m-4}, v_{4t+3}) = 2m - t + 1$.
- If $1 \le t \le m-1$, then $d(u_0, u_{4t}) = d(v_{8m-4}, u_{4t}) = 2 + t$. If $m \le t \le 2m-1$, then $d(v_{8m-4}, u_{4t}) = 2m - t$ and $d(u_0, u_{4t}) > d(v_{8m-4}, u_{4t})$.
- If $1 \le t \le m-1$, then $d(u_0, u_{4t+1}) = d(v_{8m-4}, u_{4t+1}) = 3 + t$. If $m \le t < 2m-1$, then $d(v_{8m-4}, u_{4t+1}) = 2m - t + 1$ and $d(u_0, u_{4t+1}) > d(v_{8m-4}, u_{4t+1})$.
- If $1 \le t \le m-2$, then $d(u_0, u_{4t+2}) = d(v_{8m-4}, u_{4t+2}) = 4 + t$. If $m-1 \le t < 2m-1$, then $d(v_{8m-4}, u_{4t+2}) = 2m-t+1$ and $d(u_0, u_{4t+2}) > d(v_{8m-4}, u_{4t+2})$.
- If $1 \le t \le m-2$, then $d(u_0, u_{4t+3}) = d(v_{8m-4}, u_{4t+3}) = 4 + t$. If $m-1 \le t < 2m-1$, then $d(v_{8m-4}, u_{4t+3}) = 2m - t$ and $d(u_0, u_{4t+3}) > d(v_{8m-4}, u_{4t+3})$.
- $d(u_0, u_1) = 1$, $d(v_{8m-4}, u_1) = 2m+1$, $d(u_0, u_2) = 2$, $d(v_{8m-4}, u_2) = 2m+1$, $d(u_0, u_3) = 3$, and $d(v_{8m-4}, u_3) = 2m$.

Note that $u_0 \in W_{u_0 v_{8m-4}}$ and $v_{8m-4} \in W_{v_{8m-4} u_0}$. Combined with the above discussion we arrive at $|W_{u_0 v_{8m-4}}| = 3m+7$ and $|W_{v_{8m-4} u_0}| = 5m$. Because $m \ge 4$ we may conclude that $|W_{u_0 v_{8m-4}}| < |W_{v_{8m-4} u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 8m+1$, where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+7$ and $|W_{v_{-4} u_0}| = 5m+3$.
- If $n = 8m+2$, where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+6$ and $|W_{v_{-4} u_0}| = 5m+2$.
- If $n = 8m+3$ where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+7$ and $|W_{v_{-4} u_0}| = 5m+3$.
- If $n = 8m+4$, where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+9$ and $|W_{v_{-4} u_0}| = 5m+4$.
- If $n = 8m+5$, where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+7$ and $|W_{v_{-4} u_0}| = 5m+3$.
- If $n = 8m+6$, where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+8$ and $|W_{v_{-4} u_0}| = 5m+6$.
- If $n = 8m+7$, where $m \ge 3$, then $|W_{u_0 v_{-4}}| = 3m+8$ and $|W_{v_{-4} u_0}| = 5m+6$.

In each case we have $|W_{u_0 v_{-4}}| < |W_{v_{-4} u_0}|$ as required. □

Proposition 3.3. *For any $n > 24$, the generalized Petersen graph $GP(n,4)$ is not ℓ-distance-balanced for any $3 \le \ell < \operatorname{diam}(GP(n,4))$.*

Proof. For a given fixed n, we set $D = \operatorname{diam}(GP(n,4))$.

For any $3 \le \ell < D$, we first show that there exists v_j such that $d(u_0, v_j) = \ell$ where $8 \le j \le n/2$. From [18] we recall that there exists j^* such that $d(u_0, u_{j^*}) = D$.

If $n = 8m$, where $m \ge 4$, or $n = 8m+1$, where $m \ge 3$, then from [18] we know that $j^* = 4(m-1)+2$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{4s+2}) = s+3$, where $2 \le s \le m-1$, and $d(u_0, v_{4s}) = s+1$, where $2 \le s \le m$.

If $n = 8m+2$, where $m \ge 3$, or $n = 8m+3$, where $m \ge 3$, then from [18] we know that $j^* = 4m+1$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{4s+1}) = s+2$, where $3 \le s \le m$, and $d(u_0, v_{4s}) = s+1$, where $2 \le s \le m$.

If $n = 8m+4$, where $m \ge 3$, or $n = 8m+5$, where $m \ge 3$, then from [18] we know that $j^* = 4m+2$ and $D = d(u_0, u_{j^*}) = m+4$. Note that $d(u_0, v_{4s+2}) = s+3$, where $2 \le s \le m$, and $d(u_0, v_{4s}) = s+1$, where $2 \le s \le m$.

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If $n = 8m + 6$, where $m \geq 3$, then from [18] we know that $j^* = 4m + 3$ and $D = d(u_0, u_{j^*}) = m + 4$. Note that $d(u_0, v_{4s+3}) = s + 3$, where $2 \leq s \leq m$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

If $n = 8m + 7$, where $m \geq 3$, then from [18] we know that $j^* = 4m + 2$ and $D = d(u_0, u_{j^*}) = m + 4$. Note that $d(u_0, v_{4s+2}) = s + 3$, where $2 \leq s \leq m$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

By the above discussion, there exists j, where $8 \leq j \leq n/2$, such that $d(u_0, v_j) = \ell$ for any $3 \leq \ell < D$. Define the following sets of vertices:

$$\begin{aligned}
V_1 &= \{u_i : \ 1 \leq i \leq j-1\} \cup \{v_i : \ 1 \leq i \leq j-1\}, \\
V_2 &= \{u_i : \ j+1 \leq i \leq n-1\} \cup \{v_i : \ j+1 \leq i \leq n-1\}, \\
W^1_{u_0 v_j} &= W_{u_0 v_j} \cap (V_1 \cup \{u_0, v_0, u_j, v_j\}), \\
W^1_{v_j u_0} &= W_{v_j u_0} \cap (V_1 \cup \{u_0, v_0, u_j, v_j\}), \\
W^2_{u_0 v_j} &= W_{u_0 v_j} \cap (V_2 \cup \{u_0, v_0, u_j, v_j\}), \\
W^2_{v_j u_0} &= W_{v_j u_0} \cap (V_2 \cup \{u_0, v_0, u_j, v_j\}).
\end{aligned}$$

Because $8 \leq j \leq n/2$, we have $|W^2_{u_0 v_j}| = |W^1_{u_0 v_{n-j}}|$ and $|W^2_{v_j u_0}| = |W^1_{v_{n-j} u_0}|$. So

$$\begin{aligned}
|W_{u_0 v_j}| &= |W^1_{u_0 v_j}| + |W^2_{u_0 v_j}| - 2 = |W^1_{u_0 v_j}| + |W^1_{u_0 v_{n-j}}| - 2 \quad \text{and} \\
|W_{v_j u_0}| &= |W^1_{v_j u_0}| + |W^2_{v_j u_0}| - 2 = |W^1_{v_j u_0}| + |W^1_{v_{n-j} u_0}| - 2.
\end{aligned}$$

In the following we will compute $|W^1_{u_0 v_j}|$ and $|W^1_{v_j u_0}|$ where $8 \leq j \leq n-8$. The computation is divided into eight cases, and for transparency and non-replication purposes, present only the first case in detail. Details for the other seven cases are given in Appendix B.

The computation of $|W^1_{u_0 v_{4s}}|$ and $|W^1_{v_{4s} u_0}|$, where $s \geq 5$ is odd is as follows.

- If $0 \leq t < s$, then $d(u_0, v_{4t}) = 1 + t$ and $d(v_{4s}, v_{4t}) = s - t$.
  If $0 \leq t < \frac{s-1}{2}$, then $d(u_0, v_{4t}) < d(v_{4s}, v_{4t})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{4t}) > d(v_{4s}, v_{4t})$.
- If $0 \leq t < s$, then $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_{4s}, v_{4t+1}) = s - t + 3$.
  If $0 \leq t < \frac{s+1}{2}$, then $d(u_0, v_{4t+1}) < d(v_{4s}, v_{4t+1})$.
  If $\frac{s+1}{2} < t < s$, then $d(u_0, v_{4t+1}) > d(v_{4s}, v_{4t+1})$.
- If $0 \leq t < s$, then $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_{4s}, v_{4t+2}) = s - t + 3$.
  If $0 \leq t \leq \frac{s-1}{2}$, then $d(u_0, v_{4t+2}) < d(v_{4s}, v_{4t+2})$.
  If $\frac{s+1}{2} \leq t < s$, then $d(u_0, v_{4t+2}) > d(v_{4s}, v_{4t+2})$.
- If $0 \leq t < s$, then $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_{4s}, v_{4t+3}) = s - t + 2$.
  If $0 \leq t < \frac{s-1}{2}$, then $d(u_0, v_{4t+3}) < d(v_{4s}, v_{4t+3})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{4t+3}) > d(v_{4s}, v_{4t+3})$.
- If $1 \leq t \leq s$, then $d(u_0, u_{4t}) = 2 + t$ and $d(v_{4s}, u_{4t}) = s - t + 1$.
  If $1 \leq t < \frac{s-1}{2}$, then $d(u_0, u_{4t}) < d(v_{4s}, u_{4t})$.
  If $\frac{s-1}{2} < t \leq s$, then $d(u_0, u_{4t}) > d(v_{4s}, u_{4t})$.
- If $1 \leq t < s$, then $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_{4s}, u_{4t+1}) = s - t + 2$.
  If $1 \leq t < \frac{s-1}{2}$, then $d(u_0, u_{4t+1}) < d(v_{4s}, u_{4t+1})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, u_{4t+1}) > d(v_{4s}, u_{4t+1})$.

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- If $1 \leq t < s$, then $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_{4s}, u_{4t+2}) = s - t + 2$.
  If $1 \leq t \leq \frac{s-3}{2}$, then $d(u_0, u_{4t+2}) < d(v_{4s}, u_{4t+2})$.
  If $\frac{s-1}{2} \leq t < s$, then $d(u_0, u_{4t+2}) > d(v_{4s}, u_{4t+2})$.
- If $1 \leq t < s$, then $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_{4s}, u_{4t+3}) = s - t + 1$.
  If $1 \leq t < \frac{s-3}{2}$, then $d(u_0, u_{4t+3}) < d(v_{4s}, u_{4t+3})$.
  If $\frac{s-3}{2} < t < s$, then $d(u_0, u_{4t+3}) > d(v_{4s}, u_{4t+3})$.
- $d(u_0, u_1) = 1$, $d(v_{4s}, u_1) = s + 2$, $d(u_0, u_2) = 2$, $d(v_{4s}, u_2) = s + 2$, $d(u_0, u_3) = 3$, and $d(v_{4s}, u_3) = s + 1$.

Note that $u_0 \in W^1_{u_0 v_{4s}}$ and $v_{4s} \in W^1_{v_{4s} u_0}$. Combined with the above discussion we arrive at $|W^1_{u_0 v_{4s}}| = 4s - 3$ and $|W^1_{v_{4s} u_0}| = 4s - 1$.

The conclusions in the remaining cases are as follows:

- If $s \geq 4$ and s is even, then $|W^1_{u_0 v_{4s}}| = 4s - 1$ and $|W^1_{v_{4s} u_0}| = 4s + 1$.
- If $s \geq 3$ and s is odd, then $|W^1_{u_0 v_{4s+1}}| = 4s + 1$ and $|W^1_{v_{4s+1} u_0}| = 4s + 3$.
- If $s \geq 4$ and s is even, then $|W^1_{u_0 v_{4s+1}}| = 4s - 3$ and $|W^1_{v_{4s+1} u_0}| = 4s - 1$.
- If $s \geq 5$ and s is odd, then $|W^1_{u_0 v_{4s+2}}| = 4s - 1$ and $|W^1_{v_{4s+2} u_0}| = 4s + 1$.
- If $s \geq 4$ and s is even, then $|W^1_{u_0 v_{4s+2}}| = 4s + 1$ and $|W^1_{v_{4s+2} u_0}| = 4s + 3$.
- If $s \geq 5$ and s is odd, then $|W^1_{u_0 v_{4s+3}}| = 4s + 1$ and $|W^1_{v_{4s+3} u_0}| = 4s + 3$.
- If $s \geq 4$ and s is even, then $|W^1_{u_0 v_{4s+3}}| = 4s + 1$ and $|W^1_{v_{4s+3} u_0}| = 4s + 3$.
- $|W^1_{u_0 v_8}| = 8$ and $|W^1_{v_8 u_0}| = 8$.
- $|W^1_{u_0 v_{10}}| = 11$ and $|W^1_{v_{10} u_0}| = 10$.
- $|W^1_{u_0 v_{11}}| = 10$ and $|W^1_{v_{11} u_0}| = 10$.
- $|W^1_{u_0 v_{12}}| = 10$ and $|W^1_{v_{12} u_0}| = 11$.
- $|W^1_{u_0 v_{14}}| = 12$ and $|W^1_{v_{14} u_0}| = 13$.
- $|W^1_{u_0 v_{15}}| = 14$ and $|W^1_{v_{15} u_0}| = 15$.

When $n \geq 26$, from the above computation of $|W^1_{u_0 v_j}|$ and $|W^1_{v_j u_0}|$, where $8 \leq j \leq n - 8$, for any $3 \leq \ell < D$ we know that there exists j where $d(u_0, v_j) = \ell$ and $8 \leq j \leq n/2$ such that $|W_{u_0 v_j}| < |W_{v_j u_0}|$. When $n = 25$, we have $d(u_0, v_8) = 3$, $d(u_0, v_{12}) = 4$, $d(u_0, v_{11}) = 5$, and $D(GP(25, 4)) = 6$. From the above computation of $|W^1_{u_0 v_j}|$ and $|W^1_{v_j u_0}|$, we know that $|W_{u_0 v_j}| < |W_{v_j u_0}|$ for any $j \in \{8, 11, 12\}$. ☐

4. Concluding remarks

In this paper, we prove that $GP(n, 3)$ is not ℓ-distance-balanced for $n > 16$ and $1 \leq \ell < \mathrm{diam}(GP(n, 3))$. We also prove that $GP(n, 4)$ is not ℓ-distance-balanced for $n > 24$ and $1 \leq \ell < \mathrm{diam}(GP(n, 4))$. Earlier it was proved in [20] that $GP(n, 2)$ is not ℓ-distance-balanced for $n > 11$ and $1 \leq \ell < \mathrm{diam}(GP(n, 2))$.

From Proposition 2.1 we know that $|W_{v_0 u_0}| - |W_{u_0 v_0}|$ is about $\frac{1}{3}$ times of $|V(GP(n, 3))|$. From Proposition 3.1 we know that $|W_{v_0 u_0}| - |W_{u_0 v_0}|$ is about $\frac{1}{4}$ times of $|V(GP(n, 4))|$. The above observations encourage us to try to consider Conjecture 1.2 for distance-balancedness of generalized Petersen graph $GP(n, k)$ for $k \geq 5$. The authors in [21] proved that for any $k \geq 2$ and $n > 6k^2$ $GP(n, k)$ is not distance-balanced. If Conjecture 1.2 is right, the "$6k^2$" in [21] can be greatly improved.

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Similarly, From Proposition 2.2, we know that $|W_{v_{-3}u_0}| - |W_{u_0v_{-3}}|$ is about $\frac{1}{6}$ times of $|V(GP(n,3))|$. From Proposition 3.2, we know that $|W_{v_{-4}u_0}| - |W_{u_0v_{-4}}|$ is about $\frac{1}{8}$ times of $|V(GP(n,4))|$. So we can consider Conjecture 1.2 for 2-distance-balancedness of generalized Petersen graph $GP(n,k)$ for $k \geq 5$.

The discussions on distance-balancedness and 2-distance-balancedness of $GP(n,k)$ for $k \geq 5$ may be merged into fewer cases because $|W_{v_0u_0}| - |W_{u_0v_0}|$ and $|W_{v_{-k}u_0}| - |W_{u_0v_{-k}}|$ are big relative to $|V(GP(n,k))|$. This is the work we will do.

For $3 \leq \ell < \operatorname{diam}(GP(n,k))$, the ℓ-distance-balancedness of Conjecture 1.2 for $k \geq 5$ can not easily been investigated. A new approach may be needed.

Since Miklavič and Šparl proposed the two conjectures about the ℓ-distance-balancedness of $GP(n,k)$ in 2018, there are few positive results appeared in the past several years. Up to our knowledge, the discussion is complicated even for some special pairs of (n,k). Up to now, there are only a few pairs of (n,k), for which the ℓ-distance-balancedness of $GP(n,k)$ are studied completely. So there are much many pairs of (n,k), for which the ℓ-distance-balancedness of $GP(n,k)$ will be worth studying in the future.

References

[1] A. Abiad, B. Brimkov, A. Erey, L Leshock, X. Martínez-Rivera, S. O, S.-Y. Song, J. Williford, On the Wiener index, distance cospectrality and transmission-regular graphs, Discrete Appl. Math. 230 (2017) 1–10.

[2] A. Ali, T. Došlić, Mostar index: Results and perspectives, Appl. Math. Comput. 404 (2021) 126245.

[3] K. Balakrishnan, B. Brešar, M. Changat, S. Klavžar, A. Vesel, P. Žigert Pleteršek, Equal opportunity networks, distance-balanced graphs, and Wiener game, Discrete Opt. 12 (2014) 150–154.

[4] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl, A.R. Subhamathi, Strongly distance-balanced graphs and graph products, European J. Combin. 30 (2009) 1048–1053.

[5] S. Cabello, P. Lukšič, The complexity of obtaining a distance-balanced graph, Electron. J. Combin. 18 (2011) Paper 49.

[6] M. Cavaleri, A. Donno, Distance-balanced graphs and travelling salesman problems, Ars Math. Contemp. 19 (2020) 311–324.

[7] B. Fernández, A. Hujdurović, On some problems regarding distance-balanced graphs, European J. Combin. 106 (2022) 103593.

[8] B. Fernández, Š. Miklavič, S. Penjić, On certain regular nicely distance-balanced graphs, Rev. Un. Mat. Argentina 65 (2023) 165–185.

[9] B. Frelih, Različni vidiki povezavne regularnosti v grafih, Ph.D. (in Slovene), University of Primorska, 2014.

[10] B. Frelih, Š. Miklavič, On 2-distance-balanced graphs, Ars Math. Contemp. 15 (2018) 81–95.

[11] K. Handa, Bipartite graphs with balanced (a,b)-partitions, Ars Combin. 51 (1999) 113–119.

[12] A. Ilić, S. Klavžar, M. Milanović, On distance-balanced graphs. European J. Combin. 31(2010) 733–737.

[13] J. Jerebic, S. Klavžar, D.F. Rall, Distance-balanced graphs, Ann. Combin. 12 (2008) 71–79.

[14] J. Jerebic, S. Klavžar, G. Rus, On ℓ-distance-balanced product graphs, Graphs Combin. 37 (2021) 369–379.

[15] K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, Distance-balanced graphs: symmetry conditions, Discrete Math. 306 (2006) 1881–1894.

[16] K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, The strongly distance-balanced property of the generalized Petersen graphs, Ars Math. Contemp. 2 (2009) 41–47.

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[17] K. Kutnar, Š. Miklavič, Nicely distance-balanced graphs, European J. Combin. 39 (2014) 57–67.

[18] G. Ma, J. Wang, S, Klavžar, On distance-balanced generalized Petersen graphs, Ann. Comb. 28 (2024) 329–349.

[19] Š. Miklavič, P. Šparl, On the connectivity of bipartite distance-balanced graphs, European J. Combin. 33 (2012) 237–247.

[20] Š. Miklavič, P. Šparl, ℓ-distance-balanced graphs, Discrete Appl. Math. 244 (2018) 143–154.

[21] R. Yang, X. Hou, N. Li, W. Zhong, A note on the distance-balanced property of generalized Petersen graphs, Electron. J. Combin. 16 (2009) #N33.

Appendix A

Proof of the remaining cases of Proposition 2.1. (2) When $n = 6m+1$ where $m \geq 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$.

$d(u_0, v_{3t}) = 1+t$ and $d(v_0, v_{3t}) = t$ where $1 \leq t \leq m$. $d(u_0, v_{3t+1}) = 2+t$ where $0 \leq t < m$. $d(v_0, v_{3t+1}) = 3+t$ where $0 \leq t \leq m-2$, and $d(v_0, v_{3(m-1)+1}) = m+1$. $d(u_0, v_{3t+2}) = 3+t$ and $d(v_0, v_{3t+2}) = 4+t$ where $0 \leq t < m$.

$d(u_0, u_{3t}) = 2+t$ and $d(v_0, u_{3t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{3t+1}) = 3+t$ and $d(v_0, u_{3t+1}) = 2+t$ where $1 \leq t < m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{3t+2}) = 4+t$ and $d(v_0, u_{3t+2}) = 3+t$ where $1 \leq t < m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(2m+1)+1 = 4m+3$ and $|W_{v_0u_0}| = 2(4m-2)+1 = 8m-3$. Because $m \geq 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(3) When $n = 6m+2$ where $m \geq 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$.

$d(u_0, v_{3t}) = 1+t$ and $d(v_0, v_{3t}) = t$ where $1 \leq t \leq m$. $d(u_0, v_{3t+1}) = 2+t$ and $d(v_0, v_{3t+1}) = 3+t$ where $0 \leq t \leq m$. $d(u_0, v_{3t+2}) = 3+t$ where $0 \leq t < m$. $d(v_0, v_{3t+2}) = 4+t$ where $0 \leq t \leq m-2$, and $d(v_0, v_{3(m-1)+2}) = m+1$.

$d(u_0, u_{3t}) = 2+t$ and $d(v_0, u_{3t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{3t+1}) = 3+t$ and $d(v_0, u_{3t+1}) = 2+t$ where $1 \leq t \leq m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{3t+2}) = 4+t$ and $d(v_0, u_{3t+2}) = 3+t$ where $1 \leq t < m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(2m+1)+2 = 4m+4$ and $|W_{v_0u_0}| = 2(4m-1)+2 = 8m$. Because $m \geq 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(4) When $n = 6m+3$ where $m \geq 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$.

$d(u_0, v_{3t}) = 1+t$ and $d(v_0, v_{3t}) = t$ where $1 \leq t \leq m$. $d(u_0, v_{3t+1}) = 2+t$ and $d(v_0, v_{3t+1}) = 3+t$ where $0 \leq t \leq m$. $d(u_0, v_{3t+2}) = 3+t$ and $d(v_0, v_{3t+2}) = 4+t$ where $0 \leq t < m$.

$d(u_0, u_{3t}) = 2+t$ and $d(v_0, u_{3t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{3t+1}) = 3+t$ and $d(v_0, u_{3t+1}) = 2+t$ where $1 \leq t \leq m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{3t+2}) = 4+t$ and $d(v_0, u_{3t+2}) = 3+t$ where $1 \leq t < m$.

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Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(2m+3)+1 = 4m+7$ and $|W_{v_0u_0}| = 2(4m-1)+1 = 8m-1$. Because $m \geq 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(5) When $n = 6m+4$ where $m \geq 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$.

$d(u_0, v_{3t}) = 1+t$ and $d(v_0, v_{3t}) = t$ where $1 \leq t \leq m$. $d(u_0, v_{3t+1}) = 2+t$ where $0 \leq t \leq m$. $d(v_0, v_{3t+1}) = 3+t$ where $0 \leq t \leq m-1$, and $d(v_0, v_{3m+1}) = m+1$. $d(u_0, v_{3t+2}) = 3+t$ and $d(v_0, v_{3t+2}) = 4+t$ where $0 \leq t \leq m$.

$d(u_0, u_{3t}) = 2+t$ and $d(v_0, u_{3t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{3t+1}) = 3+t$ and $d(v_0, u_{3t+1}) = 2+t$ where $1 \leq t \leq m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{3t+2}) = 4+t$ and $d(v_0, u_{3t+2}) = 3+t$ where $1 \leq t \leq m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(2m+2)+2 = 4m+6$ and $|W_{v_0u_0}| = 2 \times 4m+2 = 8m+2$. Because $m \geq 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(6) When $n = 6m+5$ where $m \geq 2$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$.

$d(u_0, v_{3t}) = 1+t$ and $d(v_0, v_{3t}) = t$ where $1 \leq t \leq m$. $d(u_0, v_{3t+1}) = 2+t$ and $d(v_0, v_{3t+1}) = 3+t$ where $0 \leq t \leq m$. $d(u_0, v_{3t+2}) = 3+t$ where $0 \leq t \leq m-1$ and $d(u_0, v_{3m+2}) = m+2$. $d(v_0, v_{3t+2}) = 4+t$ where $0 \leq t \leq m-2$, $d(v_0, v_{3(m-1)+2}) = m+2$ and $d(v_0, v_{3m+2}) = m+1$.

$d(u_0, u_{3t}) = 2+t$ and $d(v_0, u_{3t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{3t+1}) = 3+t$ and $d(v_0, u_{3t+1}) = 2+t$ where $1 \leq t \leq m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{3t+2}) = 4+t$ and $d(v_0, u_{3t+2}) = 3+t$ where $1 \leq t \leq m-1$. $d(u_0, u_{3m+2}) = m+3$ and $d(v_0, u_{3m+2}) = m+2$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(2m+2)+1 = 4m+5$ and $|W_{v_0u_0}| = 2(4m+1)+1 = 8m+3$. Because $m \geq 2$, $|W_{u_0v_0}| < |W_{v_0u_0}|$. □

Proof of the remaining cases of Proposition 2.2. (2) When $n = 6m+1$ where $m \geq 3$.

Note that $n-3 = 6m-2 = 3(2m-1)+1$.

$d(u_0, v_{3t}) = d(v_{6m-2}, v_{3t}) = 1+t$ when $0 \leq t \leq m$. $d(u_0, v_{3t}) = d(v_{6m-2}, v_{3t}) = 2m-t+2$ when $m+1 \leq t \leq 2m-1$. $d(u_0, v_{3t+1}) = 2+t$ and $d(u_0, v_{3t+1}) < d(v_{6m-2}, v_{3t+1})$ when $0 \leq t \leq m-2$. $d(v_{6m-2}, v_{3t+1}) = 2m-t-1$ and $d(u_0, v_{3t+1}) > d(v_{6m-2}, v_{3t+1})$ when $m-1 \leq t < 2m-1$. $d(u_0, v_{3t+2}) = 3+t$ and $d(u_0, v_{3t+2}) < d(v_{6m-2}, v_{3t+2})$ when $0 \leq t \leq m-1$. $d(u_0, v_{3t+2}) = d(v_{6m-2}, v_{3t+2}) = 2m-t+2$ when $m \leq t < 2m-1$.

$d(u_0, u_{3t}) = d(v_{6m-2}, u_{3t}) = 2+t$ when $1 \leq t \leq m-1$. $d(v_{6m-2}, u_{3t}) = 2m-t+1$ and $d(u_0, u_{3t}) > d(v_{6m-2}, u_{3t})$ when $m \leq t \leq 2m-1$. $d(u_0, u_1) = 1$ and $d(v_{6m-2}, u_1) = 2m$. $d(u_0, u_{3t+1}) = d(v_{6m-2}, u_{3t+1}) = 3+t$ when $1 \leq t \leq m-2$. $d(v_{6m-2}, u_{3t+1}) = 2m-t$ and $d(u_0, u_{3t+1}) > d(v_{6m-2}, u_{3t+1})$ when $m-1 \leq t \leq 2m-1$. $d(u_0, u_2) = 2$ and $d(v_{6m-2}, u_2) = 2m+1$. $d(u_0, u_{3t+2}) =$

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$d(v_{6m-2}, u_{3t+2}) = 4+t$ when $1 \le t \le m-2$. $d(v_{6m-2}, u_{3t+2}) = 2m-t+1$ and $d(u_0, u_{3t+2}) > d(v_{6m-2}, u_{3t+2})$ when $m-1 \le t < 2m-1$.

Note that $u_0 \in W_{u_0v_{6m-2}}$ and $v_{6m-2} \in W_{v_{6m-2}u_0}$. Combined with the above discussion, $|W_{u_0v_{6m-2}}| = 2m+4$ and $|W_{v_{6m-2}u_0}| = 4m+2$. Because $m \ge 3$, $|W_{u_0v_{6m-2}}| < |W_{v_{6m-2}u_0}|$.

(3) When $n = 6m+2$ where $m \ge 3$.

Note that $n-3 = 6m-1 = 3(2m-1)+2$.

$d(u_0, v_{3t}) = d(v_{6m-1}, v_{3t}) = 1+t$ when $0 \le t \le m+1$. $d(u_0, v_{3t}) = d(v_{6m-1}, v_{3t}) = 2m-t+3$ when $m+2 \le t \le 2m-1$. $d(u_0, v_{3t+1}) = 2+t$ and $d(u_0, v_{3t+1}) < d(v_{6m-1}, v_{3t+1})$ when $0 \le t \le m-1$. $d(u_0, v_{3t+1}) = d(v_{6m-1}, v_{3t+1}) = 2m-t+2$ when $m \le t \le 2m-1$. $d(u_0, v_{3t+2}) = 3+t$ and $d(u_0, v_{3t+2}) < d(v_{6m-1}, v_{3t+2})$ when $0 \le t \le m-3$. $d(u_0, v_{3(m-2)+2}) = d(v_{6m-1}, v_{3(m-2)+2}) = m+1$. $d(v_{6m-1}, v_{3t+2}) = 2m-t-1$ and $d(u_0, v_{3t+2}) > d(v_{6m-1}, v_{3t+2})$ when $m-1 \le t < 2m-1$.

$d(u_0, u_{3t}) = d(v_{6m-1}, u_{3t}) = 2+t$ when $1 \le t \le m$. $d(v_{6m-1}, u_{3t}) = 2m-t+2$ and $d(u_0, u_{3t}) > d(v_{6m-1}, u_{3t})$ when $m+1 \le t \le 2m-1$. $d(u_0, u_1) = 1$ and $d(v_{6m-1}, u_1) = 2m+1$. $d(u_0, u_{3t+1}) = d(v_{6m-1}, u_{3t+1}) = 3+t$ when $1 \le t \le m-1$. $d(v_{6m-1}, u_{3t+1}) = 2m-t+1$ and $d(u_0, u_{3t+1}) > d(v_{6m-1}, u_{3t+1})$ when $m \le t \le 2m-1$. $d(u_0, u_2) = 2$ and $d(v_{6m-1}, u_2) = 2m$. $d(u_0, u_{3t+2}) = d(v_{6m-1}, u_{3t+2}) = 4+t$ when $1 \le t \le m-2$. $d(v_{6m-1}, u_{3t+2}) = 2m-t$ and $d(u_0, u_{3t+2}) > d(v_{6m-1}, u_{3t+2})$ when $m-1 \le t \le 2m-1$.

Note that $u_0 \in W_{u_0v_{6m-1}}$ and $v_{6m-1} \in W_{v_{6m-1}u_0}$. Combined with the above discussion, $|W_{u_0v_{6m-1}}| = 2m+3$ and $|W_{v_{6m-1}u_0}| = 4m+1$. Because $m \ge 3$, $|W_{u_0v_{6m-1}}| < |W_{v_{6m-1}u_0}|$.

(4) When $n = 6m+3$ where $m \ge 3$.

Note that $n-3 = 6m = 3 \times 2m$.

$d(u_0, v_{3t}) = d(v_{6m}, v_{3t}) = 1+t$ when $0 \le t \le m-1$. $d(v_{6m}, v_{3t}) = 2m-t$ and $d(u_0, v_{3t}) > d(v_{6m}, v_{3t})$ when $m \le t < 2m$. $d(u_0, v_{3t+1}) = 2+t$ and $d(u_0, v_{3t+1}) < d(v_{6m}, v_{3t+1})$ when $0 \le t \le m$. $d(u_0, v_{3t+1}) = d(v_{6m}, v_{3t+1}) = 2m-t+3$ when $m+1 \le t < 2m$. $d(u_0, v_{3t+2}) = 3+t$ and $d(u_0, v_{3t+2}) < d(v_{6m}, v_{3t+2})$ when $0 \le t \le m-1$. $d(u_0, v_{3t+2}) = d(v_{6m}, v_{3t+2}) = 2m-t+2$ when $m \le t < 2m$.

$d(u_0, u_{3t}) = d(v_{6m}, u_{3t}) = 2+t$ when $1 \le t \le m-1$. $d(v_{6m}, u_{3t}) = 2m-t+1$ and $d(u_0, u_{3t}) > d(v_{6m}, u_{3t})$ when $m \le t \le 2m$. $d(u_0, u_1) = 1$ and $d(v_{6m}, u_1) = 2m+2$. $d(u_0, u_{3t+1}) = d(v_{6m}, u_{3t+1}) = 3+t$ when $1 \le t \le m-1$. $d(v_{6m}, u_{3t+1}) = 2m-t+2$ and $d(u_0, u_{3t+1}) > d(v_{6m}, u_{3t+1})$ when $m \le t < 2m$. $d(u_0, u_2) = 2$ and $d(v_{6m}, u_2) = 2m+1$. $d(u_0, u_{3t+2}) = d(v_{6m}, u_{3t+2}) = 4+t$ when $1 \le t \le m-2$. $d(v_{6m}, u_{3t+2}) = 2m-t+1$ and $d(u_0, u_{3t+2}) > d(v_{6m}, u_{3t+2})$ when $m-1 \le t < 2m$.

Note that $u_0 \in W_{u_0v_{6m}}$ and $v_{6m} \in W_{v_{6m}u_0}$. Combined with the above discussion, $|W_{u_0v_{6m}}| = 2m+6$ and $|W_{v_{6m}u_0}| = 4m+3$. Because $m \ge 3$, $|W_{u_0v_{6m}}| < |W_{v_{6m}u_0}|$.

(5) When $n = 6m+4$ where $m \ge 3$.

Note that $n-3 = 6m+1 = 3 \times 2m+1$.

$d(u_0, v_{3t}) = d(v_{6m+1}, v_{3t}) = 1+t$ when $0 \le t \le m+1$. $d(u_0, v_{3t}) = d(v_{6m+1}, v_{3t}) = 2m-t+3$ when $m+2 \le t \le 2m$. $d(u_0, v_{3t+1}) = 2+t$ and $d(u_0, v_{3t+1}) < d(v_{6m+1}, v_{3t+1})$ when $0 \le t \le m-2$. $d(u_0, v_{3(m-1)+1}) = d(v_{6m+1}, v_{3(m-1)+1}) =$

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$m+1$. $d(v_{6m+1}, v_{3t+1}) = 2m-t$ and $d(u_0, v_{3t+1}) > d(v_{6m+1}, v_{3t+1})$ when $m \le t < 2m$. $d(u_0, v_{3t+2}) = 3+t$ and $d(u_0, v_{3t+2}) < d(v_{6m+1}, v_{3t+2})$ when $0 \le t \le m-1$. $d(u_0, v_{3t+2}) = d(v_{6m+1}, v_{3t+2}) = 2m-t+3$ when $m \le t < 2m$.

$d(u_0, u_{3t}) = d(v_{6m+1}, u_{3t}) = 2+t$ when $1 \le t \le m$. $d(v_{6m+1}, u_{3t}) = 2m-t+2$ and $d(u_0, u_{3t}) > d(v_{6m+1}, u_{3t})$ when $m+1 \le t \le 2m$. $d(u_0, u_1) = 1$ and $d(v_{6m+1}, u_1) = 2m+1$. $d(u_0, u_{3t+1}) = d(v_{6m+1}, u_{3t+1}) = 3+t$ when $1 \le t \le m-1$. $d(v_{6m+1}, u_{3t+1}) = 2m-t+1$ and $d(u_0, u_{3t+1}) > d(v_{6m+1}, u_{3t+1})$ when $m \le t \le 2m$. $d(u_0, u_2) = 2$ and $d(v_{6m}, u_2) = 2m+2$. $d(u_0, u_{3t+2}) = d(v_{6m+1}, u_{3t+2}) = 4+t$ when $1 \le t \le m-1$. $d(v_{6m+1}, u_{3t+2}) = 2m-t+2$ and $d(u_0, u_{3t+2}) > d(v_{6m+1}, u_{3t+2})$ when $m \le t < 2m$.

Note that $u_0 \in W_{u_0v_{6m+1}}$ and $v_{6m+1} \in W_{v_{6m+1}u_0}$. Combined with the above discussion, $|W_{u_0v_{6m+1}}| = 2m+4$ and $|W_{v_{6m+1}u_0}| = 4m+2$. Because $m \ge 3$, $|W_{u_0v_{6m+1}}| < |W_{v_{6m+1}u_0}|$.

(6) When $n = 6m+5$ where $m \ge 2$.

Note that $n-3 = 6m+2 = 3 \times 2m + 2$.

$d(u_0, v_{3t}) = d(v_{6m+2}, v_{3t}) = 1+t$ when $0 \le t \le m+1$. $d(u_0, v_{3t}) = d(v_{6m+2}, v_{3t}) = 2m-t+4$ when $m+2 \le t \le 2m$. $d(u_0, v_{3t+1}) = 2+t$ and $d(u_0, v_{3t+1}) < d(v_{6m+2}, v_{3t+1})$ when $0 \le t \le m$. $d(u_0, v_{3t+1}) = d(v_{6m+2}, v_{3t+1}) = 2m-t+3$ when $m+1 \le t \le 2m$. $d(u_0, v_{3t+2}) = 3+t$ and $d(u_0, v_{3t+2}) < d(v_{6m+2}, v_{3t+2})$ when $0 \le t \le m-2$. $d(v_{6m+2}, v_{3t+2}) = 2m-t$ and $d(u_0, v_{3t+2}) > d(v_{6m+1}, v_{3t+2})$ when $m-1 \le t < 2m$.

$d(u_0, u_{3t}) = d(v_{6m+2}, u_{3t}) = 2+t$ when $1 \le t \le m$. $d(v_{6m+2}, u_{3t}) = 2m-t+3$ and $d(u_0, u_{3t}) > d(v_{6m+2}, u_{3t})$ when $m+1 \le t \le 2m$. $d(u_0, u_1) = 1$ and $d(v_{6m+2}, u_1) = 2m+2$. $d(u_0, u_{3t+1}) = d(v_{6m+2}, u_{3t+1}) = 3+t$ when $1 \le t \le m-1$. $d(v_{6m+2}, u_{3t+1}) = 2m-t+2$ and $d(u_0, u_{3t+1}) > d(v_{6m+2}, u_{3t+1})$ when $m \le t \le 2m$. $d(u_0, u_2) = 2$ and $d(v_{6m}, u_2) = 2m+1$. $d(u_0, u_{3t+2}) = d(v_{6m+2}, u_{3t+2}) = 4+t$ when $1 \le t \le m-2$. $d(v_{6m+2}, u_{3t+2}) = 2m-t+1$ and $d(u_0, u_{3t+2}) > d(v_{6m+2}, u_{3t+2})$ when $m-1 \le t \le 2m$.

Note that $u_0 \in W_{u_0v_{6m+2}}$ and $v_{6m+2} \in W_{v_{6m+2}u_0}$. Combined with the above discussion, $|W_{u_0v_{6m+2}}| = 2m+5$ and $|W_{v_{6m+2}u_0}| = 4m+5$. Because $m \ge 2$, $|W_{u_0v_{6m+2}}| < |W_{v_{6m+2}u_0}|$. □

Proof of the remaining cases of Proposition 2.3. (1a) The computation of $|W^1_{u_0v_{3s}}|$ and $|W^1_{v_{3s}u_0}|$ when $s = 3$.

$d(u_0, v_0) = 1$ and $d(v_9, v_0) = 3$. $d(u_0, v_3) = d(v_9, v_3) = 2$. $d(u_0, v_6) = 3$ and $d(v_9, v_6) = 1$. $d(u_0, v_1) = 2$ and $d(v_9, v_1) = 6$. $d(u_0, v_4) = 3$ and $d(v_9, v_4) = 5$. $d(u_0, v_7) = 4$ and $d(v_9, v_7) = 4$. $d(u_0, v_2) = 3$ and $d(v_9, v_2) = 5$. $d(u_0, v_5) = 4$ and $d(v_9, v_5) = 4$. $d(u_0, v_8) = 5$ and $d(v_9, v_8) = 3$. So $v_0, v_1, v_4, v_2 \in W^1_{u_0v_9}$ and $v_6, v_8 \in W^1_{v_9u_0}$.

$d(u_0, u_3) = 3$ and $d(v_9, u_3) = 3$. $d(u_0, u_6) = 4$ and $d(v_9, u_6) = 2$. $d(u_0, u_9) = 5$ and $d(v_9, u_9) = 1$. $d(u_0, u_1) = 1$ and $d(v_9, u_1) = 5$. $d(u_0, u_4) = 4$ and $d(v_9, u_4) = 4$. $d(u_0, u_7) = 5$ and $d(v_9, u_7) = 3$. $d(u_0, u_2) = 2$ and $d(v_9, u_2) = 4$. $d(u_0, u_5) = 5$ and $d(v_9, u_5) = 3$. $d(u_0, u_8) = 6$ and $d(v_9, u_8) = 2$. So $u_1, u_2 \in W^1_{u_0v_9}$ and $u_6, u_9, u_7, u_5, u_8 \in W^1_{v_9u_0}$.

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Note that $u_0 \in W^1_{u_0v_9}$ and $v_9 \in W^1_{v_9u_0}$. Combined with the above discussion, $|W^1_{u_0v_9}| = 7$ and $|W^1_{v_9u_0}| = 8$.

(1b) Computation of $|W^1_{u_0v_{3s}}|$ and $|W^1_{v_{3s}u_0}|$ when s is even and $s \geq 2$.

When $s = 2$,

$d(u_0, v_0) = 1$ and $d(v_6, v_0) = 2$. $d(u_0, v_3) = 2$ and $d(v_6, v_3) = 1$. $d(u_0, v_1) = 2$ and $d(v_6, v_1) = 5$. $d(u_0, v_4) = 3$ and $d(v_6, v_4) = 4$. $d(u_0, v_2) = 3$ and $d(v_6, v_2) = 4$. $d(u_0, v_5) = 4$ and $d(v_6, v_5) = 3$. So $v_0, v_1, v_2, v_4 \in W^1_{u_0v_6}$ and $v_3, v_5 \in W^1_{v_6u_0}$.

$d(u_0, u_3) = 3$ and $d(v_6, u_3) = 2$. $d(u_0, u_6) = 4$ and $d(v_6, u_6) = 1$. $d(u_0, u_1) = 1$ and $d(v_6, u_1) = 4$. $d(u_0, u_4) = 4$ and $d(v_6, u_4) = 3$. $d(u_0, u_2) = 2$ and $d(v_6, u_2) = 3$. $d(u_0, u_5) = 5$ and $d(v_6, u_5) = 2$. So $u_1, u_2 \in W^1_{u_0v_6}$ and $u_3, u_4, u_5, u_6 \in W^1_{v_6u_0}$.

Note that $u_0 \in W^1_{u_0v_6}$ and $v_6 \in W^1_{v_6u_0}$. Combined with the above discussion, $|W^1_{u_0v_6}| = 7$ and $|W^1_{v_6u_0}| = 7$.

When $s \geq 4$,

$d(u_0, v_{3t}) = 1 + t$ and $d(v_{3s}, v_{3t}) = s - t$ where $0 \leq t < s$. When $0 \leq t \leq \frac{s-2}{2}$, $d(u_0, v_{3t}) < d(v_{3s}, v_{3t})$. When $\frac{s}{2} \leq t < s$, $d(u_0, v_{3t}) > d(v_{3s}, v_{3t})$. $d(u_0, v_{3t+1}) = 2 + t$ and $d(v_{3s}, v_{3t+1}) = s - t + 3$ where $0 \leq t < s$. When $0 \leq t \leq \frac{s}{2}$, $d(u_0, v_{3t+1}) < d(v_{3s}, v_{3t+1})$. When $\frac{s+2}{2} \leq t < s$, $d(u_0, v_{3t+1}) > d(v_{3s}, v_{3t+1})$. $d(u_0, v_{3t+2}) = 3 + t$ and $d(v_{3s}, v_{3t+2}) = s - t + 2$ where $0 \leq t < s$. When $0 \leq t \leq \frac{s-2}{2}$, $d(u_0, v_{3t+2}) < d(v_{3s}, v_{3t+2})$. When $\frac{s}{2} \leq t < s$, $d(u_0, v_{3t+2}) > d(v_{3s}, v_{3t+2})$.

$d(u_0, u_{3t}) = 2 + t$ and $d(v_{3s}, u_{3t}) = s - t + 1$ where $1 \leq t \leq s$. When $1 \leq t \leq \frac{s-2}{2}$, $d(u_0, u_{3t}) < d(v_{3s}, u_{3t})$. When $\frac{s}{2} \leq t \leq s$, $d(u_0, u_{3t}) > d(v_{3s}, u_{3t})$. $d(u_0, u_1) = 1$ and $d(v_{3s}, u_1) = s + 2$. $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_{3s}, u_{3t+1}) = s - t + 2$ where $1 \leq t < s$. When $1 \leq t \leq \frac{s-2}{2}$, $d(u_0, u_{3t+1}) < d(v_{3s}, u_{3t+1})$. When $\frac{s}{2} \leq t < s$, $d(u_0, u_{3t+1}) > d(v_{3s}, u_{3t+1})$. $d(u_0, u_2) = 2$ and $d(v_{3s}, u_2) = s + 1$. $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_{3s}, u_{3t+2}) = s - t + 1$ where $1 \leq t < s$. When $1 \leq t \leq \frac{s-4}{2}$, $d(u_0, u_{3t+2}) < d(v_{3s}, u_{3t+2})$. When $\frac{s-2}{2} \leq t < s$, $d(u_0, u_{3t+2}) > d(v_{3s}, u_{3t+2})$.

Note that $u_0 \in W^1_{u_0v_{3s}}$ and $v_{3s} \in W^1_{v_{3s}u_0}$. Combined with the above discussion, $|W^1_{u_0v_{3s}}| = 3s$ and $|W^1_{v_{3s}u_0}| = 3s + 2$.

(2a) Computation of $|W^1_{u_0v_{3s+1}}|$ and $|W^1_{v_{3s+1}u_0}|$ when s is odd and $s \geq 3$.

$d(u_0, v_{3t}) = 1 + t$ and $d(v_{3s+1}, v_{3t}) = s - t + 3$ where $0 \leq t \leq s$. When $0 \leq t \leq \frac{s+1}{2}$, $d(u_0, v_{3t}) < d(v_{3s+1}, v_{3t})$. When $\frac{s+3}{2} \leq t \leq s$, $d(u_0, v_{3t}) > d(v_{3s+1}, v_{3t})$. $d(u_0, v_{3t+1}) = 2 + t$ and $d(v_{3s+1}, v_{3t+1}) = s - t$ where $0 \leq t < s$. When $0 \leq t \leq \frac{s-3}{2}$, $d(u_0, v_{3t+1}) < d(v_{3s+1}, v_{3t+1})$. When $\frac{s-1}{2} \leq t < s$, $d(u_0, v_{3t+1}) > d(v_{3s+1}, v_{3t+1})$. $d(u_0, v_{3t+2}) = 3 + t$ and $d(v_{3s+1}, v_{3t+2}) = s - t + 3$ where $0 \leq t < s$. When $0 \leq t \leq \frac{s-1}{2}$, $d(u_0, v_{3t+2}) < d(v_{3s+1}, v_{3t+2})$. When $\frac{s+1}{2} \leq t < s$, $d(u_0, v_{3t+2}) > d(v_{3s+1}, v_{3t+2})$.

$d(u_0, u_{3t}) = 2 + t$ and $d(v_{3s+1}, u_{3t}) = s - t + 2$ where $1 \leq t \leq s$. When $1 \leq t \leq \frac{s-1}{2}$, $d(u_0, u_{3t}) < d(v_{3s+1}, u_{3t})$. When $\frac{s+1}{2} \leq t \leq s$, $d(u_0, u_{3t}) > d(v_{3s+1}, u_{3t})$. $d(u_0, u_1) = 1$ and $d(v_{3s+1}, u_1) = s + 1$. $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_{3s+1}, u_{3t+1}) = s - t + 1$ where $1 \leq t \leq s$. When $1 \leq t \leq \frac{s-3}{2}$, $d(u_0, u_{3t+1}) < d(v_{3s+1}, u_{3t+1})$. When $\frac{s-1}{2} \leq t \leq s$, $d(u_0, u_{3t+1}) > d(v_{3s+1}, u_{3t+1})$. $d(u_0, u_2) = 2$ and $d(v_{3s+1}, u_2) = s + 2$. $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_{3s+1}, u_{3t+2}) = s - t + 2$ where $1 \leq t < s$. When

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$1 \le t \le \frac{s-3}{2}$, $d(u_0, u_{3t+2}) < d(v_{3s+1}, u_{3t+2})$. When $\frac{s-1}{2} \le t < s$, $d(u_0, u_{3t+2}) > d(v_{3s+1}, u_{3t+2})$.

Note that $u_0 \in W^1_{u_0 v_{3s+1}}$ and $v_{3s+1} \in W^1_{v_{3s+1} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{3s+1}}| = 3s + 1$ and $|W^1_{v_{3s+1} u_0}| = 3s + 3$.

(2b) Computation of $|W^1_{u_0 v_{3s+1}}|$ and $|W^1_{v_{3s+1} u_0}|$ when s is even and $s \ge 2$.

When $s = 2$.

$d(u_0, v_0) = 1$ and $d(v_7, v_0) = 5$. $d(u_0, v_3) = 2$ and $d(v_7, v_3) = 4$. $d(u_0, v_6) = 3$ and $d(v_7, v_6) = 3$. $d(u_0, v_1) = 2$ and $d(v_7, v_1) = 2$. $d(u_0, v_4) = 3$ and $d(v_7, v_4) = 1$. $d(u_0, v_2) = 3$ and $d(v_7, v_2) = 5$. $d(u_0, v_5) = 4$ and $d(v_7, v_5) = 4$. So $v_0, v_2, v_3 \in W^1_{u_0 v_7}$ and $v_4 \in W^1_{v_7 u_0}$.

$d(u_0, u_3) = 3$ and $d(v_7, u_3) = 3$. $d(u_0, u_6) = 4$ and $d(v_7, u_6) = 2$. $d(u_0, u_1) = 1$ and $d(v_7, u_1) = 3$. $d(u_0, u_4) = 4$ and $d(v_7, u_4) = 2$. $d(u_0, u_7) = 5$ and $d(v_7, u_7) = 1$. $d(u_0, u_2) = 2$ and $d(v_7, u_2) = 4$. $d(u_0, u_5) = 5$ and $d(v_7, u_5) = 4$. So $u_1, u_2 \in W^1_{u_0 v_7}$ and $u_4, u_5, u_6, u_7 \in W^1_{v_7 u_0}$.

Note that $u_0 \in W^1_{u_0 v_7}$ and $v_7 \in W^1_{v_7 u_0}$. Combined with the above discussion, $|W^1_{u_0 v_7}| = 6$ and $|W^1_{v_7 u_0}| = 6$.

When $s \ge 4$.

$d(u_0, v_{3t}) = 1 + t$ and $d(v_{3s+1}, v_{3t}) = s - t + 3$ where $0 \le t \le s$. When $0 \le t < \frac{s+2}{2}$, $d(u_0, v_{3t}) < d(v_{3s+1}, v_{3t})$. When $\frac{s+2}{2} < t \le s$, $d(u_0, v_{3t}) > d(v_{3s+1}, v_{3t})$. $d(u_0, v_{3t+1}) = 2+t$ and $d(v_{3s+1}, v_{3t+1}) = s-t$ where $0 \le t < s$. When $0 \le t < \frac{s-2}{2}$, $d(u_0, v_{3t+1}) < d(v_{3s+1}, v_{3t+1})$. When $\frac{s-2}{2} < t < s$, $d(u_0, v_{3t+1}) > d(v_{3s+1}, v_{3t+1})$. $d(u_0, v_{3t+2}) = 3+t$ and $d(v_{3s+1}, v_{3t+2}) = s-t+3$ where $0 \le t < s$. When $0 \le t < \frac{s}{2}$, $d(u_0, v_{3t+2}) < d(v_{3s+1}, v_{3t+2})$. When $\frac{s}{2} < t < s$, $d(u_0, v_{3t+2}) > d(v_{3s+1}, v_{3t+2})$.

$d(u_0, u_{3t}) = 2+t$ and $d(v_{3s+1}, u_{3t}) = s-t+2$ where $1 \le t \le s$. When $1 \le t < \frac{s}{2}$, $d(u_0, u_{3t}) < d(v_{3s+1}, u_{3t})$. When $\frac{s}{2} < t \le s$, $d(u_0, u_{3t}) > d(v_{3s+1}, u_{3t})$. $d(u_0, u_1) = 1$ and $d(v_{3s+1}, u_1) = s + 1$. $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_{3s+1}, u_{3t+1}) = s - t + 1$ where $1 \le t \le s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{3t+1}) < d(v_{3s+1}, u_{3t+1})$. When $\frac{s-2}{2} < t \le s$, $d(u_0, u_{3t+1}) > d(v_{3s+1}, u_{3t+1})$. $d(u_0, u_2) = 2$ and $d(v_{3s+1}, u_2) = s+2$. $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_{3s+1}, u_{3t+2}) = s - t + 2$ where $1 \le t < s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{3t+2}) < d(v_{3s+1}, u_{3t+2})$. When $\frac{s-2}{2} < t < s$, $d(u_0, u_{3t+2}) > d(v_{3s+1}, u_{3t+2})$.

Note that $u_0 \in W^1_{u_0 v_{3s+1}}$ and $v_{3s+1} \in W^1_{v_{3s+1} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{3s+1}}| = 3s - 2$ and $|W^1_{v_{3s+1} u_0}| = 3s$.

(3a) Computation of $|W^1_{u_0 v_{3s+2}}|$ and $|W^1_{v_{3s+2} u_0}|$ when s is odd and $s \ge 3$.

When $s = 3$.

$d(u_0, v_0) = 1$ and $d(v_{11}, v_0) = 7$. $d(u_0, v_3) = 2$ and $d(v_{11}, v_3) = 6$. $d(u_0, v_6) = 3$ and $d(v_{11}, v_6) = 5$. $d(u_0, v_9) = d(v_{11}, v_9) = 4$. $d(u_0, v_1) = 2$ and $d(v_{11}, v_1) = 6$. $d(u_0, v_4) = 3$ and $d(v_{11}, v_4) = 5$. $d(u_0, v_7) = 4$ and $d(v_{11}, v_7) = 4$. $d(u_0, v_{10}) = 5$ and $d(v_{11}, v_{10}) = 3$. $d(u_0, v_2) = 3$ and $d(v_{11}, v_2) = 3$. $d(u_0, v_5) = 4$ and $d(v_{11}, v_5) = 2$. $d(u_0, v_8) = 5$ and $d(v_{11}, v_8) = 1$. So $v_0, v_1, v_3, v_4, v_6 \in W^1_{u_0 v_{11}}$ and $v_5, v_8, v_{10} \in W^1_{v_{11} u_0}$.

$d(u_0, u_3) = 3$ and $d(v_{11}, u_3) = 5$. $d(u_0, u_6) = 4$ and $d(v_{11}, u_6) = 4$. $d(u_0, u_9) = 5$ and $d(v_{11}, u_9) = 3$. $d(u_0, u_1) = 1$ and $d(v_{11}, u_1) = 5$. $d(u_0, u_4) = 4$ and

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$d(v_{11},u_4)=4$. $d(u_0,u_7)=5$ and $d(v_{11},u_7)=3$. $d(u_0,u_{10})=6$ and $d(v_{11},u_{10})=2$. $d(u_0,u_2)=2$ and $d(v_{11},u_2)=4$. $d(u_0,u_5)=5$ and $d(v_{11},u_5)=3$. $d(u_0,u_8)=6$ and $d(v_{11},u_8)=2$. $d(u_0,u_{11})=7$ and $d(v_{11},u_{11})=1$. So $u_1,u_2,u_3\in W^1_{u_0v_{11}}$ and $u_5,u_7,u_8,u_9,u_{10},u_{11}\in W^1_{v_{11}u_0}$.

Note that $u_0\in W^1_{u_0v_{11}}$ and $v_{11}\in W^1_{v_{11}u_0}$. Combined with the above discussion, $|W^1_{u_0v_{11}}|=9$ and $|W^1_{v_{11}u_0}|=10$.

When $s\geq 5$.

$d(u_0,v_{3t})=1+t$ and $d(v_{3s+2},v_{3t})=s-t+4$ where $0\leq t\leq s$. When $0\leq t<\frac{s+3}{2}$, $d(u_0,v_{3t})<d(v_{3s+2},v_{3t})$. When $\frac{s+3}{2}<t\leq s$, $d(u_0,v_{3t})>d(v_{3s+2},v_{3t})$. $d(u_0,v_{3t+1})=2+t$ and $d(v_{3s+2},v_{3t+1})=s-t+3$ where $0\leq t\leq s$. When $0\leq t<\frac{s+1}{2}$, $d(u_0,v_{3t+1})<d(v_{3s+2},v_{3t+1})$. When $\frac{s+1}{2}<t\leq s$, $d(u_0,v_{3t+1})>d(v_{3s+2},v_{3t+1})$. $d(u_0,v_{3t+2})=3+t$ and $d(v_{3s+2},v_{3t+2})=s-t$ where $0\leq t<s$. When $0\leq t<\frac{s-3}{2}$, $d(u_0,v_{3t+2})<d(v_{3s+2},v_{3t+2})$. When $\frac{s-3}{2}<t<s$, $d(u_0,v_{3t+2})>d(v_{3s+2},v_{3t+2})$.

$d(u_0,u_{3t})=2+t$ and $d(v_{3s+2},u_{3t})=s-t+3$ where $1\leq t\leq s$. When $1\leq t<\frac{s+1}{2}$, $d(u_0,u_{3t})<d(v_{3s+2},u_{3t})$. When $\frac{s+1}{2}<t\leq s$, $d(u_0,u_{3t})>d(v_{3s+2},u_{3t})$. $d(u_0,u_1)=1$ and $d(v_{3s+2},u_1)=s+2$. $d(u_0,u_{3t+1})=3+t$ and $d(v_{3s+2},u_{3t+1})=s-t+2$ where $1\leq t\leq s$. When $1\leq t<\frac{s-1}{2}$, $d(u_0,u_{3t+1})<d(v_{3s+2},u_{3t+1})$. When $\frac{s-1}{2}<t\leq s$, $d(u_0,u_{3t+1})>d(v_{3s+2},u_{3t+1})$. $d(u_0,u_2)=2$ and $d(v_{3s+2},u_2)=s+1$. $d(u_0,u_{3t+2})=4+t$ and $d(v_{3s+2},u_{3t+2})=s-t+1$ where $1\leq t\leq s$. When $1\leq t<\frac{s-3}{2}$, $d(u_0,u_{3t+2})<d(v_{3s+2},u_{3t+2})$. When $\frac{s-3}{2}<t\leq s$, $d(u_0,u_{3t+2})>d(v_{3s+2},u_{3t+2})$.

Note that $u_0\in W^1_{u_0v_{3s+2}}$ and $v_{3s+2}\in W^1_{v_{3s+2}u_0}$. Combined with the above discussion, $|W^1_{u_0v_{3s+2}}|=3s-1$ and $|W^1_{v_{3s+2}u_0}|=3s+1$.

(3b) Computation of $|W^1_{u_0v_{3s+2}}|$ and $|W^1_{v_{3s+2}u_0}|$ when s is even and $s\geq 2$.

When $s=2$.

$d(u_0,v_0)=1$ and $d(v_8,v_0)=6$. $d(u_0,v_3)=2$ and $d(v_8,v_3)=5$. $d(u_0,v_6)=3$ and $d(v_8,v_6)=4$. $d(u_0,v_1)=2$ and $d(v_8,v_1)=5$. $d(u_0,v_4)=3$ and $d(v_8,v_4)=4$. $d(u_0,v_7)=4$ and $d(v_8,v_7)=3$. $d(u_0,v_2)=3$ and $d(v_8,v_2)=2$. $d(u_0,v_5)=4$ and $d(v_8,v_5)=1$. So $v_0,v_1,v_3,v_4,v_6\in W^1_{u_0v_8}$ and $v_2,v_5,v_7\in W^1_{v_8u_0}$.

$d(u_0,u_3)=3$ and $d(v_8,u_3)=4$. $d(u_0,u_6)=4$ and $d(v_8,u_6)=3$. $d(u_0,u_1)=1$ and $d(v_8,u_1)=4$. $d(u_0,u_4)=4$ and $d(v_8,u_4)=3$. $d(u_0,u_7)=5$ and $d(v_8,u_7)=2$. $d(u_0,u_2)=2$ and $d(v_8,u_2)=3$. $d(u_0,u_5)=5$ and $d(v_8,u_5)=2$. $d(u_0,u_8)=6$ and $d(v_8,u_8)=1$. So $u_1,u_2,u_3\in W^1_{u_0v_8}$ and $u_4,u_5,u_6,u_7,u_8\in W^1_{v_8u_0}$.

Note that $u_0\in W^1_{u_0v_8}$ and $v_8\in W^1_{v_8u_0}$. Combined with the above discussion, $|W^1_{u_0v_8}|=9$ and $|W^1_{v_8u_0}|=9$.

When $s\geq 4$.

$d(u_0,v_{3t})=1+t$ and $d(v_{3s+2},v_{3t})=s-t+4$ where $0\leq t\leq s$. When $0\leq t\leq\frac{s+2}{2}$, $d(u_0,v_{3t})<d(v_{3s+2},v_{3t})$. When $\frac{s+4}{2}\leq t\leq s$, $d(u_0,v_{3t})>d(v_{3s+2},v_{3t})$. $d(u_0,v_{3t+1})=2+t$ and $d(v_{3s+2},v_{3t+1})=s-t+3$ where $0\leq t\leq s$. When $0\leq t\leq\frac{s}{2}$, $d(u_0,v_{3t+1})<d(v_{3s+2},v_{3t+1})$. When $\frac{s+2}{2}\leq t\leq s$, $d(u_0,v_{3t+1})>d(v_{3s+2},v_{3t+1})$. $d(u_0,v_{3t+2})=3+t$ and $d(v_{3s+2},v_{3t+2})=s-t$ where $0\leq t<s$. When $0\leq t\leq\frac{s-4}{2}$, $d(u_0,v_{3t+2})<d(v_{3s+2},v_{3t+2})$. When $\frac{s-2}{2}\leq t<s$, $d(u_0,v_{3t+2})>d(v_{3s+2},v_{3t+2})$.

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$d(u_0, u_{3t}) = 2 + t$ and $d(v_{3s+2}, u_{3t}) = s - t + 3$ where $1 \le t \le s$. When $1 \le t \le \frac{s}{2}$, $d(u_0, u_{3t}) < d(v_{3s+2}, u_{3t})$. When $\frac{s+2}{2} \le t \le s$, $d(u_0, u_{3t}) > d(v_{3s+2}, u_{3t})$. $d(u_0, u_1) = 1$ and $d(v_{3s+2}, u_1) = s + 2$. $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_{3s+2}, u_{3t+1}) = s-t+2$ where $1 \le t \le s$. When $1 \le t \le \frac{s-2}{2}$, $d(u_0, u_{3t+1}) < d(v_{3s+2}, u_{3t+1})$. When $\frac{s}{2} \le t \le s$, $d(u_0, u_{3t+1}) > d(v_{3s+2}, u_{3t+1})$. $d(u_0, u_2) = 2$ and $d(v_{3s+2}, u_2) = s + 1$. $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_{3s+2}, u_{3t+2}) = s - t + 1$ where $1 \le t \le s$. When $1 \le t \le \frac{s-4}{2}$, $d(u_0, u_{3t+2}) < d(v_{3s+2}, u_{3t+2})$. When $\frac{s-2}{2} \le t \le s$, $d(u_0, u_{3t+2}) > d(v_{3s+2}, u_{3t+2})$.

Note that $u_0 \in W^1_{u_0 v_{3s+2}}$ and $v_{3s+2} \in W^1_{v_{3s+2} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{3s+2}}| = 3s + 2$ and $|W^1_{v_{3s+2} u_0}| = 3s + 4$. □

Appendix B

Proof of the remaining cases of Proposition 3.1. (2) When $n = 8m+1$ where $m \ge 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$ where $1 \le t \le m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3+t$ where $0 \le t \le m-2$. $d(u_0, v_{4(m-1)+1}) = d(v_0, v_{4(m-1)+1}) = m+1$. $d(u_0, v_{4t+2}) = 3+t$ and $d(v_0, v_{4t+2}) = 4+t$ where $0 \le t < m$. $d(u_0, v_{4t+3}) = 3+t$ and $d(v_0, v_{4t+3}) = 4 + t$ where $0 \le t < m$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$ where $1 \le t \le m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$ where $1 \le t < m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$ where $1 \le t < m$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$ where $1 \le t < m$.

Note that $u_0 \in W_{u_0 v_0}$ and $v_0 \in W_{v_0 u_0}$. Combined with the above discussion, $|W_{u_0 v_0}| = 2(3m+1)+1 = 6m+3$ and $|W_{v_0 u_0}| = 2(5m-3)+1 = 10m-5$. Because $m \ge 3$, $|W_{u_0 v_0}| < |W_{v_0 u_0}|$.

(3) When $n = 8m + 2$ where $m \ge 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$ where $1 \le t \le m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$ where $0 \le t \le m$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$ where $0 \le t \le m - 2$. $d(u_0, v_{4(m-1)+2}) = m + 2$ and $d(v_0, v_{4(m-1)+2}) = m + 1$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_0, v_{4t+3}) = 4 + t$ where $0 \le t < m$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$ where $1 \le t \le m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$ where $1 \le t \le m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$ where $1 \le t < m$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$ where $1 \le t < m$.

Note that $u_0 \in W_{u_0 v_0}$ and $v_0 \in W_{v_0 u_0}$. Combined with the above discussion, $|W_{u_0 v_0}| = 2(3m+1)+2 = 6m+4$ and $|W_{v_0 u_0}| = 2(5m-2)+2 = 10m-2$. Because $m \ge 3$, $|W_{u_0 v_0}| < |W_{v_0 u_0}|$.

(4) When $n = 8m + 3$ where $m \ge 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$.

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$d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$ where $1 \le t \le m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$ where $0 \le t \le m$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$ where $0 \le t < m$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_0, v_{4t+3}) = 4 + t$ where $0 \le t \le m - 2$. $d(u_0, v_{4(m-1)+3}) = m + 2$ and $d(v_0, v_{4(m-1)+3}) = m + 1$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$ where $1 \le t \le m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$ where $1 \le t \le m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$ where $1 \le t < m$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$ where $1 \le t < m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(3m+2)+1 = 6m+5$ and $|W_{v_0u_0}| = 2(5m-1)+1 = 10m-1$. Because $m \ge 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(5) When $n = 8m + 4$ where $m \ge 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$ where $1 \le t \le m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$ where $0 \le t \le m$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$ where $0 \le t \le m$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_0, v_{4t+3}) = 4 + t$ where $0 \le t < m$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$ where $1 \le t \le m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$ where $1 \le t \le m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$ where $1 \le t \le m$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$ where $1 \le t < m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(3m+3)+2 = 6m+8$ and $|W_{v_0u_0}| = 2(5m-2)+2 = 10m-2$. Because $m \ge 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(6) When $n = 8m + 5$ where $m \ge 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$ where $1 \le t \le m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$ where $0 \le t \le m - 1$. $d(u_0, v_{4m+1}) = m + 2$ and $d(v_0, v_{4m+1}) = m + 1$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$ where $0 \le t \le m$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_0, v_{4t+3}) = 4 + t$ where $0 \le t < m$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$ where $1 \le t \le m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$ where $1 \le t \le m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$ where $1 \le t \le m$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$ where $1 \le t < m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(3m + 3) + 1 = 6m + 7$ and $|W_{v_0u_0}| = 2 \times 5m + 1 = 10m + 1$. Because $m \ge 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(7) When $n = 8m + 6$ where $m \ge 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \le i \le \frac{n}{2}$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$ where $1 \le t \le m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$ where $0 \le t \le m$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$

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where $0 \leq t \leq m-2$. $d(u_0, v_{4(m-1)+2}) = d(v_0, v_{4(m-1)+2}) = m+2$. $d(u_0, v_{4m+2}) = m+2$ and $d(v_0, v_{4m+2}) = m+1$. $d(u_0, v_{4t+3}) = 3+t$ and $d(v_0, v_{4t+3}) = 4+t$ where $0 \leq t \leq m$.

$d(u_0, u_{4t}) = 2+t$ and $d(v_0, u_{4t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3+t$ and $d(v_0, u_{4t+1}) = 2+t$ where $1 \leq t \leq m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4+t$ and $d(v_0, u_{4t+2}) = 3+t$ where $1 \leq t \leq m-1$. $d(u_0, u_{4m+2}) = m+3$ and $d(v_0, u_{4m+2}) = m+2$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4+t$ and $d(v_0, u_{4t+3}) = 3+t$ where $1 \leq t \leq m$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(3m+2)+2 = 6m+6$ and $|W_{v_0u_0}| = 2 \times 5m + 2 = 10m+2$. Because $m \geq 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$.

(8) When $n = 8m+7$ where $m \geq 3$.

By symmetry, we just need to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$.

$d(u_0, v_{4t}) = 1+t$ and $d(v_0, v_{4t}) = t$ where $1 \leq t \leq m$. $d(u_0, v_{4t+1}) = 2+t$ and $d(v_0, v_{4t+1}) = 3+t$ where $0 \leq t \leq m$. $d(u_0, v_{4t+2}) = 3+t$ and $d(v_0, v_{4t+2}) = 4+t$ where $0 \leq t \leq m$. $d(u_0, v_{4t+3}) = 3+t$ and $d(v_0, v_{4t+3}) = 4+t$ where $0 \leq t \leq m-2$. $d(u_0, v_{4(m-1)+3}) = d(v_0, v_{4(m-1)+3}) = m+2$. $d(u_0, v_{4m+3}) = m+2$ and $d(v_0, v_{4m+3}) = m+1$.

$d(u_0, u_{4t}) = 2+t$ and $d(v_0, u_{4t}) = 1+t$ where $1 \leq t \leq m$. $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$. $d(u_0, u_{4t+1}) = 3+t$ and $d(v_0, u_{4t+1}) = 2+t$ where $1 \leq t \leq m$. $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$. $d(u_0, u_{4t+2}) = 4+t$ and $d(v_0, u_{4t+2}) = 3+t$ where $1 \leq t \leq m$. $d(u_0, u_3) = 3$ and $d(v_0, u_3) = 3$. $d(u_0, u_{4t+3}) = 4+t$ and $d(v_0, u_{4t+3}) = 3+t$ where $1 \leq t \leq m-1$. $d(u_0, u_{4m+3}) = m+3$ and $d(v_0, u_{4m+3}) = m+2$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion, $|W_{u_0v_0}| = 2(3m+3)+1 = 6m+7$ and $|W_{v_0u_0}| = 2(5m+1)+1 = 10m+3$. Because $m \geq 3$, $|W_{u_0v_0}| < |W_{v_0u_0}|$. □

Proof of the remaining cases of Proposition 3.2. (2) When $n = 8m+1$ where $m \geq 3$.

Note that $n-4 = 8m-3 = 4(2m-1)+1$.

$d(u_0, v_{4t}) = d(v_{8m-3}, v_{4t}) = 1+t$ when $0 \leq t \leq m$. $d(u_0, v_{4t}) = d(v_{8m-3}, v_{4t}) = 2m-t+2$ when $m+1 \leq t \leq 2m-1$. $d(u_0, v_{4t+1}) = 2+t$ and $d(u_0, v_{4t+1}) < d(v_{8m-3}, v_{4t+1})$ when $0 \leq t \leq m-2$. $d(v_{8m-3}, v_{4t+1}) = 2m-t-1$ and $d(u_0, v_{4t+1}) > d(v_{8m-3}, v_{4t+1})$ when $m-1 \leq t < 2m-1$. $d(u_0, v_{4t+2}) = 3+t$ and $d(u_0, v_{4t+2}) < d(v_{8m-3}, v_{4t+2})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+2}) = d(v_{8m-3}, v_{4t+2}) = 2m-t+2$ when $m \leq t < 2m-1$. $d(u_0, v_{4t+3}) = 3+t$ and $d(u_0, v_{4t+3}) < d(v_{8m-3}, v_{4t+3})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+3}) = d(v_{8m-3}, v_{4t+3}) = 2m-t+2$ when $m \leq t < 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m-3}, u_{4t}) = 2+t$ when $1 \leq t \leq m-1$. $d(v_{8m-3}, u_{4t}) = 2m-t+1$ and $d(u_0, u_{4t}) > d(v_{8m-3}, u_{4t})$ when $m \leq t \leq 2m-1$. $d(u_0, u_1) = 1$ and $d(v_{8m-3}, u_1) = 2m$. $d(u_0, u_{4t+1}) = d(v_{8m-3}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m-2$. $d(v_{8m-3}, u_{4t+1}) = 2m-t$ and $d(u_0, u_{4t+1}) > d(v_{8m-3}, u_{4t+1})$ when $m-1 \leq t \leq 2m-1$. $d(u_0, u_2) = 2$ and $d(v_{8m-3}, u_2) = 2m+1$. $d(u_0, u_{4t+2}) = d(v_{8m-3}, u_{4t+2}) = 4+t$ when $1 \leq t \leq m-2$. $d(v_{8m-3}, u_{4t+2}) = 2m-t+1$ and $d(u_0, u_{4t+2}) > d(v_{8m-3}, u_{4t+2})$ when $m-1 \leq t < 2m-1$. $d(u_0, u_3) = 3$ and $d(v_{8m-3}, u_3) = 2m+1$. $d(u_0, u_{4t+3}) = d(v_{8m-3}, u_{4t+3}) = 4+t$ when $1 \leq t \leq m-2$.

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$d(v_{8m-3}, u_{4t+3}) = 2m-t+1$ and $d(u_0, u_{4t+3}) > d(v_{8m-3}, u_{4t+3})$ when $m-1 \leq t < 2m-1$.

Note that $u_0 \in W_{u_0v_{8m-3}}$ and $v_{8m-3} \in W_{v_{8m-3}u_0}$. Combined with the above discussion, $|W_{u_0v_{8m-3}}| = 3m+7$ and $|W_{v_{8m-3}u_0}| = 5m+3$. Because $m \geq 3$, $|W_{u_0v_{8m-3}}| < |W_{v_{8m-3}u_0}|$.

(3) When $n = 8m+2$ where $m \geq 3$.

Note that $n-4 = 8m-2 = 4(2m-1)+2$.

$d(u_0, v_{4t}) = d(v_{8m-2}, v_{4t}) = 1+t$ when $0 \leq t \leq m+1$. $d(u_0, v_{4t}) = d(v_{8m-2}, v_{4t}) = 2m-t+3$ when $m+2 \leq t \leq 2m-1$. $d(u_0, v_{4t+1}) = 2+t$ and $d(u_0, v_{4t+1}) < d(v_{8m-2}, v_{4t+1})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+1}) = d(v_{8m-2}, v_{4t+1}) = 2m-t+2$ when $m \leq t \leq 2m-1$. $d(u_0, v_{4t+2}) = 3+t$ and $d(u_0, v_{4t+2}) < d(v_{8m-2}, v_{4t+2})$ when $0 \leq t < m-2$. $d(u_0, v_{4(m-2)+2}) = d(v_{8m-2}, v_{4(m-2)+2}) = m+1$. $d(v_{8m-2}, v_{4t+2}) = 2m-t-1$ and $d(u_0, v_{4t+2}) > d(v_{8m-2}, v_{4t+2})$ when $m-2 < t < 2m-1$. $d(u_0, v_{4t+3}) = 3+t$ and $d(u_0, v_{4t+3}) < d(v_{8m-2}, v_{4t+3})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+3}) = d(v_{8m-2}, v_{4t+3}) = 2m-t+2$ when $m \leq t < 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m-2}, u_{4t}) = 2+t$ when $1 \leq t \leq m$. $d(v_{8m-2}, u_{4t}) = 2m-t+2$ and $d(u_0, u_{4t}) > d(v_{8m-2}, u_{4t})$ when $m+1 \leq t \leq 2m-1$. $d(u_0, u_1) = 1$ and $d(v_{8m-2}, u_1) = 2m+1$. $d(u_0, u_{4t+1}) = d(v_{8m-2}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m-1$. $d(v_{8m-2}, u_{4t+1}) = 2m-t+1$ and $d(u_0, u_{4t+1}) > d(v_{8m-2}, u_{4t+1})$ when $m \leq t \leq 2m-1$. $d(u_0, u_2) = 2$ and $d(v_{8m-2}, u_2) = 2m$. $d(u_0, u_{4t+2}) = d(v_{8m-2}, u_{4t+2}) = 4+t$ when $1 \leq t \leq m-2$. $d(v_{8m-2}, u_{4t+2}) = 2m-t$ and $d(u_0, u_{4t+2}) > d(v_{8m-2}, u_{4t+2})$ when $m-1 \leq t \leq 2m-1$. $d(u_0, u_3) = 3$ and $d(v_{8m-2}, u_3) = 2m+1$. $d(u_0, u_{4t+3}) = d(v_{8m-2}, u_{4t+3}) = 4+t$ when $1 \leq t \leq m-2$. $d(v_{8m-2}, u_{4t+3}) = 2m-t+1$ and $d(u_0, u_{4t+3}) > d(v_{8m-2}, u_{4t+3})$ when $m-1 \leq t < 2m-1$.

Note that $u_0 \in W_{u_0v_{8m-2}}$ and $v_{8m-2} \in W_{v_{8m-2}u_0}$. Combined with the above discussion, $|W_{u_0v_{8m-2}}| = 3m+6$ and $|W_{v_{8m-2}u_0}| = 5m+2$. Because $m \geq 3$, $|W_{u_0v_{8m-2}}| < |W_{v_{8m-2}u_0}|$.

(4) When $n = 8m+3$ where $m \geq 3$.

Note that $n-4 = 8m-1 = 4(2m-1)+3$.

$d(u_0, v_{4t}) = d(v_{8m-1}, v_{4t}) = 1+t$ when $0 \leq t \leq m+1$. $d(u_0, v_{4t}) = d(v_{8m-1}, v_{4t}) = 2m-t+3$ when $m+2 \leq t \leq 2m-1$. $d(u_0, v_{4t+1}) = 2+t$ and $d(u_0, v_{4t+1}) < d(v_{8m-1}, v_{4t+1})$ when $0 \leq t \leq m$. $d(u_0, v_{4t+1}) = d(v_{8m-1}, v_{4t+1}) = 2m-t+3$ when $m+1 \leq t \leq 2m-1$. $d(u_0, v_{4t+2}) = 3+t$ and $d(u_0, v_{4t+2}) < d(v_{8m-1}, v_{4t+2})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+2}) = d(v_{8m-1}, v_{4t+2}) = 2m-t+2$ when $m \leq t \leq 2m-1$. $d(u_0, v_{4t+3}) = 3+t$ and $d(u_0, v_{4t+3}) < d(v_{8m-1}, v_{4t+3})$ when $0 \leq t < m-2$. $d(u_0, v_{4(m-2)+3}) = d(v_{8m-1}, v_{4(m-2)+3}) = m+1$. $d(v_{8m-1}, v_{4t+3}) = 2m-t-1$ and $d(u_0, v_{4t+3}) > d(v_{8m-1}, v_{4t+3})$ when $m-2 < t < 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m-1}, u_{4t}) = 2+t$ when $1 \leq t \leq m$. $d(v_{8m-1}, u_{4t}) = 2m-t+2$ and $d(u_0, u_{4t}) > d(v_{8m-1}, u_{4t})$ when $m+1 \leq t \leq 2m-1$. $d(u_0, u_1) = 1$ and $d(v_{8m-1}, u_1) = 2m+2$. $d(u_0, u_{4t+1}) = d(v_{8m-1}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m-1$. $d(v_{8m-1}, u_{4t+1}) = 2m-t+2$ and $d(u_0, u_{4t+1}) > d(v_{8m-1}, u_{4t+1})$ when $m \leq t \leq 2m-1$. $d(u_0, u_2) = 2$ and $d(v_{8m-1}, u_2) = 2m+1$. $d(u_0, u_{4t+2}) = d(v_{8m-1}, u_{4t+2}) = 4+t$ when $1 \leq t \leq m-2$. $d(v_{8m-1}, u_{4t+2}) = 2m-t+1$ and $d(u_0, u_{4t+2}) > d(v_{8m-1}, u_{4t+2})$ when $m-1 \leq t \leq 2m-1$. $d(u_0, u_3) = 3$ and

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$d(v_{8m-1}, u_3) = 2m$. $d(u_0, u_{4t+3}) = d(v_{8m-1}, u_{4t+3}) = 4 + t$ when $1 \leq t \leq m-2$. $d(v_{8m-1}, u_{4t+3}) = 2m - t$ and $d(u_0, u_{4t+3}) > d(v_{8m-1}, u_{4t+3})$ when $m-1 \leq t \leq 2m-1$.

Note that $u_0 \in W_{u_0v_{8m-1}}$ and $v_{8m-1} \in W_{v_{8m-1}u_0}$. Combined with the above discussion, $|W_{u_0v_{8m-1}}| = 3m + 7$ and $|W_{v_{8m-1}u_0}| = 5m + 3$. Because $m \geq 3$, $|W_{u_0v_{8m-1}}| < |W_{v_{8m-1}u_0}|$.

(5) When $n = 8m + 4$ where $m \geq 3$.

Note that $n - 4 = 8m = 4 \times 2m$.

$d(u_0, v_{4t}) = d(v_{8m}, v_{4t}) = 1 + t$ when $0 \leq t \leq m-1$. $d(v_{8m}, v_{4t}) = 2m - t$ and $d(u_0, v_{4t}) > d(v_{8m}, v_{4t})$ when $m \leq t \leq 2m-1$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(u_0, v_{4t+1}) < d(v_{8m}, v_{4t+1})$ when $0 \leq t \leq m$. $d(u_0, v_{4t+1}) = d(v_{8m}, v_{4t+1}) = 2m - t + 3$ when $m+1 \leq t \leq 2m-1$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(u_0, v_{4t+2}) < d(v_{8m}, v_{4t+2})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+2}) = d(v_{8m}, v_{4t+2}) = 2m - t + 3$ when $m \leq t \leq 2m-1$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(u_0, v_{4t+3}) < d(v_{8m}, v_{4t+3})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+3}) = d(v_{8m}, v_{4t+3}) = 2m - t + 2$ when $m \leq t \leq 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m}, u_{4t}) = 2+t$ when $1 \leq t \leq m-1$. $d(v_{8m}, u_{4t}) = 2m-t+1$ and $d(u_0, u_{4t}) > d(v_{8m}, u_{4t})$ when $m \leq t \leq 2m$. $d(u_0, u_1) = 1$ and $d(v_{8m}, u_1) = 2m+2$. $d(u_0, u_{4t+1}) = d(v_{8m}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m-1$. $d(v_{8m}, u_{4t+1}) = 2m-t+2$ and $d(u_0, u_{4t+1}) > d(v_{8m}, u_{4t+1})$ when $m \leq t \leq 2m-1$. $d(u_0, u_2) = 2$ and $d(v_{8m}, u_2) = 2m + 2$. $d(u_0, u_{4t+2}) = d(v_{8m}, u_{4t+2}) = 4 + t$ when $1 \leq t \leq m-1$. $d(v_{8m}, u_{4t+2}) = 2m-t+2$ and $d(u_0, u_{4t+2}) > d(v_{8m}, u_{4t+2})$ when $m \leq t \leq 2m-1$. $d(u_0, u_3) = 3$ and $d(v_{8m}, u_3) = 2m + 1$. $d(u_0, u_{4t+3}) = d(v_{8m}, u_{4t+3}) = 4 + t$ when $1 \leq t \leq m-2$. $d(v_{8m}, u_{4t+3}) = 2m - t + 1$ and $d(u_0, u_{4t+3}) > d(v_{8m}, u_{4t+3})$ when $m-1 \leq t \leq 2m-1$.

Note that $u_0 \in W_{u_0v_{8m}}$ and $v_{8m} \in W_{v_{8m}u_0}$. Combined with the above discussion, $|W_{u_0v_{8m}}| = 3m + 9$ and $|W_{v_{8m}u_0}| = 5m + 4$. Because $m \geq 3$, $|W_{u_0v_{8m}}| < |W_{v_{8m}u_0}|$.

(6) When $n = 8m + 5$ where $m \geq 3$.

Note that $n - 4 = 8m + 1 = 4 \times 2m + 1$.

$d(u_0, v_{4t}) = d(v_{8m+1}, v_{4t}) = 1+t$ when $0 \leq t \leq m+1$. $d(u_0, v_{4t}) = d(v_{8m+1}, v_{4t}) = 2m - t + 3$ when $m+2 \leq t \leq 2m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(u_0, v_{4t+1}) < d(v_{8m+1}, v_{4t+1})$ when $0 \leq t \leq m-2$. $d(u_0, v_{4(m-1)+1}) = d(v_{8m+1}, v_{4(m-1)+1}) = m+1$. $d(v_{8m+1}, v_{4t+1}) = 2m - t$ and $d(u_0, v_{4t+1}) > d(v_{8m+1}, v_{4t+1})$ when $m \leq t \leq 2m-1$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(u_0, v_{4t+2}) < d(v_{8m+1}, v_{4t+2})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+2}) = d(v_{8m+1}, v_{4t+2}) = 2m - t + 3$ when $m \leq t \leq 2m-1$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(u_0, v_{4t+3}) < d(v_{8m+1}, v_{4t+3})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+3}) = d(v_{8m+1}, v_{4t+3}) = 2m - t + 3$ when $m \leq t \leq 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m+1}, u_{4t}) = 2 + t$ when $1 \leq t \leq m$. $d(v_{8m+1}, u_{4t}) = 2m - t + 2$ and $d(u_0, u_{4t}) > d(v_{8m+1}, u_{4t})$ when $m+1 \leq t \leq 2m$. $d(u_0, u_1) = 1$ and $d(v_{8m+1}, u_1) = 2m+1$. $d(u_0, u_{4t+1}) = d(v_{8m+1}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m-1$. $d(v_{8m+1}, u_{4t+1}) = 2m - t + 1$ and $d(u_0, u_{4t+1}) > d(v_{8m+1}, u_{4t+1})$ when $m \leq t \leq 2m$. $d(u_0, u_2) = 2$ and $d(v_{8m+1}, u_2) = 2m + 2$. $d(u_0, u_{4t+2}) = d(v_{8m+1}, u_{4t+2}) = 4 + t$ when $1 \leq t \leq m-1$. $d(v_{8m+1}, u_{4t+2}) = 2m - t + 2$ and $d(u_0, u_{4t+2}) > d(v_{8m+1}, u_{4t+2})$ when $m \leq t \leq 2m-1$. $d(u_0, u_3) = 3$ and $d(v_{8m+1}, u_3) = 2m + 2$.

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$d(u_0, u_{4t+3}) = d(v_{8m+1}, u_{4t+3}) = 4 + t$ when $1 \leq t \leq m-1$. $d(v_{8m+1}, u_{4t+3}) = 2m - t + 2$ and $d(u_0, u_{4t+3}) > d(v_{8m+1}, u_{4t+3})$ when $m \leq t \leq 2m-1$.

Note that $u_0 \in W_{u_0 v_{8m+1}}$ and $v_{8m+1} \in W_{v_{8m+1} u_0}$. Combined with the above discussion, $|W_{u_0 v_{8m+1}}| = 3m + 7$ and $|W_{v_{8m+1} u_0}| = 5m + 3$. Because $m \geq 3$, $|W_{u_0 v_{8m+1}}| < |W_{v_{8m+1} u_0}|$.

(7) When $n = 8m + 6$ where $m \geq 3$.

Note that $n - 4 = 8m + 2 = 4 \times 2m + 2$.

$d(u_0, v_{4t}) = d(v_{8m+2}, v_{4t}) = 1+t$ when $0 \leq t \leq m+1$. $d(u_0, v_{4t}) = d(v_{8m+2}, v_{4t}) = 2m - t + 4$ when $m + 2 \leq t \leq 2m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(u_0, v_{4t+1}) < d(v_{8m+2}, v_{4t+1})$ when $0 \leq t \leq m$. $d(u_0, v_{4t+1}) = d(v_{8m+2}, v_{4t+1}) = 2m - t + 3$ when $m + 1 \leq t \leq 2m$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(u_0, v_{4t+2}) < d(v_{8m+2}, v_{4t+2})$ when $0 \leq t \leq m-2$. $d(v_{8m+2}, v_{4t+2}) = 2m - t$ and $d(u_0, v_{4t+2}) > d(v_{8m+2}, v_{4t+2})$ when $m - 1 \leq t \leq 2m-1$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(u_0, v_{4t+3}) < d(v_{8m+2}, v_{4t+3})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+3}) = d(v_{8m+2}, v_{4t+3}) = 2m - t + 3$ when $m \leq t \leq 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m+2}, u_{4t}) = 2 + t$ when $1 \leq t \leq m$. $d(v_{8m+2}, u_{4t}) = 2m - t + 3$ and $d(u_0, u_{4t}) > d(v_{8m+2}, u_{4t})$ when $m + 1 \leq t \leq 2m$. $d(u_0, u_1) = 1$ and $d(v_{8m+2}, u_1) = 2m+2$. $d(u_0, u_{4t+1}) = d(v_{8m+2}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m-1$. $d(v_{8m+2}, u_{4t+1}) = 2m - t + 2$ and $d(u_0, u_{4t+1}) > d(v_{8m+2}, u_{4t+1})$ when $m \leq t \leq 2m$. $d(u_0, u_2) = 2$ and $d(v_{8m+2}, u_2) = 2m + 1$. $d(u_0, u_{4t+2}) = d(v_{8m+2}, u_{4t+2}) = 4 + t$ when $1 \leq t \leq m-2$. $d(v_{8m+2}, u_{4t+2}) = 2m - t + 1$ and $d(u_0, u_{4t+2}) > d(v_{8m+2}, u_{4t+2})$ when $m - 1 \leq t \leq 2m$. $d(u_0, u_3) = 3$ and $d(v_{8m+2}, u_3) = 2m + 2$. $d(u_0, u_{4t+3}) = d(v_{8m+2}, u_{4t+3}) = 4 + t$ when $1 \leq t \leq m-1$. $d(v_{8m+2}, u_{4t+3}) = 2m - t + 2$ and $d(u_0, u_{4t+3}) > d(v_{8m+2}, u_{4t+3})$ when $m \leq t \leq 2m-1$.

Note that $u_0 \in W_{u_0 v_{8m+2}}$ and $v_{8m+2} \in W_{v_{8m+2} u_0}$. Combined with the above discussion, $|W_{u_0 v_{8m+2}}| = 3m + 8$ and $|W_{v_{8m+2} u_0}| = 5m + 6$. Because $m \geq 3$, $|W_{u_0 v_{8m+2}}| < |W_{v_{8m+2} u_0}|$.

(8) When $n = 8m + 7$ where $m \geq 3$.

Note that $n - 4 = 8m + 3 = 4 \times 2m + 3$.

$d(u_0, v_{4t}) = d(v_{8m+3}, v_{4t}) = 1+t$ when $0 \leq t \leq m+1$. $d(u_0, v_{4t}) = d(v_{8m+3}, v_{4t}) = 2m - t + 4$ when $m + 2 \leq t \leq 2m$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(u_0, v_{4t+1}) < d(v_{8m+3}, v_{4t+1})$ when $0 \leq t \leq m$. $d(u_0, v_{4t+1}) = d(v_{8m+3}, v_{4t+1}) = 2m - t + 4$ when $m + 1 \leq t \leq 2m$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(u_0, v_{4t+2}) < d(v_{8m+3}, v_{4t+2})$ when $0 \leq t \leq m-1$. $d(u_0, v_{4t+2}) = d(v_{8m+3}, v_{4t+2}) = 2m - t + 3$ when $m \leq t \leq 2m$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(u_0, v_{4t+3}) < d(v_{8m+3}, v_{4t+3})$ when $0 \leq t \leq m-2$. $d(v_{8m+3}, v_{4t+3}) = 2m - t$ and $d(u_0, v_{4t+3}) > d(v_{8m+3}, v_{4t+3})$ when $m - 1 \leq t \leq 2m-1$.

$d(u_0, u_{4t}) = d(v_{8m+3}, u_{4t}) = 2+t$ when $1 \leq t \leq m$. $d(v_{8m+3}, u_{4t}) = 2m-t+3$ and $d(u_0, u_{4t}) > d(v_{8m+3}, u_{4t})$ when $m+1 \leq t \leq 2m$. $d(u_0, u_1) = 1$ and $d(v_{8m+3}, u_1) = 2m+3$. $d(u_0, u_{4t+1}) = d(v_{8m+3}, u_{4t+1}) = 3+t$ when $1 \leq t \leq m$. $d(v_{8m+3}, u_{4t+1}) = 2m-t+3$ and $d(u_0, u_{4t+1}) > d(v_{8m+3}, u_{4t+1})$ when $m+1 \leq t \leq 2m$. $d(u_0, u_2) = 2$ and $d(v_{8m+3}, u_2) = 2m + 2$. $d(u_0, u_{4t+2}) = d(v_{8m+3}, u_{4t+2}) = 4 + t$ when $1 \leq t \leq m-1$. $d(v_{8m+3}, u_{4t+2}) = 2m - t + 2$ and $d(u_0, u_{4t+2}) > d(v_{8m+3}, u_{4t+2})$ when $m \leq t \leq 2m$. $d(u_0, u_3) = 3$ and $d(v_{8m+3}, u_3) = 2m + 1$. $d(u_0, u_{4t+3}) =$

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$d(v_{8m+3}, u_{4t+3}) = 4 + t$ when $1 \le t \le m - 2$. $d(v_{8m+3}, u_{4t+3}) = 2m - t + 1$ and $d(u_0, u_{4t+3}) > d(v_{8m+3}, u_{4t+3})$ when $m - 1 \le t \le 2m$.

Note that $u_0 \in W_{u_0v_{8m+3}}$ and $v_{8m+3} \in W_{v_{8m+3}u_0}$. Combined with the above discussion, $|W_{u_0v_{8m+3}}| = 3m + 8$ and $|W_{v_{8m+3}u_0}| = 5m + 6$. Because $m \ge 3$, $|W_{u_0v_{8m+3}}| < |W_{v_{8m+3}u_0}|$. □

Proof of the remaining cases of Proposition 3.3. (1a) The computation of $|W^1_{u_0v_{4s}}|$ and $|W^1_{v_{4s}u_0}|$ when $s = 3$.

$d(u_0, v_0) = 1$ and $d(v_{12}, v_0) = 3$. $d(u_0, v_4) = d(v_{12}, v_4) = 2$. $d(u_0, v_8) = 3$ and $d(v_{12}, v_8) = 1$. $d(u_0, v_1) = 2$ and $d(v_{12}, v_1) = 6$. $d(u_0, v_5) = 3$ and $d(v_{12}, v_5) = 5$. $d(u_0, v_9) = 4$ and $d(v_{12}, v_9) = 4$. $d(u_0, v_2) = 3$ and $d(v_{12}, v_2) = 6$. $d(u_0, v_6) = 4$ and $d(v_{12}, v_6) = 5$. $d(u_0, v_{10}) = 5$ and $d(v_{12}, v_{10}) = 4$. $d(u_0, v_3) = 3$ and $d(v_{12}, v_3) = 5$. $d(u_0, v_7) = 4$ and $d(v_{12}, v_7) = 4$. $d(u_0, v_{11}) = 5$ and $d(v_{12}, v_{11}) = 3$. So $v_0, v_1, v_2, v_3, v_5, v_6 \in W^1_{u_0v_{12}}$ and $v_8, v_{10}, v_{11} \in W^1_{v_{12}u_0}$.

$d(u_0, u_4) = 3$ and $d(v_{12}, u_4) = 3$. $d(u_0, u_8) = 4$ and $d(v_{12}, u_8) = 2$. $d(u_0, u_{12}) = 5$ and $d(v_{12}, u_{12}) = 1$. $d(u_0, u_1) = 1$ and $d(v_{12}, u_1) = 5$. $d(u_0, u_5) = 4$ and $d(v_{12}, u_5) = 4$. $d(u_0, u_9) = 5$ and $d(v_{12}, u_9) = 3$. $d(u_0, u_2) = 2$ and $d(v_{12}, u_2) = 5$. $d(u_0, u_6) = 5$ and $d(v_{12}, u_6) = 4$. $d(u_0, u_{10}) = 6$ and $d(v_{12}, u_{10}) = 3$. $d(u_0, u_3) = 3$ and $d(v_{12}, u_3) = 4$. $d(u_0, u_7) = 5$ and $d(v_{12}, u_7) = 3$. $d(u_0, u_{11}) = 6$ and $d(v_{12}, u_{11}) = 2$. So $u_1, u_2, u_3 \in W^1_{u_0v_{12}}$ and $u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12} \in W^1_{v_{12}u_0}$.

Note that $u_0 \in W^1_{u_0v_{12}}$ and $v_{12} \in W^1_{v_{12}u_0}$. Combined with the above discussion, $|W^1_{u_0v_{12}}| = 10$ and $|W^1_{v_{12}u_0}| = 11$.

(1b) Computation of $|W^1_{u_0v_{4s}}|$ and $|W^1_{v_{4s}u_0}|$ when s is even and $s \ge 2$.

When $s = 2$,

$d(u_0, v_0) = 1$ and $d(v_8, v_0) = 2$. $d(u_0, v_4) = 2$ and $d(v_8, v_4) = 1$. $d(u_0, v_1) = 2$ and $d(v_8, v_1) = 5$. $d(u_0, v_5) = 3$ and $d(v_8, v_5) = 4$. $d(u_0, v_2) = 3$ and $d(v_8, v_2) = 5$. $d(u_0, v_6) = 4$ and $d(v_8, v_6) = 4$. $d(u_0, v_3) = 3$ and $d(v_8, v_3) = 4$. $d(u_0, v_7) = 4$ and $d(v_8, v_7) = 3$. So $v_0, v_1, v_2, v_3, v_5 \in W^1_{u_0v_8}$ and $v_4, v_7 \in W^1_{v_8u_0}$.

$d(u_0, u_4) = 3$ and $d(v_8, u_4) = 2$. $d(u_0, u_8) = 4$ and $d(v_8, u_8) = 1$. $d(u_0, u_1) = 1$ and $d(v_8, u_1) = 4$. $d(u_0, u_5) = 4$ and $d(v_8, u_5) = 3$. $d(u_0, u_2) = 2$ and $d(v_8, u_2) = 4$. $d(u_0, u_6) = 5$ and $d(v_8, u_6) = 3$. $d(u_0, u_3) = 3$ and $d(v_8, u_3) = 3$. $d(u_0, u_7) = 5$ and $d(v_8, u_7) = 2$. So $u_1, u_2 \in W^1_{u_0v_8}$ and $u_4, u_5, u_6, u_7, u_8 \in W^1_{v_8u_0}$.

Note that $u_0 \in W^1_{u_0v_8}$ and $v_8 \in W^1_{v_8u_0}$. Combined with the above discussion, $|W^1_{u_0v_8}| = 8$ and $|W^1_{v_8u_0}| = 8$.

When $s \ge 4$,

$d(u_0, v_{4t}) = 1 + t$ and $d(v_{4s}, v_{4t}) = s - t$ where $0 \le t < s$. When $0 \le t \le \frac{s-2}{2}$, $d(u_0, v_{4t}) < d(v_{4s}, v_{4t})$. When $\frac{s}{2} \le t < s$, $d(u_0, v_{4t}) > d(v_{4s}, v_{4t})$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_{4s}, v_{4t+1}) = s - t + 3$ where $0 \le t < s$. When $0 \le t \le \frac{s}{2}$, $d(u_0, v_{4t+1}) < d(v_{4s}, v_{4t+1})$. When $\frac{s+2}{2} \le t < s$, $d(u_0, v_{4t+1}) > d(v_{4s}, v_{4t+1})$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_{4s}, v_{4t+2}) = s - t + 3$ where $0 \le t < s$. When $0 \le t < \frac{s}{2}$, $d(u_0, v_{4t+2}) < d(v_{4s}, v_{4t+2})$. When $\frac{s}{2} < t < s$, $d(u_0, v_{4t+2}) > d(v_{4s}, v_{4t+2})$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_{4s}, v_{4t+3}) = s - t + 2$ where $0 \le t < s$. When $0 \le t \le \frac{s-2}{2}$, $d(u_0, v_{4t+3}) < d(v_{4s}, v_{4t+3})$. When $\frac{s}{2} \le t < s$, $d(u_0, v_{4t+3}) > d(v_{4s}, v_{4t+3})$.

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$d(u_0, u_{4t}) = 2+t$ and $d(v_{4s}, u_{4t}) = s-t+1$ where $1 \le t \le s$. When $1 \le t \le \frac{s-2}{2}$, $d(u_0, u_{4t}) < d(v_{4s}, u_{4t})$. When $\frac{s}{2} \le t \le s$, $d(u_0, u_{4t}) > d(v_{4s}, u_{4t})$. $d(u_0, u_1) = 1$ and $d(v_{4s}, u_1) = s + 2$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_{4s}, u_{4t+1}) = s - t + 2$ where $1 \le t < s$. When $1 \le t \le \frac{s-2}{2}$, $d(u_0, u_{4t+1}) < d(v_{4s}, u_{4t+1})$. When $\frac{s}{2} \le t < s$, $d(u_0, u_{4t+1}) > d(v_{4s}, u_{4t+1})$. $d(u_0, u_2) = 2$ and $d(v_{4s}, u_2) = s + 2$. $d(u_0, u_{4t+2}) = 4+t$ and $d(v_{4s}, u_{4t+2}) = s-t+2$ where $1 \le t < s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{4t+2}) < d(v_{4s}, u_{4t+2})$. When $\frac{s-2}{2} < t < s$, $d(u_0, u_{4t+2}) > d(v_{4s}, u_{4t+2})$. $d(u_0, u_3) = 3$ and $d(v_{4s}, u_3) = s + 1$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_{4s}, u_{4t+3}) = s - t + 1$ where $1 \le t < s$. When $1 \le t \le \frac{s-4}{2}$, $d(u_0, u_{4t+3}) < d(v_{4s}, u_{4t+3})$. When $\frac{s-2}{2} \le t < s$, $d(u_0, u_{4t+3}) > d(v_{4s}, u_{4t+3})$.

Note that $u_0 \in W^1_{u_0 v_{4s}}$ and $v_{4s} \in W^1_{v_{4s} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{4s}}| = 4s - 1$ and $|W^1_{v_{4s} u_0}| = 4s + 1$.

(2a) Computation of $|W^1_{u_0 v_{4s+1}}|$ and $|W^1_{v_{4s+1} u_0}|$ when s is odd and $s \ge 3$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_{4s+1}, v_{4t}) = s - t + 3$ where $0 \le t \le s$. When $0 \le t \le \frac{s+1}{2}$, $d(u_0, v_{4t}) < d(v_{4s+1}, v_{4t})$. When $\frac{s+3}{2} \le t \le s$, $d(u_0, v_{4t}) > d(v_{4s+1}, v_{4t})$. $d(u_0, v_{4t+1}) = 2+t$ and $d(v_{4s+1}, v_{4t+1}) = s-t$ where $0 \le t < s$. When $0 \le t \le \frac{s-3}{2}$, $d(u_0, v_{4t+1}) < d(v_{4s+1}, v_{4t+1})$. When $\frac{s-1}{2} \le t < s$, $d(u_0, v_{4t+1}) > d(v_{4s+1}, v_{4t+1})$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_{4s+1}, v_{4t+2}) = s - t + 3$ where $0 \le t < s$. When $0 \le t \le \frac{s-1}{2}$, $d(u_0, v_{4t+2}) < d(v_{4s+1}, v_{4t+2})$. When $\frac{s+1}{2} \le t < s$, $d(u_0, v_{4t+2}) > d(v_{4s+1}, v_{4t+2})$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_{4s+1}, v_{4t+3}) = s - t + 3$ where $0 \le t < s$. When $0 \le t \le \frac{s-1}{2}$, $d(u_0, v_{4t+3}) < d(v_{4s+1}, v_{4t+3})$. When $\frac{s+1}{2} \le t < s$, $d(u_0, v_{4t+3}) > d(v_{4s+1}, v_{4t+3})$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_{4s+1}, u_{4t}) = s - t + 2$ where $1 \le t \le s$. When $1 \le t \le \frac{s-1}{2}$, $d(u_0, u_{4t}) < d(v_{4s+1}, u_{4t})$. When $\frac{s+1}{2} \le t \le s$, $d(u_0, u_{4t}) > d(v_{4s+1}, u_{4t})$. $d(u_0, u_1) = 1$ and $d(v_{4s+1}, u_1) = s + 1$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_{4s+1}, u_{4t+1}) = s-t+1$ where $1 \le t \le s$. When $1 \le t \le \frac{s-3}{2}$, $d(u_0, u_{4t+1}) < d(v_{4s+1}, u_{4t+1})$. When $\frac{s-1}{2} \le t \le s$, $d(u_0, u_{4t+1}) > d(v_{4s+1}, u_{4t+1})$. $d(u_0, u_2) = 2$ and $d(v_{4s+1}, u_2) = s + 2$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_{4s+1}, u_{4t+2}) = s - t + 2$ where $1 \le t < s$. When $1 \le t \le \frac{s-3}{2}$, $d(u_0, u_{4t+2}) < d(v_{4s+1}, u_{4t+2})$. When $\frac{s-1}{2} \le t < s$, $d(u_0, u_{4t+2}) > d(v_{4s+1}, u_{4t+2})$. $d(u_0, u_3) = 3$ and $d(v_{4s+1}, u_3) = s + 2$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_{4s+1}, u_{4t+3}) = s - t + 2$ where $1 \le t < s$. When $1 \le t \le \frac{s-3}{2}$, $d(u_0, u_{4t+3}) < d(v_{4s+1}, u_{4t+3})$. When $\frac{s-1}{2} \le t < s$, $d(u_0, u_{4t+3}) > d(v_{4s+1}, u_{4t+3})$.

Note that $u_0 \in W^1_{u_0 v_{4s+1}}$ and $v_{4s+1} \in W^1_{v_{4s+1} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{4s+1}}| = 4s + 1$ and $|W^1_{v_{4s+1} u_0}| = 4s + 3$.

(2b) Computation of $|W^1_{u_0 v_{4s+1}}|$ and $|W^1_{v_{4s+1} u_0}|$ when s is even and $s \ge 4$.

When $s \ge 4$,

$d(u_0, v_{4t}) = 1 + t$ and $d(v_{4s+1}, v_{4t}) = s - t + 3$ where $0 \le t \le s$. When $0 \le t < \frac{s+2}{2}$, $d(u_0, v_{4t}) < d(v_{4s+1}, v_{4t})$. When $\frac{s+2}{2} < t \le s$, $d(u_0, v_{4t}) > d(v_{4s+1}, v_{4t})$. $d(u_0, v_{4t+1}) = 2+t$ and $d(v_{4s+1}, v_{4t+1}) = s-t$ where $0 \le t < s$. When $0 \le t < \frac{s-2}{2}$, $d(u_0, v_{4t+1}) < d(v_{4s+1}, v_{4t+1})$. When $\frac{s-2}{2} < t < s$, $d(u_0, v_{4t+1}) > d(v_{4s+1}, v_{4t+1})$. $d(u_0, v_{4t+2}) = 3+t$ and $d(v_{4s+1}, v_{4t+2}) = s-t+3$ where $0 \le t < s$. When $0 \le t < \frac{s}{2}$, $d(u_0, v_{4t+2}) < d(v_{4s+1}, v_{4t+2})$. When $\frac{s}{2} < t < s$, $d(u_0, v_{4t+2}) > d(v_{4s+1}, v_{4t+2})$.

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$d(u_0, v_{4t+3}) = 3+t$ and $d(v_{4s+1}, v_{4t+3}) = s-t+3$ where $0 \le t < s$. When $0 \le t < \frac{s}{2}$, $d(u_0, v_{4t+3}) < d(v_{4s+1}, v_{4t+3})$. When $\frac{s}{2} < t < s$, $d(u_0, v_{4t+3}) > d(v_{4s+1}, v_{4t+3})$.

$d(u_0, u_{4t}) = 2+t$ and $d(v_{4s+1}, u_{4t}) = s-t+2$ where $1 \le t \le s$. When $1 \le t < \frac{s}{2}$, $d(u_0, u_{4t}) < d(v_{4s+1}, u_{4t})$. When $\frac{s}{2} < t \le s$, $d(u_0, u_{4t}) > d(v_{4s+1}, u_{4t})$. $d(u_0, u_1) = 1$ and $d(v_{4s+1}, u_1) = s+1$. $d(u_0, u_{4t+1}) = 3+t$ and $d(v_{4s+1}, u_{4t+1}) = s-t+1$ where $1 \le t \le s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{4t+1}) < d(v_{4s+1}, u_{4t+1})$. When $\frac{s-2}{2} < t \le s$, $d(u_0, u_{4t+1}) > d(v_{4s+1}, u_{4t+1})$. $d(u_0, u_2) = 2$ and $d(v_{4s+1}, u_2) = s+2$. $d(u_0, u_{4t+2}) = 4+t$ and $d(v_{4s+1}, u_{4t+2}) = s-t+2$ where $1 \le t < s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{4t+2}) < d(v_{4s+1}, u_{4t+2})$. When $\frac{s-2}{2} < t < s$, $d(u_0, u_{4t+2}) > d(v_{4s+1}, u_{4t+2})$. $d(u_0, u_3) = 3$ and $d(v_{4s+1}, u_3) = s+2$. $d(u_0, u_{4t+3}) = 4+t$ and $d(v_{4s+1}, u_{4t+3}) = s-t+2$ where $1 \le t < s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{4t+3}) < d(v_{4s+1}, u_{4t+3})$. When $\frac{s-2}{2} < t < s$, $d(u_0, u_{4t+3}) > d(v_{4s+1}, u_{4t+3})$.

Note that $u_0 \in W^1_{u_0 v_{4s+1}}$ and $v_{4s+1} \in W^1_{v_{4s+1} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{4s+1}}| = 4s-3$ and $|W^1_{v_{4s+1} u_0}| = 4s-1$.

(3a) Computation of $|W^1_{u_0 v_{4s+2}}|$ and $|W^1_{v_{4s+2} u_0}|$ when s is odd and $s \ge 3$.

When $s = 3$,

$d(u_0, v_0) = 1$ and $d(v_{14}, v_0) = 7$. $d(u_0, v_4) = 2$ and $d(v_{14}, v_4) = 6$. $d(u_0, v_8) = 3$ and $d(v_{14}, v_8) = 5$. $d(u_0, v_{12}) = 4$ and $d(v_{14}, v_{12}) = 4$. $d(u_0, v_1) = 2$ and $d(v_{14}, v_1) = 6$. $d(u_0, v_5) = 3$ and $d(v_{14}, v_5) = 5$. $d(u_0, v_9) = 4$ and $d(v_{14}, v_9) = 4$. $d(u_0, v_{13}) = 5$ and $d(v_{14}, v_{13}) = 3$. $d(u_0, v_2) = 3$ and $d(v_{14}, v_2) = 3$. $d(u_0, v_6) = 4$ and $d(v_{14}, v_6) = 2$. $d(u_0, v_{10}) = 5$ and $d(v_{14}, v_{10}) = 1$. $d(u_0, v_3) = 3$ and $d(v_{14}, v_3) = 6$. $d(u_0, v_7) = 4$ and $d(v_{14}, v_7) = 5$. $d(u_0, v_{11}) = 5$ and $d(v_{14}, v_{11}) = 4$. So $v_0, v_1, v_3, v_4, v_5, v_7, v_8 \in W^1_{u_0 v_{14}}$ and $v_6, v_{10}, v_{11}, v_{13} \in W^1_{v_{14} u_0}$.

$d(u_0, u_4) = 3$ and $d(v_{14}, u_4) = 5$. $d(u_0, u_8) = 4$ and $d(v_{14}, u_8) = 4$. $d(u_0, u_{12}) = 5$ and $d(v_{14}, u_{12}) = 3$. $d(u_0, u_1) = 1$ and $d(v_{14}, u_1) = 5$. $d(u_0, u_5) = 4$ and $d(v_{14}, u_5) = 4$. $d(u_0, u_9) = 5$ and $d(v_{14}, u_9) = 3$. $d(u_0, u_{13}) = 6$ and $d(v_{14}, u_{13}) = 2$. $d(u_0, u_2) = 2$ and $d(v_{14}, u_2) = 4$. $d(u_0, u_6) = 5$ and $d(v_{14}, u_6) = 3$. $d(u_0, u_{10}) = 6$ and $d(v_{14}, u_{10}) = 2$. $d(u_0, u_{14}) = 7$ and $d(v_{14}, u_{14}) = 1$. $d(u_0, u_3) = 3$ and $d(v_{14}, u_3) = 5$. $d(u_0, u_7) = 5$ and $d(v_{14}, u_7) = 4$. $d(u_0, u_{11}) = 6$ and $d(v_{14}, u_{11}) = 3$. So $u_1, u_2, u_3, u_4 \in W^1_{u_0 v_{14}}$ and $u_6, u_7, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14} \in W^1_{v_{14} u_0}$.

Note that $u_0 \in W^1_{u_0 v_{14}}$ and $v_{14} \in W^1_{v_{14} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{14}}| = 12$ and $|W^1_{v_{14} u_0}| = 13$.

When $s \ge 5$,

$d(u_0, v_{4t}) = 1+t$ and $d(v_{4s+2}, v_{4t}) = s-t+4$ where $0 \le t \le s$. When $0 \le t < \frac{s+3}{2}$, $d(u_0, v_{4t}) < d(v_{4s+2}, v_{4t})$. When $\frac{s+3}{2} < t \le s$, $d(u_0, v_{4t}) > d(v_{4s+2}, v_{4t})$. $d(u_0, v_{4t+1}) = 2+t$ and $d(v_{4s+2}, v_{4t+1}) = s-t+3$ where $0 \le t \le s$. When $0 \le t < \frac{s+1}{2}$, $d(u_0, v_{4t+1}) < d(v_{4s+2}, v_{4t+1})$. When $\frac{s+1}{2} < t \le s$, $d(u_0, v_{4t+1}) > d(v_{4s+2}, v_{4t+1})$. $d(u_0, v_{4t+2}) = 3+t$ and $d(v_{4s+2}, v_{4t+2}) = s-t$ where $0 \le t < s$. When $0 \le t < \frac{s-3}{2}$, $d(u_0, v_{4t+2}) < d(v_{4s+2}, v_{4t+2})$. When $\frac{s-3}{2} < t < s$, $d(u_0, v_{4t+2}) > d(v_{4s+2}, v_{4t+2})$. $d(u_0, v_{4t+3}) = 3+t$ and $d(v_{4s+2}, v_{4t+3}) = s-t+3$ where $0 \le t < s$. When $0 \le t \le \frac{s-1}{2}$, $d(u_0, v_{4t+3}) < d(v_{4s+2}, v_{4t+3})$. When $\frac{s+1}{2} \le t < s$, $d(u_0, v_{4t+3}) > d(v_{4s+2}, v_{4t+3})$.

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$d(u_0,u_{4t})=2+t$ and $d(v_{4s+2},u_{4t})=s-t+3$ where $1\le t\le s$. When $1\le t<\frac{s+1}{2}$, $d(u_0,u_{4t})<d(v_{4s+2},u_{4t})$. When $\frac{s+1}{2}<t\le s$, $d(u_0,u_{4t})>d(v_{4s+2},u_{4t})$. $d(u_0,u_1)=1$ and $d(v_{4s+2},u_1)=s+2$. $d(u_0,u_{4t+1})=3+t$ and $d(v_{4s+2},u_{4t+1})=s-t+2$ where $1\le t\le s$. When $1\le t<\frac{s-1}{2}$, $d(u_0,u_{4t+1})<d(v_{4s+2},u_{4t+1})$. When $\frac{s-1}{2}<t\le s$, $d(u_0,u_{4t+1})>d(v_{4s+2},u_{4t+1})$. $d(u_0,u_2)=2$ and $d(v_{4s+2},u_2)=s+1$. $d(u_0,u_{4t+2})=4+t$ and $d(v_{4s+2},u_{4t+2})=s-t+1$ where $1\le t\le s$. When $1\le t<\frac{s-3}{2}$, $d(u_0,u_{4t+2})<d(v_{4s+2},u_{4t+2})$. When $\frac{s-3}{2}<t\le s$, $d(u_0,u_{4t+2})>d(v_{4s+2},u_{4t+2})$. $d(u_0,u_3)=3$ and $d(v_{4s+2},u_3)=s+2$. $d(u_0,u_{4t+3})=4+t$ and $d(v_{4s+2},u_{4t+3})=s-t+2$ where $1\le t<s$. When $1\le t\le\frac{s-3}{2}$, $d(u_0,u_{4t+3})<d(v_{4s+2},u_{4t+3})$. When $\frac{s-1}{2}\le t<s$, $d(u_0,u_{4t+3})>d(v_{4s+2},u_{4t+3})$.

Note that $u_0\in W^1_{u_0v_{4s+2}}$ and $v_{4s+2}\in W^1_{v_{4s+2}u_0}$. Combined with the above discussion, $|W^1_{u_0v_{4s+2}}|=4s-1$ and $|W^1_{v_{4s+2}u_0}|=4s+1$.

(3b) Computation of $|W^1_{u_0v_{4s+2}}|$ and $|W^1_{v_{4s+2}u_0}|$ when s is even and $s\ge 2$.

When $s=2$,

$d(u_0,v_0)=1$ and $d(v_{10},v_0)=6$. $d(u_0,v_4)=2$ and $d(v_{10},v_4)=5$. $d(u_0,v_8)=3$ and $d(v_{10},v_8)=4$. $d(u_0,v_1)=2$ and $d(v_{10},v_1)=5$. $d(u_0,v_5)=3$ and $d(v_{10},v_5)=4$. $d(u_0,v_9)=4$ and $d(v_{10},v_9)=3$. $d(u_0,v_2)=3$ and $d(v_{10},v_2)=2$. $d(u_0,v_6)=4$ and $d(v_{10},v_6)=1$. $d(u_0,v_3)=3$ and $d(v_{10},v_3)=5$. $d(u_0,v_7)=4$ and $d(v_{10},v_7)=4$. So $v_0,v_1,v_3,v_4,v_5,v_8\in W^1_{u_0v_{10}}$ and $v_2,v_6,v_9\in W^1_{v_{10}u_0}$.

$d(u_0,u_4)=3$ and $d(v_{10},u_4)=4$. $d(u_0,u_8)=4$ and $d(v_{10},u_8)=3$. $d(u_0,u_1)=1$ and $d(v_{10},u_1)=4$. $d(u_0,u_5)=4$ and $d(v_{10},u_5)=3$. $d(u_0,u_9)=5$ and $d(v_{10},u_9)=2$. $d(u_0,u_2)=2$ and $d(v_{10},u_2)=3$. $d(u_0,u_6)=5$ and $d(v_{10},u_6)=2$. $d(u_0,u_{10})=6$ and $d(v_{10},u_{10})=1$. $d(u_0,u_3)=3$ and $d(v_{10},u_3)=4$. $d(u_0,u_7)=5$ and $d(v_{10},u_7)=3$. So $u_1,u_2,u_3,u_4\in W^1_{u_0v_{10}}$ and $u_5,u_6,u_7,u_8,u_9,u_{10}\in W^1_{v_{10}u_0}$.

Note that $u_0\in W^1_{u_0v_{10}}$ and $v_8\in W^1_{v_{10}u_0}$. Combined with the above discussion, $|W^1_{u_0v_{10}}|=11$ and $|W^1_{v_{10}u_0}|=10$.

When $s\ge 4$,

$d(u_0,v_{4t})=1+t$ and $d(v_{4s+2},v_{4t})=s-t+4$ where $0\le t\le s$. When $0\le t\le\frac{s+2}{2}$, $d(u_0,v_{4t})<d(v_{4s+2},v_{4t})$. When $\frac{s+4}{2}\le t\le s$, $d(u_0,v_{4t})>d(v_{4s+2},v_{4t})$. $d(u_0,v_{4t+1})=2+t$ and $d(v_{4s+2},v_{4t+1})=s-t+3$ where $0\le t\le s$. When $0\le t\le\frac{s}{2}$, $d(u_0,v_{4t+1})<d(v_{4s+2},v_{4t+1})$. When $\frac{s+2}{2}\le t\le s$, $d(u_0,v_{4t+1})>d(v_{4s+2},v_{4t+1})$. $d(u_0,v_{4t+2})=3+t$ and $d(v_{4s+2},v_{4t+2})=s-t$ where $0\le t<s$. When $0\le t\le\frac{s-4}{2}$, $d(u_0,v_{4t+2})<d(v_{4s+2},v_{4t+2})$. When $\frac{s-2}{2}\le t<s$, $d(u_0,v_{4t+2})>d(v_{4s+2},v_{4t+2})$. $d(u_0,v_{4t+3})=3+t$ and $d(v_{4s+2},v_{4t+3})=s-t+3$ where $0\le t<s$. When $0\le t<\frac{s}{2}$, $d(u_0,v_{4t+3})<d(v_{4s+2},v_{4t+3})$. When $\frac{s}{2}<t<s$, $d(u_0,v_{4t+3})>d(v_{4s+2},v_{4t+3})$.

$d(u_0,u_{4t})=2+t$ and $d(v_{4s+2},u_{4t})=s-t+3$ where $1\le t\le s$. When $1\le t\le\frac{s}{2}$, $d(u_0,u_{4t})<d(v_{4s+2},u_{4t})$. When $\frac{s+2}{2}\le t\le s$, $d(u_0,u_{4t})>d(v_{4s+2},u_{4t})$. $d(u_0,u_1)=1$ and $d(v_{4s+2},u_1)=s+2$. $d(u_0,u_{4t+1})=3+t$ and $d(v_{4s+2},u_{4t+1})=s-t+2$ where $1\le t\le s$. When $1\le t\le\frac{s-2}{2}$, $d(u_0,u_{4t+1})<d(v_{4s+2},u_{4t+1})$. When $\frac{s}{2}\le t\le s$, $d(u_0,u_{4t+1})>d(v_{4s+2},u_{4t+1})$. $d(u_0,u_2)=2$ and $d(v_{4s+2},u_2)=s+1$. $d(u_0,u_{4t+2})=4+t$ and $d(v_{4s+2},u_{4t+2})=s-t+1$ where $1\le t\le s$. When $1\le t\le\frac{s-4}{2}$, $d(u_0,u_{4t+2})<d(v_{4s+2},u_{4t+2})$. When $\frac{s-2}{2}\le t\le s$, $d(u_0,u_{4t+2})>d(v_{4s+2},u_{4t+2})$. $d(u_0,u_3)=3$ and $d(v_{4s+2},u_3)=s+2$. $d(u_0,u_{4t+3})=4+t$ and

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$d(v_{4s+2}, u_{4t+3}) = s-t+2$ where $1 \le t < s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{4t+3}) < d(v_{4s+2}, u_{4t+3})$. When $\frac{s-2}{2} < t < s$, $d(u_0, u_{4t+3}) > d(v_{4s+2}, u_{4t+3})$.

Note that $u_0 \in W^1_{u_0 v_{4s+2}}$ and $v_{4s+2} \in W^1_{v_{4s+2} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{4s+2}}| = 4s+1$ and $|W^1_{v_{4s+2} u_0}| = 4s+3$.

(4a) Computation of $|W^1_{u_0 v_{4s+3}}|$ and $|W^1_{v_{4s+3} u_0}|$ when s is odd and $s \ge 3$.

When $s = 3$,

$d(u_0, v_0) = 1$ and $d(v_{15}, v_0) = 7$. $d(u_0, v_4) = 2$ and $d(v_{15}, v_4) = 6$. $d(u_0, v_8) = 3$ and $d(v_{15}, v_8) = 5$. $d(u_0, v_{12}) = 4$ and $d(v_{15}, v_{12}) = 4$. $d(u_0, v_1) = 2$ and $d(v_{15}, v_1) = 7$. $d(u_0, v_5) = 3$ and $d(v_{15}, v_5) = 6$. $d(u_0, v_9) = 4$ and $d(v_{15}, v_9) = 5$. $d(u_0, v_{13}) = 5$ and $d(v_{15}, v_{13}) = 4$. $d(u_0, v_2) = 3$ and $d(v_{15}, v_2) = 6$. $d(u_0, v_6) = 4$ and $d(v_{15}, v_6) = 5$. $d(u_0, v_{10}) = 5$ and $d(v_{15}, v_{10}) = 4$. $d(u_0, v_{14}) = 6$ and $d(v_{15}, v_{14}) = 3$. $d(u_0, v_3) = 3$ and $d(v_{15}, v_3) = 3$. $d(u_0, v_7) = 4$ and $d(v_{15}, v_7) = 2$. $d(u_0, v_{11}) = 5$ and $d(v_{15}, v_{11}) = 1$. So $v_0, v_1, v_2, v_4, v_5, v_6, v_8, v_9 \in W^1_{u_0 v_{15}}$ and $v_7, v_{10}, v_{11}, v_{13}, v_{14} \in W^1_{v_{15} u_0}$.

$d(u_0, u_4) = 3$ and $d(v_{15}, u_4) = 5$. $d(u_0, u_8) = 4$ and $d(v_{15}, u_8) = 4$. $d(u_0, u_{12}) = 5$ and $d(v_{15}, u_{12}) = 3$. $d(u_0, u_1) = 1$ and $d(v_{15}, u_1) = 6$. $d(u_0, u_5) = 4$ and $d(v_{15}, u_5) = 5$. $d(u_0, u_9) = 5$ and $d(v_{15}, u_9) = 4$. $d(u_0, u_{13}) = 6$ and $d(v_{15}, u_{13}) = 3$. $d(u_0, u_2) = 2$ and $d(v_{15}, u_2) = 5$. $d(u_0, u_6) = 5$ and $d(v_{15}, u_6) = 4$. $d(u_0, u_{10}) = 6$ and $d(v_{15}, u_{10}) = 3$. $d(u_0, u_{14}) = 7$ and $d(v_{15}, u_{14}) = 2$. $d(u_0, u_3) = 3$ and $d(v_{15}, u_3) = 4$. $d(u_0, u_7) = 5$ and $d(v_{15}, u_7) = 3$. $d(u_0, u_{11}) = 6$ and $d(v_{15}, u_{11}) = 2$. $d(u_0, u_{15}) = 7$ and $d(v_{15}, u_{15}) = 1$. So $u_1, u_2, u_3, u_4, u_5 \in W^1_{u_0 v_{15}}$ and $u_6, u_7, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ are in $W^1_{v_{15} u_0}$.

Note that $u_0 \in W^1_{u_0 v_{15}}$ and $v_{15} \in W^1_{v_{15} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{15}}| = 14$ and $|W^1_{v_{15} u_0}| = 15$.

When $s \ge 5$,

$d(u_0, v_{4t}) = 1+t$ and $d(v_{4s+3}, v_{4t}) = s-t+4$ where $0 \le t \le s$. When $0 \le t < \frac{s+3}{2}$, $d(u_0, v_{4t}) < d(v_{4s+3}, v_{4t})$. When $\frac{s+3}{2} < t \le s$, $d(u_0, v_{4t}) > d(v_{4s+3}, v_{4t})$. $d(u_0, v_{4t+1}) = 2+t$ and $d(v_{4s+3}, v_{4t+1}) = s-t+4$ where $0 \le t \le s$. When $0 \le t \le \frac{s+1}{2}$, $d(u_0, v_{4t+1}) < d(v_{4s+3}, v_{4t+1})$. When $\frac{s+3}{2} \le t \le s$, $d(u_0, v_{4t+1}) > d(v_{4s+3}, v_{4t+1})$. $d(u_0, v_{4t+2}) = 3+t$ and $d(v_{4s+3}, v_{4t+2}) = s-t+3$ where $0 \le t \le s$. When $0 \le t \le \frac{s-1}{2}$, $d(u_0, v_{4t+2}) < d(v_{4s+3}, v_{4t+2})$. When $\frac{s+1}{2} \le t \le s$, $d(u_0, v_{4t+2}) > d(v_{4s+3}, v_{4t+2})$. $d(u_0, v_{4t+3}) = 3+t$ and $d(v_{4s+3}, v_{4t+3}) = s-t$ where $0 \le t < s$. When $0 \le t < \frac{s-3}{2}$, $d(u_0, v_{4t+3}) < d(v_{4s+3}, v_{4t+3})$. When $\frac{s-3}{2} < t < s$, $d(u_0, v_{4t+3}) > d(v_{4s+3}, v_{4t+3})$.

$d(u_0, u_{4t}) = 2+t$ and $d(v_{4s+3}, u_{4t}) = s-t+3$ where $1 \le t \le s$. When $1 \le t < \frac{s+1}{2}$, $d(u_0, u_{4t}) < d(v_{4s+3}, u_{4t})$. When $\frac{s+1}{2} < t \le s$, $d(u_0, u_{4t}) > d(v_{4s+3}, u_{4t})$. $d(u_0, u_1) = 1$ and $d(v_{4s+3}, u_1) = s+3$. $d(u_0, u_{4t+1}) = 3+t$ and $d(v_{4s+3}, u_{4t+1}) = s-t+3$ where $1 \le t \le s$. When $1 \le t \le \frac{s-1}{2}$, $d(u_0, u_{4t+1}) < d(v_{4s+3}, u_{4t+1})$. When $\frac{s+1}{2} \le t \le s$, $d(u_0, u_{4t+1}) > d(v_{4s+3}, u_{4t+1})$. $d(u_0, u_2) = 2$ and $d(v_{4s+3}, u_2) = s+2$. $d(u_0, u_{4t+2}) = 4+t$ and $d(v_{4s+3}, u_{4t+2}) = s-t+2$ where $1 \le t \le s$. When $1 \le t \le \frac{s-3}{2}$, $d(u_0, u_{4t+2}) < d(v_{4s+3}, u_{4t+2})$. When $\frac{s-1}{2} \le t \le s$, $d(u_0, u_{4t+2}) > d(v_{4s+3}, u_{4t+2})$. $d(u_0, u_3) = 3$ and $d(v_{4s+3}, u_3) = s+1$. $d(u_0, u_{4t+3}) = 4+t$ and

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$d(v_{4s+3}, u_{4t+3}) = s - t + 1$ where $1 \le t \le s$. When $1 \le t < \frac{s-3}{2}$, $d(u_0, u_{4t+3}) < d(v_{4s+3}, u_{4t+3})$. When $\frac{s-3}{2} < t \le s$, $d(u_0, u_{4t+3}) > d(v_{4s+3}, u_{4t+3})$.

Note that $u_0 \in W^1_{u_0 v_{4s+3}}$ and $v_{4s+3} \in W^1_{v_{4s+3} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{4s+3}}| = 4s + 1$ and $|W^1_{v_{4s+3} u_0}| = 4s + 3$.

(4b) Computation of $|W^1_{u_0 v_{4s+3}}|$ and $|W^1_{v_{4s+3} u_0}|$ when s is even.

When $s = 2$.

$d(u_0, v_0) = 1$ and $d(v_{11}, v_0) = 6$. $d(u_0, v_4) = 2$ and $d(v_{11}, v_4) = 5$. $d(u_0, v_8) = 3$ and $d(v_{11}, v_8) = 4$. $d(u_0, v_1) = 2$ and $d(v_{11}, v_1) = 6$. $d(u_0, v_5) = 3$ and $d(v_{11}, v_5) = 5$. $d(u_0, v_9) = 4$ and $d(v_{11}, v_9) = 4$. $d(u_0, v_2) = 3$ and $d(v_{11}, v_2) = 5$. $d(u_0, v_6) = 4$ and $d(v_{11}, v_6) = 4$. $d(u_0, v_{10}) = 5$ and $d(v_{11}, v_{10}) = 3$. $d(u_0, v_3) = 3$ and $d(v_{11}, v_3) = 2$. $d(u_0, v_7) = 4$ and $d(v_{11}, v_7) = 1$. So $v_0, v_1, v_2, v_4, v_5, v_8 \in W^1_{u_0 v_{11}}$ and $v_3, v_7, v_{10} \in W^1_{v_{11} u_0}$.

$d(u_0, u_4) = 3$ and $d(v_{11}, u_4) = 4$. $d(u_0, u_8) = 4$ and $d(v_{11}, u_8) = 3$. $d(u_0, u_1) = 1$ and $d(v_{11}, u_1) = 5$. $d(u_0, u_5) = 4$ and $d(v_{11}, u_5) = 4$. $d(u_0, u_9) = 5$ and $d(v_{11}, u_9) = 3$. $d(u_0, u_2) = 2$ and $d(v_{11}, u_2) = 4$. $d(u_0, u_6) = 5$ and $d(v_{11}, u_6) = 3$. $d(u_0, u_{10}) = 6$ and $d(v_{11}, u_{10}) = 2$. $d(u_0, u_3) = 3$ and $d(v_{11}, u_3) = 3$. $d(u_0, u_7) = 5$ and $d(v_{11}, u_7) = 2$. $d(u_0, u_{11}) = 6$ and $d(v_{11}, u_{11}) = 1$. So $u_1, u_2, u_4 \in W^1_{u_0 v_{11}}$ and $u_6, u_7, u_8, u_9, u_{10}, u_{11} \in W^1_{v_{11} u_0}$.

Note that $u_0 \in W^1_{u_0 v_{11}}$ and $v_{11} \in W^1_{v_{11} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{11}}| = 10$ and $|W^1_{v_{11} u_0}| = 10$.

When $s \ge 4$.

$d(u_0, v_{4t}) = 1 + t$ and $d(v_{4s+3}, v_{4t}) = s - t + 4$ where $0 \le t \le s$. When $0 \le t \le \frac{s+2}{2}$, $d(u_0, v_{4t}) < d(v_{4s+3}, v_{4t})$. When $\frac{s+4}{2} \le t \le s$, $d(u_0, v_{4t}) > d(v_{4s+3}, v_{4t})$. $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_{4s+3}, v_{4t+1}) = s - t + 4$ where $0 \le t \le s$. When $0 \le t < \frac{s+2}{2}$, $d(u_0, v_{4t+1}) < d(v_{4s+3}, v_{4t+1})$. When $\frac{s+2}{2} < t \le s$, $d(u_0, v_{4t+1}) > d(v_{4s+3}, v_{4t+1})$. $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_{4s+3}, v_{4t+2}) = s - t + 3$ where $0 \le t \le s$. When $0 \le t < \frac{s}{2}$, $d(u_0, v_{4t+2}) < d(v_{4s+3}, v_{4t+2})$. When $\frac{s}{2} < t \le s$, $d(u_0, v_{4t+2}) > d(v_{4s+3}, v_{4t+2})$. $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_{4s+3}, v_{4t+3}) = s - t$ where $0 \le t < s$. When $0 \le t \le \frac{s-4}{2}$, $d(u_0, v_{4t+3}) < d(v_{4s+3}, v_{4t+3})$. When $\frac{s-2}{2} \le t < s$, $d(u_0, v_{4t+3}) > d(v_{4s+3}, v_{4t+3})$.

$d(u_0, u_{4t}) = 2 + t$ and $d(v_{4s+3}, u_{4t}) = s - t + 3$ where $1 \le t \le s$. When $1 \le t \le \frac{s}{2}$, $d(u_0, u_{4t}) < d(v_{4s+3}, u_{4t})$. When $\frac{s+2}{2} \le t \le s$, $d(u_0, u_{4t}) > d(v_{4s+3}, u_{4t})$. $d(u_0, u_1) = 1$ and $d(v_{4s+3}, u_1) = s + 3$. $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_{4s+3}, u_{4t+1}) = s - t + 3$ where $1 \le t \le s$. When $1 \le t < \frac{s}{2}$, $d(u_0, u_{4t+1}) < d(v_{4s+3}, u_{4t+1})$. When $\frac{s}{2} < t \le s$, $d(u_0, u_{4t+1}) > d(v_{4s+3}, u_{4t+1})$. $d(u_0, u_2) = 2$ and $d(v_{4s+3}, u_2) = s + 2$. $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_{4s+3}, u_{4t+2}) = s - t + 2$ where $1 \le t \le s$. When $1 \le t < \frac{s-2}{2}$, $d(u_0, u_{4t+2}) < d(v_{4s+3}, u_{4t+2})$. When $\frac{s-2}{2} < t \le s$, $d(u_0, u_{4t+2}) > d(v_{4s+3}, u_{4t+2})$. $d(u_0, u_3) = 3$ and $d(v_{4s+3}, u_3) = s + 1$. $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_{4s+3}, u_{4t+3}) = s - t + 1$ where $1 \le t \le s$. When $1 \le t \le \frac{s-4}{2}$, $d(u_0, u_{4t+3}) < d(v_{4s+3}, u_{4t+3})$. When $\frac{s-2}{2} \le t \le s$, $d(u_0, u_{4t+3}) > d(v_{4s+3}, u_{4t+3})$.

Note that $u_0 \in W^1_{u_0 v_{4s+3}}$ and $v_{4s+3} \in W^1_{v_{4s+3} u_0}$. Combined with the above discussion, $|W^1_{u_0 v_{4s+3}}| = 4s + 1$ and $|W^1_{v_{4s+3} u_0}| = 4s + 3$. □

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