# On idomatic partitions of direct products of complete graphs

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**Abstract**

Independent dominating sets in the direct product of four complete graphs are considered. Possible types of such sets are classified. The sets in which every pair of vertices agree in exactly one coordinate, called $T_1$-sets, are explicitly described. It is proved that the direct product of four complete graphs admits an idomatic partition into $T_1$-sets if and only if each factor has at least three vertices and the orders of at least two factors are divisible by 3.

**Key words**: independent set; dominating set; idomatic partition; direct product of graph; complete graph

## 1 Introduction

The direct product of graphs is the product in the category of graphs and has been extensively investigated since the 1960's, see [9] for results up to 2000. Rather surprisingly, many fundamental discoveries about the direct product were obtained recently. Let us briefly mention some of them.

Cancellation properties of the direct product were studied by Lovász [12] who proved that nonbipartite factors can always be canceled. The cancellation for bipartite factors was finally clarified in [6] so that the situations in which cancellation holds or fails are now completely described. Hammack [5] also proved a conjecture from [10] and thus finished a clarification of the structure of the direct product of two bipartite graphs. From the algorithmic point of view, Imrich [8] designed the first polynomial algorithm for factorization of nonbipartite graphs.

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Very recently, the original approach was simplified in [7] where the computational complexity is given for the first time.

In many respects the direct product is the most difficult among the standard graph products. For instance, while it is not difficult to see that the direct product is connected if and only if both factors are connected and at least one is not bipartite [18], it is a difficult problem to determine the exact connectivity of direct products [1]. Probably the most challenging open problem related to the direct product of graphs is Hedetniemi's conjecture. It asserts that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. Despite many efforts it is still widely open, see the surveys [13, 15, 19]. The strongest general known result goes back to El-Zahar and Sauer who proved that the conjecture holds for 4-chromatic graphs [4]. (Other types of colorings were also considered on the direct product, see, for instance, [14].)

Colorings are just partitions into independent sets. In this paper we focus on partitions into independent sets that are at the same time dominating sets. Such partitions are called idomatic partitions. It was proved in [3] that the only idomatic partitions of the direct product of two complete graphs consist of parts in which one coordinate is fixed and the other arbitrary. The authors also posed the problem of characterizing idomatic partitions of direct products of at least three complete graphs. The problem for three factors was solved in [16]. Roughly speaking, there are only two types of independent dominating sets that can be combined into idomatic partitions. Here we consider products of four factors in which case the variety of types (see the next section for the definition of a type) is already bigger.

The rest of the paper is organized as follows. In Section 2 we fix notation and terminology and introduce possible types of independent dominating sets. Then, in Section 3, all our results are presented. We show that among seven potential types, four are not possible and for three there exist independent dominating sets. Then a characterization of independent dominating sets in which every pair of vertices agree in exactly one coordinate is given. Each such set necessary contains nine vertices whose coordinates can be explicitly described. Our last result asserts that an idomatic partition into such sets exists if and only if the order of at least two factors is divisible by 3. The proofs are given is Section 4. The concluding section contains several comments and problems.

## 2 Preliminaries

The *direct product* $G \times H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$ and edges $(g_1, h_1)(g_2, h_2)$, where $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. This graph product is commutative and associative, hence it extends naturally to more than two factors. The direct product of graphs $G_1, \ldots, G_n$ will be denoted $\times_{i=1}^{n} G_i$. It is well-known (cf. [9]) that the direct product of vertex-transitive graphs is vertex-transitive. In the rest we will frequently and implicitly use this fact.

A fall coloring of a graph $G$ is a partition of $V(G)$ into color classes that are (as usual) independent sets, and every vertex $u$ has at least one neighbor in each of the other color classes. Not every graph contains a fall coloring, consider for instance the 5-cycle, so the basic question here is which graphs admit such colorings. It is not difficult to see that fall colorings of $G$ coincide with partitions of the vertices of $G$ into independent dominating sets. Such a partition is in this context known as an *idomatic partition*.

Fall colorings were introduced in [3], but a closely related concept was studied back in 1976 by Cockayne and Hedetniemi [2]: they were interested in the largest number of independent dominating sets contained in a given graph. Fall colorings were further studied in [11] where it is proved that a strongly chordal graph $G$ has a fall coloring if and only if the clique number of $G$ equals the minimum degree in $G$ plus one.

As already mentioned, Valencia-Pabon [16] characterized idomatic partitions of direct products of three complete graphs. In that case, there are two types of independent dominating sets that can be combined into idomatic partitions. To deal with four factors, the following concepts will be useful.

For a complete graph $K_n$, $n \geq 1$, we will always assume $V(K_n) = [n] = \{0, 1, \ldots, n-1\}$. Let $G = \times_{i=1}^{k} K_{n_i}$ and let $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ be vertices of $G$. Then let

$$e(u, v) = |\{i \mid u_i = v_i\}|$$

be the number of coordinates in which $u$ and $v$ coincide. Note that the function $e$ can be defined for direct products of arbitrary graphs. Observe also that $e(u, v) = k - H(u, v)$, where $H(u, v)$ is the Hamming distance between the vectors $u$ and $v$, that is, the number of coordinates in which they differ. However, since the distance in the direct product is not the Hamming distance, we prefer to use $e(u, v)$.

With this notation we can state that $u$ and $v$ are adjacent in $G = \times_{i=1}^{k} K_{n_i}$ if and only if $e(u, v) = 0$. Therefore $I \subseteq V(G)$ is independent if and only if $e(u, v) > 0$ for any $u, v \in I$. Note also that $e(u, v) \leq k - 1$ holds for any $u \neq v$.

Now comes the key definition. Let $X \subseteq V(G)$ be an independent set of $G = \times_{i=1}^{k} K_{n_i}$. Let

$$\{e(u, v) \mid u, v \in X, u \neq v\} = \{j_1, \ldots, j_r\}.$$

Then we say that $X$ is a $T_{j_1,\ldots,j_r}$*-set*.

Let $I \subseteq V(G)$, $G = \times_{i=1}^{3} K_{n_i}$, be an independent and dominating set of $G$. Then $I$ can only be a $T_1$-set, a $T_2$-set, or a $T_{1,2}$-set. Valencia-Pabon [16] showed that there is no such $T_2$-set and described the other two types as well as determined when they form idomatic partitions. (The sporadic example from [3] is a $T_1$-set.)

## 3 Results

We consider direct products of four complete graphs and index the factors from 0 to 3. We first exclude the following four types.

**Proposition 3.1** *Let $I$ be an independent dominating set of $G = \times_{i=0}^{3} K_{n_i}$, $n_i \geq 2$. Then $I$ is not $T_2$, $T_3$, $T_{2,3}$, and $T_{1,3}$.*

So we are left with the possible types $T_1, T_{1,2}, T_{1,2,3}$ and all three of them are achievable. We have already mentioned that for two and three factors one can construct independent dominating sets by fixing one coordinate. This is true for any number of factors, in particular the vertex subset of $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$, $n_i \geq 2$, defined with

$$I = [n_0] \times [n_1] \times [n_2] \times \{i\}\,,$$

where $i \in [n_3]$, is independent and dominating. Moreover, $I$ is a $T_{1,2,3}$-set. Of course, we could fix any of the four coordinates in the above construction. But there are additional sporadic independent dominating sets that are $T_{1,2,3}$. Consider $K_2 \times K_2 \times K_2 \times K_2$, then

$$\begin{aligned} I_1 &= \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),\\ &\quad (0,0,0,1),(1,1,0,0),(1,0,1,0),(1,0,0,1)\} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \{(1,1,1,1),(0,1,1,1),(1,0,1,1),(1,1,0,1),\\ &\quad (1,1,1,0),(0,0,1,1),(0,1,0,1),(0,1,1,0)\} \end{aligned}$$

are both dominating $T_{1,2,3}$-sets. In addition, they form an idomatic partition.

We next give an idomatic partition into $T_{1,2}$-sets. Let $G = K_2 \times K_2 \times K_2 \times K_4$ and consider the set

$$\begin{aligned} I_1 &= \{(0,0,0,0),(1,1,1,0),(0,1,1,1),(1,0,0,1),\\ &\quad (1,0,1,2),(0,1,0,2),(1,1,0,3),(0,0,1,3)\}. \end{aligned}$$

Clearly, $I_1$ is a $T_{1,2}$-set. But it is also dominating. To see it, note first that $G$ consists of four connected components isomorphic to $K_2 \times K_4$ which is in turn isomorphic to the 3-cube $Q_3$. Then consecutive pairs of vertices dominate the four copies of $Q_3$ respectively. Finally, the sets $I_1, I_2, I_3, I_4$, where

$$\begin{aligned} I_2 &= \{(u_1 + 1 \bmod 2, u_2, u_3, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\}\,, \\ I_3 &= \{(u_1, u_2 + 1 \bmod 2, u_3, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\}\,, \\ I_4 &= \{(u_1, u_2, u_3 + 1 \bmod 2, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\}\,, \end{aligned}$$

form an idomatic partition of $G$.

In our first main result we characterize $T_1$-sets as follows.

**Theorem 3.2** *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$, $n_i \geq 2$, and let $I$ be a $T_1$-set of $G$. Then $I$ is a dominating set if and only if $n_i \geq 3$ and $I$ is of the form*

$$\begin{aligned} I = \{&(\alpha_0, \alpha_1, \alpha_2, \alpha_3), (\alpha_0, \beta_1, \beta_2, \beta_3), (\alpha_0, \gamma_1, \gamma_2, \gamma_3), \\ &(\beta_0, \alpha_1, \beta_2, \gamma_3), (\beta_0, \gamma_1, \alpha_2, \beta_3), (\beta_0, \beta_1, \gamma_2, \alpha_3), \\ &(\gamma_0, \alpha_1, \gamma_2, \beta_3), (\gamma_0, \beta_1, \alpha_2, \gamma_3), (\gamma_0, \gamma_1, \beta_2, \alpha_3)\}\,, \end{aligned}$$

*where $\alpha_i, \beta_i, \gamma_i \in K_{n_i}$ are pairwise different, $0 \leq i \leq 3$.*

In the second main result we characterize the products which admit idomatic partitions into $T_1$-sets.

**Theorem 3.3** *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$, $n_i \geq 2$. Then $G$ has an idomatic partition into $T_1$-sets if and only if $n_i \geq 3$ and there exist indices $j, k \in [4]$, $j \neq k$, such that $3 | n_j$ and $3 | n_k$.*

# 4 Proofs

**Proof of Proposition 3.1**. Assume on the contrary that $I$ is a $T_2$-set or a $T_{2,3}$-set. Since $I$ is dominating, $|I| > 2$. Assume $(0,0,0,0) \in I$. Using vertex-transitivity and the commutativity of the direct product we may also assume that $(0,0,1,1) \in I$. Since $e((0,0,1,1),(1,0,0,0)) = 1$, we get $(1,0,0,0) \notin I$. Hence there exists a vertex $(a,b,c,d) \in I$ such that $(a,b,c,d)$ dominates $(1,0,0,0)$. This in particular implies that $b, c, d \neq 0$. But then $e((a,b,c,d),(0,0,0,0)) \leq 1$, a contradiction.

Suppose next that $I$ is a $T_3$-set. We may assume that $\{(0,0,0,0),(0,0,0,1)\} \subseteq I$. Clearly, $(0,0,1,1) \notin I$, hence there exists a vertex $(a,b,c,d) \in I$ that dominates $(0,0,1,1)$. But then $a, b \neq 0$ and thus $e((a,b,c,d),(0,0,0,0)) \leq 2$.

In the last case let $I$ be a $T_{1,3}$ set. Again, assume that $\{(0,0,0,0),(0,0,0,1)\} \subseteq I$. From $e((0,0,0,1),(0,0,1,0)) = 2$ we deduce that $(0,0,1,0) \notin I$. Hence there is a vertex $(a,b,c,d) \in I$ that dominates $(0,0,1,0)$. The elements $a$, $b$, and

$d$ are different from 0, therefore $c = 0$ due to the independence of $I$. Moreover, $e((0,0,0,1),(a,b,0,1)) = 2$, hence $d \neq 1$. Since we already know that $d \neq 0$, the fourth factor must contain more than 2 vertices, for otherwise no element from $I$ would dominate $(0,0,1,0)$. We next consider the vertex $(0,0,0,d)$, $d \notin \{0,1\}$. Suppose that $(0,0,0,d) \notin I$. $I$ is dominating, so there exists a vertex $(e,f,g,h) \in I$, such that $e,f,g \neq 0$ and $h \neq d$. But then the vertex $(e,f,g,h)$ is adjacent with at least one of the vertices $(0,0,0,0),(0,0,0,1) \in I$, a contradiction. Thus we must have $(0,0,0,d) \in I$. The proof is concluded by observing that $e((0,0,0,d),(a,b,0,d)) = 2$. □

**Proof of Theorem 3.2**. We will prove the theorem in two steps. In the first, major step, we assume that $n_i \geq 3$, $0 \leq i \leq 3$.

So let $I$ be a dominating $T_1$-set of $G$ and without loss of generality assume that $(0,0,0,0) \in I$. Clearly, $|I| > 1$, hence we may further assume that $(0,\beta_1,\beta_2,\beta_3) \in I$, where $\beta_1,\beta_2,\beta_3 \neq 0$.

If $w \neq 0$ then $(0,0,0,w) \notin I$, hence there exists a vertex $(\beta_0,b,c,d) \in I$, where $\beta_0,b,c \neq 0$ and $d \neq w$. Since the latter vertex is not adjacent to $(0,0,0,0)$ we get $d = 0$. In addition, it is also not adjacent to $(0,\beta_1,\beta_2,\beta_3)$, hence either $b = \beta_1$ and $c \neq \beta_2$, or $b \neq \beta_1$ and $c = \beta_2$. In the first case set $c = \gamma_2$, in the second $b = \gamma_1$. This gives the following two possibilities:

$$A_1 = \{(0,0,0,0),(0,\beta_1,\beta_2,\beta_3),(\beta_0,\beta_1,\gamma_2,0)\} \subseteq I$$

and

$$A_2 = \{(0,0,0,0),(0,\beta_1,\beta_2,\beta_3),(\beta_0,\gamma_1,\beta_2,0)\} \subseteq I.$$

Similarly, the vertex $(0,y,0,0)$ does not belong to $I$, hence there exists $(a,b,c,d) \in I$ (for some $a,b,c,d$ different from above) that dominates $(0,y,0,0)$. As $(0,0,0,0) \in I$, we infer that $b = 0$. Now comparing $(a,0,c,d)$ with $(0,\beta_1,\beta_2,\beta_3)$ we obtain either (i) $(a,0,c,d) = (a,0,\beta_2,d), d \neq \beta_3$, or (ii) $(a,0,c,d) = (a,0,c,\beta_3), c \neq \beta_2$.

Consider case (i). Comparing $(a,0,\beta_2,d)$ $(d \neq \beta_3)$ with the third vertex of $A_1$ we get $a = \beta_0$. Since $0 \neq d \neq \beta_3$ we can set $d = \gamma_3$. Similarly, comparing $(a,0,\beta_2,d)$ with the third vertex of $A_2$ we obtain $0 \neq a \neq \beta_0$ and $0 \neq d \neq \beta_3$, hence we can set $a = \gamma_0$ and $d = \gamma_3$.

Consider case (ii). In this case $(a,0,c,\beta_3)$ $(c \neq \beta_2)$ is a candidate for a vertex from $I$. Considering the possibility of $A_1$, we must have either $a \neq \beta_0$ and $c = \gamma_2$, or $a = \beta_0$ and $\beta_2 \neq c \neq \gamma_2$. In the latter option set $c = \delta_2$. Note that when $n_2 = 3$ this is not possible since there are only three coordinates available for the $K_{n_2}$ factor. Comparing $(a,0,c,\beta_3)$ with the third vertex of $A_2$ we find that $a = \beta_0$. Since $0 \neq c \neq \beta_2$ we can set $c = \gamma_2$. Hence we have altogether obtained

5 possible subsets of $I$:

$$\begin{aligned}
B_1 &= A_1 \cup \{(\beta_0, 0, \beta_2, \gamma_3)\},\\
B_2 &= A_2 \cup \{(\gamma_0, 0, \beta_2, \gamma_3)\},\\
B_3 &= A_1 \cup \{(\gamma_0, 0, \gamma_2, \beta_3)\},\\
B_4 &= A_1 \cup \{(\beta_0, 0, \delta_2, \beta_3)\},\\
B_5 &= A_2 \cup \{(\beta_0, 0, \gamma_2, \beta_3)\}.
\end{aligned}$$

We next take into account that $(0,0,z,0) \notin I$ for any $z \neq 0$. Hence there is a vertex $(a,b,c,d) \in I$, $a,b,d \neq 0$, $c \neq z$. Comparing it with $(0,0,0,0)$ we get $c=0$. Since it is not adjacent to $(0,\beta_1,\beta_2,\beta_3)$, either (i) $b=\beta_1$ and $d \neq \beta_3$, or (ii) $b \neq \beta_1$ and $d = \beta_3$.

We first consider case (i), that is, we will compare the vertex $(a,\beta_1,0,d)$ with $B_1,\dots,B_5$. Since $e((a,\beta_1,0,d),(\beta_0,\beta_1,\gamma_2,0)) = 1$ we infer $a \neq \beta_0$. Hence, as in the set $B_1$ no $\gamma_0$ is present we can set $a = \gamma_0$. Considering further the vertex $(\beta_0,0,\beta_2,\gamma_3) \in B_1$ we find that $d = \gamma_3$, for otherwise $e((\gamma_0,\beta_1,0,d),(\beta_0,0,\beta_2,\gamma_3))$ $= 0$. Set

$$C_1 = B_1 \cup \{(\gamma_0,\beta_1,0,\gamma_3)\}.$$

Next, compare $(a,\beta_1,0,d)$ with the vertices from $B_2$. Then $a = \beta_0$ because it is not adjacent to $(\beta_0,\gamma_1,\beta_2,0)$. In addition, $d = \gamma_3$ as $(\beta_0,\beta_1,0,d)$ is not adjacent to $(\gamma_0,0,\beta_2,\gamma_3)$. So we have the following possibility:

$$C_2 = B_2 \cup \{(\beta_0,\beta_1,0,\gamma_3)\}.$$

Next, compare $(a,\beta_1,0,d)$ with $(\gamma_0,0,\gamma_2,\beta_3) \in B_3$. Since $d \neq \beta_3$ we must have $a = \gamma_0$, while $d$ can be denoted with $\gamma_3$, yielding

$$C_3 = B_3 \cup \{(\gamma_0,\beta_1,0,\gamma_3)\}.$$

Now compare $(a,\beta_1,0,d)$ with $(\beta_0,0,\delta_2,\beta_3) \in B_4$. Since $d \neq \beta_3$ we have $a = \beta_0$. But then $e((\beta_0,\beta_1,0,d),(\beta_0,\beta_1,\gamma_2,0)) = 2$, a contradiction.

Finally, compare $(a,\beta_1,0,d)$ with $(\beta_0,\gamma_1,\beta_2,0) \in B_5$. Since $e((a,\beta_1,0,d),$ $(\beta_0,\gamma_1,\beta_2,0)) = 1$, we have $a = \beta_0$. As $0 \neq d \neq \beta_3$ and $\gamma_3$ is not present in $B_5$ we can set $d = \gamma_3$ thus yielding

$$C_4 = B_5 \cup \{(\beta_0,\beta_1,0,\gamma_3)\}.$$

We next consider case (ii) by comparing $(a,b,0,\beta_3)(b \neq \beta_1)$ with the sets $B_1,\dots,B_5$. The arguments are similar as above. In particular, also this vertex is not compatible with $B_4$. The other four cases give the following possibilities:

$$\begin{aligned}
C_5 &= B_1 \cup \{(\beta_0,\gamma_1,0,\beta_3)\},\\
C_6 &= B_2 \cup \{(\gamma_0,\gamma_1,0,\beta_3)\},\\
C_7 &= B_3 \cup \{(\beta_0,\gamma_1,0,\beta_3)\},\\
C_8 &= B_5 \cup \{(\gamma_0,\gamma_1,0,\beta_3)\},
\end{aligned}$$

where $\gamma_1$ is introduced into $C_5$ and $C_7$, and $\gamma_0$ into $C_8$.

To complete the proof of necessity part of the proof we claim that in any of the above 8 cases the set $I$ is as stated in the lemma. More precisely, if one of the sets $C_1, C_3, C_5, C_7$ is contained in $I$, then the set

$$\begin{gathered}\{(0,0,0,0),(0,\beta_1,\beta_2,\beta_3),(0,\gamma_1,\gamma_2,\gamma_3),\\ (\beta_0,0,\beta_2,\gamma_3),(\beta_0,\beta_1,\gamma_2,0),(\beta_0,\gamma_1,0,\beta_3),\\ (\gamma_0,0,\gamma_2,\beta_3),(\gamma_0,\beta_1,0,\gamma_3),(\gamma_0,\gamma_1,\beta_2,0)\}\end{gathered}$$

is contained in $I$. And if one of $C_2, C_4, C_6, C_8$ is contained in $I$, then eventually the set

$$\begin{gathered}\{(0,0,0,0),(0,\beta_1,\beta_2,\beta_3),(0,\gamma_1,\gamma_2,\gamma_3),\\ (\beta_0,0,\gamma_2,\beta_3),(\beta_0,\beta_1,0,\gamma_3),(\beta_0,\gamma_1,\beta_2,0),\\ (\gamma_0,0,\beta_2,\gamma_3),(\gamma_0,\beta_1,\gamma_2,0),(\gamma_0,\gamma_1,0,\beta_3)\}\end{gathered}$$

is contained in $I$. Note that both of these sets are as claimed in the statement of the lemma, which can be seen by observing that exchanging $\beta_0$ with $\gamma_0$ in the second set yields the first one. Observe also that such a set is a maximal independent set with respect to the property that $e(u,v) = 1$ for any of its different vertices $u$ and $v$.

We are going to prove the above claim for $C_1$. The still missing vertices are

$$(0,\gamma_1,\gamma_2,\gamma_3),(\beta_0,\gamma_1,0,\beta_3),(\gamma_0,0,\gamma_2,\beta_3) \ \text{ and } \ (\gamma_0,\gamma_1,\beta_2,0)\,,$$

where $\gamma_1 \in V(K_{n_1}) \setminus \{0,\beta_1\}$ has not yet been introduced.

Assume $(\gamma_0,0,\gamma_2,\beta_3) \notin I$. Then there exists $(a,b,c,d) \in I$ such that $a \neq \gamma_0, b \neq 0, c \neq \gamma_2$ and $d \neq \beta_3$. Since $e((a,b,c,d),(0,0,0,0)) = 1$, exactly one of $a, c$ and $d$ equals 0. If $a = 0$, then we compare $(0,b,c,d)$ with $(\beta_0,\beta_1,\gamma_2,0) \in C_1$ to realize that they can only be equal in the second coordinate, that is, $b = \beta_1$. It follows that $(0,\beta_1,c,d) = (0,\beta_1,\beta_2,\beta_3) \in C_1$, which is not possible because $d \neq \beta_3$. Suppose $c = 0$. Since $e((a,b,0,d),(0,\beta_1,\beta_2,\beta_3)) = 1$, we have $b = \beta_1$. Then $(a,\beta_1,0,d) = (\gamma_0,\beta_1,0,\gamma_3) \in C_1$, another contradiction. Suppose finally $d = 0$. Since $e((a,b,c,0),(\gamma_0,\beta_1,0,\gamma_3)) = 1$, we get $b = \beta_1$ and thus $(a,b,c,0) = (\beta_0,\beta_1,\gamma_2,0) \in C_1$, the final contradiction. We conclude that $(\gamma_0,0,\gamma_2,\beta_3) \in I$.

To prove that the remaining three vertices necessarily lie in $I$, some more efforts are needed. Clearly, $(0,0,\gamma_2,\gamma_3) \notin I$. Note also that this vertex is not dominated with any of the vertices from $C_1 \cup \{(\gamma_0,0,\gamma_2,\beta_3)\}$. Hence there exists $(a,b,c,d) \in I$ with $a, b \neq 0, c \neq \gamma_2$, and $d \neq \gamma_3$. Since $e((a,b,c,d),(0,0,0,0)) = 1$, either $c = 0$ or $d = 0$. Assume first $c = 0$. Comparing $(a,b,0,d)$ with $(\beta_0,0,\beta_2,\gamma_3)$ yields $a = \beta_0$. Since $(\beta_0,b,0,d)$ and $(\beta_0,\beta_1,\gamma_2,0)$ already coincide in one position and differ in two positions, we find that $b \neq \beta_1$. Hence we can set $b = \gamma_1$.

Comparing $(\beta_0, \gamma_1, 0, d)$ with $(0, \beta_1, \beta_2, \beta_3)$ gives $d = \beta_3$. We conclude that

$$C_{1.1} = C_1 \cup \{(\gamma_0, 0, \beta_2, \gamma_3)\} \cup \{(\beta_0, \gamma_1, 0, \beta_3)\} \subseteq I.$$

Assume next $d = 0$. Then comparing $(a, b, c, 0)$ with $(\gamma_0, 0, \gamma_2, \beta_3)$ we get $a = \gamma_0$. Since $(\gamma_0, b, c, 0)$ and $(\beta_0, \beta_1, \gamma_2, 0) \in C_1$ already coincide in one position and differ in two positions, $b \neq \beta_1$. Hence we can introduce $b = \gamma_1$. We next find that $c = \beta_2$ by comparing $(\gamma_0, \gamma_1, c, 0)$ with $(\beta_0, 0, \beta_2, \gamma_3)$. So we have another possibility for a subset of $I$:

$$C_{1.2} = C_1 \cup \{(\gamma_0, 0, \beta_2, \gamma_3)\} \cup \{(\gamma_0, \gamma_1, \beta_2, 0)\} \subseteq I.$$

Suppose $C_{1.1} \subseteq I$. Then we need to prove that $(\gamma_0, \gamma_1, \beta_2, 0)$ and $(0, \gamma_1, \gamma_2, \gamma_3)$ belong to $I$. Assume on the contrary that $(\gamma_0, \gamma_1, \beta_2, 0) \notin I$. Then there is a vertex $(a, b, c, d) \in I$ such that $a \neq \gamma_0, b \neq \gamma_1, c \neq \beta_2$, and $d \neq 0$. Since $e((a, b, c, d), (0, 0, 0, 0)) = 1$ and as $d \neq 0$, exactly one of $a, b, c$ equals 0. In the first case compare $(0, b, c, d)$ with $(\beta_0, \gamma_1, 0, \beta_3) \in C_{1.1}$. It follows that $d = \beta_3$. Then, since $e((0, b, c, \beta_3), (0, \beta_1, \beta_2, \beta_3)) \geq 2$, these two vertices must be the same. But this means that $c = \beta_2$, a contradiction. Assume next $b = 0$. Then comparing $(a, 0, c, d)$ with $(\gamma_0, \beta_1, 0, \gamma_3) \in C_{1.1}$ we find that $d = \gamma_3$. It follows that $(a, 0, c, \gamma_3) = (\beta_0, 0, \beta_2, \gamma_3)$, another contradiction because $c \neq \beta_2$. In the last case, $c = 0$, compare $(a, b, 0, d)$ with $(\gamma_0, 0, \gamma_2, \beta_3) \in C_{1.1}$ to see that $d = \beta_3$. Therefore $(a, b, 0, \beta_3) = (\beta_0, \gamma_1, 0, \beta_3)$, the final contradiction, since $b \neq \gamma_1$. We conclude that $(\gamma_0, \gamma_1, \beta_2, 0) \in I$.

We proceed similarly for the vertex $(0, \gamma_1, \gamma_2, \gamma_3)$ and assume that it does not belong to $I$. Then there exists $(a, b, c, d) \in I$, such that $a \neq 0, b \neq \gamma_1, c \neq \gamma_2$, and $d \neq \gamma_3$. Since $e((a, b, c, d), (0, 0, 0, 0)) = 1$, exactly one of $b, c, d$ equals 0. In the first case compare $(a, 0, c, d)$ with $(\beta_0, \beta_1, \gamma_2, 0) \in C_{1.1}$ to get $a = \beta_0$. But then $(\beta_0, 0, c, d) = (\beta_0, 0, \beta_2, \gamma_3)$, which is not possible since $d \neq \gamma_3$. In the second case we have $a = \beta_0$ by considering $(a, b, 0, d)$ and $(\beta_0, 0, \beta_2, \gamma_3) \in C_{1.1}$. Then $(\beta_0, b, 0, d) = (\beta_0, \gamma_1, 0, \beta_3)$, which is not possible since $b \neq \gamma_1$. Finally, if $d = 0$, we have $a = \gamma_0$ (compare $(a, b, c, 0)$ with $(\gamma_0, 0, \gamma_2, \beta_3) \in C_{1.1}$), therefore $(\gamma_0, b, c, 0) = (\gamma_0, \gamma_1, \beta_2, 0)$. Another contradiction, since $b \neq \gamma_1$. We conclude that $(0, \gamma_1, \gamma_2, \gamma_3) \in I$ and hence $I$ is as required.

Assume $C_{1.2} \subseteq I$. Then we need to show that $(\beta_0, \gamma_1, 0, \beta_3)$ and $(0, \gamma_1, \gamma_2, \gamma_3)$ belong to $I$. Suppose $(\beta_0, \gamma_1, 0, \beta_3) \notin I$. Then there exists $(a, b, c, d) \in I$, such that $a \neq \beta_0, b \neq \gamma_1, c \neq 0$, $d \neq \beta_3$, and exactly one of $a, b, d$ equals 0. If $a = 0$ then $c = \beta_2$ by considering $(0, b, c, d)$ and $(\gamma_0, \gamma_1, \beta_2, 0)$. But then $(0, b, \beta_2, d) = (0, \beta_1, \beta_2, \beta_3)$, contradicting $d \neq \beta_3$. If $b = 0$, then consider $(a, 0, c, d)$ and $(0, \beta_1, \beta_2, \beta_3)$ to get $c = \beta_2$. Now $(a, 0, \beta_2, d) = (\beta_0, 0, \beta_2, \gamma_3)$, contradicting $a \neq \beta_0$. Finally, if $d = 0$, then $(a, b, c, 0)$ and $(\beta_0, 0, \beta_2, \gamma_3)$ give $c = \beta_2$. But then $(a, b, \beta_2, 0) = (\gamma_0, \gamma_1, \beta_2, 0)$, which contradicts $b \neq \gamma_1$. Therefore, $(\beta_0, \gamma_1, 0, \beta_3) \in I$.

Suppose $(0, \gamma_1, \gamma_2, \gamma_3) \notin I$. Then we have $(a, b, c, d) \in I$, where $a \neq 0, b \neq \gamma_1, c \neq \gamma_2$, $d \neq \gamma_3$, and one of $b, c, d$ is 0. If $b = 0$ then $a = \gamma_0$ (consider $(a, 0, c, d)$ and $(\gamma_0, \beta_1, 0, \gamma_3)$), and therefore $(\gamma_0, 0, c, d) = (\gamma_0, 0, \gamma_2, \beta_3)$, contradicting $c \neq \gamma_2$. If $c = 0$ we get $a = \beta_0$ (consider $(a, b, 0, d)$ and $(\beta_0, 0, \beta_2, \gamma_3)$), and so $(\beta_0, b, 0, d) = (\beta_0, \gamma_1, 0, \beta_3)$, contradicting $b \neq \gamma_1$. Finally, for $d = 0$ we have $a = \gamma_0$ (consider $(a, b, c, 0)$ and $(\gamma_0, 0, \gamma_2, \beta_3)$). Now $(\gamma_0, b, c, 0) = (\gamma_0, \gamma_1, \beta_2, 0)$, contradicting $b \neq \gamma_1$. We conclude that $(0, \gamma_1, \gamma_2, \gamma_3) \in I$ and hence also in this case $I$ is as claimed.

The proofs for the remaining 7 cases, that is, for the sets $C_i$, $i = 2, 3, ..., 8$, go along the same lines as the above proof for $C_1$. In each of these sets we miss four vertices and exactly one of the integers $\gamma_0, \gamma_1, \gamma_2$, and $\gamma_3$. (In the above case, $\gamma_1$ was missing.) Now, exactly one among the four missing vertices does not contain the missing integer (above such a vertex is $(\gamma_0, 0, \gamma_2, \beta_3)$). Then we prove for this vertex that belongs to $I$. We continue by selecting a vertex not in $I$ that coincides with the previous six vertices in at least one position, and coincides in at least one position also with one of the three missing vertices. (Above the vertex $(0, 0, \gamma_2, \gamma_3)$ played this role.) Now we proceed as above and detect the remaining two vertices. We note that the order in which we prove the inclusion of these two vertices is essential because the inclusion of one of them is needed to prove the inclusion of the other.

Conversely, assume that $I$ contains 9 vertices as described in the statement. We need to prove that an arbitrary vertex $x = (\delta_0, \delta_1, \delta_2, \delta_3) \notin I$ is dominated by at least one vertex from $I$.

Note first that for each $i$, each of the $\alpha_i, \beta_i$, and $\gamma_i$, appears in exactly three vertices from $I$. If, say, $u$, $v$, and $w$ contain $\alpha_i$, then $e(u, v) = 1$, $e(u, w) = 1$, $e(v, w) = 1$, and, moreover, $u$, $v$, and $w$ coincide in the position where $\alpha_i$ stands. We now distinguish four cases.

Suppose $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 0$. Then $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ dominates $x$.

Let $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 3$ and let $\delta_k \neq \alpha_k$, where $k \in \{0, 1, 2, 3\}$. Then $x$ is dominated by the two vertices that contain $\alpha_k$ and do not contain $\alpha_i$ for $i \neq k$.

The next case is $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 2$. Let $\delta_k \neq \alpha_k$ and $\delta_\ell \neq \alpha_\ell$, where $k, \ell \in \{0, 1, 2, 3\}$. Consider the two vertices that contain $\alpha_k$ and none of the remaining $\alpha_i$'s. These two vertices can coincide with $x$ only in $\delta_\ell$. Since they differ in the corresponding position (which is because they agree in $\alpha_k$), one of them dominates $x$.

The last case to consider is when $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 1$. Let $\delta_k = \alpha_k$. Then $\delta_i \neq \alpha_i$ for $i \neq k$. Assume that no vertex from $I$ dominates $x$. Let $I_1$ be the set of the three vertices from $I$ that contain $\beta_k$ and let $I_2$ be the set of the three vertices that contain $\gamma_k$. Since $x$ is dominated by no vertex from $I_1 \cup I_2$, we have three positions to agree with each of them. For such a fixed position, $x$ can agree with at most one vertex from $I_1$ and with at most one vertex from $I_2$. Hence only in the optimal case we can have $e(x, u) \geq 1$ for each $u \in I_1 \cup I_2$. This in particular implies that $\delta_i \in \{\beta_i, \gamma_i\}$ for any $i \neq k$. No two of the $\delta_i$'s, where

$i \neq k$, appear in the same vertex from $I_1 \cup I_2$, for otherwise we would have two vertices from this set that would agree on two positions. Note further that for any two integers $r$ and $s$ that appear as coordinates of vertices from $I$, there is exactly one vertex in $I$ that contains $r$ and $s$. There are three pairs of integers of the form $\{\delta_i, \delta_j\}$, where $i, j \neq k$. None of these two such integers appear in the same vertex from $I_1 \cup I_2$. Hence these pairs must appear on the two vertices from $I \setminus (I_1 \cup I_2 \cup \{(0,0,0,0)\})$. Therefore, two of these pairs appear on the same vertex, but this means that $x$ is equal to one them, the final contradiction which completes the first step of the proof.

It remains to prove that if $2 \in \{n_i, 0 \leq i \leq 3\}$, then $G$ contains no dominating $T_1$-set. Suppose $I$ is such a set and assume $(0,0,0,0) \in I$. We may also without loss of generality assume $n_3 = 2$. Since $(1,0,0,0) \notin I$, there exists a vertex $(a,b,c,d) \in I$ such that $a \neq 1$ and $b,c,d \neq 0$. Hence $d = 1$ and because $e((a,b,c,1),(0,0,0,0)) = 1$ we have $a = 0$. Thus

$$\{(0,0,0,0),(0,b,c,1)\} \subseteq I.$$

Consider next $(0,1,0,0) \notin I$. Then there exists $(e,f,g,h) \in I$ that dominates $(0,1,0,0)$. Similarly as above we get $h = 1$ and $f = 0$. Since $e((e,0,g,1),(0,0,0,0)) = 1$ and $e((e,0,g,1),(0,b,c,1)) = 1$ we also have $g \neq 0$ and $g \neq c$. If $n_2 = 2$ this is a contradiction because no vertex of $I$ dominates $(0,1,0,0)$. So let $n_2 > 2$. Then

$$\{(0,0,0,0),(0,b,c,1),(e,0,g,1)\} \subseteq I.$$

Now $(0,0,1,0) \notin I$, hence there is a vertex $(k,l,m,n) \in I$, such that $k,l,n \neq 0$ and $m \neq 1$. Similarly as above we find that

$$\{(0,0,0,0),(0,b,c,1),(e,0,g,1),(k,l,0,1)\} \subseteq I.$$

If $n_0 = 2$ or $n_1 = 2$ we have a contradiction because $0,e,k$ are pairwise different as well as are $0,b,l$. So let $n_0 \geq 3$ and $n_1 \geq 3$. Since $(0,l,g,1) \notin I$, there is a vertex $(x,y,z,w) \in I$ with $x \neq 0$, $y \neq l$, $z \neq g$, and $w = 0$. The set $I$ is $T_1$ hence $(x,y,z,0) = (e,l,c,0)$ or $(x,y,z,0) = (k,b,g,0)$. But both possibilities are impossible because $y \neq l$ and $z \neq g$, respectively. □

**Proof of Theorem 3.3**. Note first that by Theorem 3.2, $n_i \geq 3$ is a necessary condition for the existence of an idomatic partition into $T_1$-sets. Hence assume in the rest that this is the case.

Suppose that $G$ has an idomatic partition into $T_1$-sets. By Theorem 3.2, each part (every $T_1$-set) in an idomatic partition into $T_1$-sets has nine vertices, and thus 9 is a divisor of $n_0n_1n_2n_3$, so there exists at least one $j \in [4]$ such that $3|n_j$. By the commutativity of the direct product, we can assume that $j = 3$. Let $I_\ell$

be a $T_1$-set of our idomatic partition. By Theorem 3.2, $I_\ell$ is of the form

$$\{(\alpha_0,\alpha_1,\alpha_2,\alpha_3),(\alpha_0,\beta_1,\beta_2,\beta_3),(\alpha_0,\gamma_1,\gamma_2,\gamma_3),\\ (\beta_0,\alpha_1,\beta_2,\gamma_3),(\beta_0,\gamma_1,\alpha_2,\beta_3),(\beta_0,\beta_1,\gamma_2,\alpha_3),\\ (\gamma_0,\alpha_1,\gamma_2,\beta_3),(\gamma_0,\beta_1,\alpha_2,\gamma_3),(\gamma_0,\gamma_1,\beta_2,\alpha_3)\}$$

for some pairwise different $\alpha_i,\beta_i,\gamma_i \in [n_i]$, $0 \le i \le 3$. The number of vertices of the form $(x,y,z,\alpha_3)$ with fixed $\alpha_3 \in [n_3]$ in $G$ is exactly $n_0n_1n_2$. In every $T_1$-set of the partition in which $\alpha_3$ occurs, there are exactly three vertices $(x,y,z,\alpha_3),(x',y',z',\alpha_3)$ and $(x'',y'',z'',\alpha_3)$, where $x$, $x'$, and $x''$ are pairwise different and so are $y$, $y'$, and $y''$. Hence we must be able to partition the $n_0n_1n_2$ vertices containing $\alpha_3$ into triples of vertices described above. Therefore, $3|n_0n_1n_2$.

Conversely, assume that there exist indices $j,k \in [4]$, $j \neq k$, such that $3|n_j$ and $3|n_k$. The graph $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$ can be seen as the Cayley graph associated with the direct product group $\mathcal{G} = Z_{n_0} \times Z_{n_1} \times Z_{n_2} \times Z_{n_3}$ with connector set $\big([n_0]\setminus\{0\}\big)\times\big([n_1]\setminus\{0\}\big)\times\big([n_2]\setminus\{0\}\big)\times\big([n_3]\setminus\{0\}\big)$, where $Z_{n_i}$ denotes the additive cyclic group of the integers modulo $n_i$. Using the commutativity of the direct product again, we can assume that $j = 2$ and $k = 3$. Let $a_j$ be an element of order $\frac{n_j}{3}$ in the group $Z_{n_j}$ for $j \in \{2,3\}$. Let $H_0 = \langle(1,0,0,0)\rangle$ denote the cyclic subgroup of $\mathcal{G}$ generated by the element $(1,0,0,0)$. Similarly, let $H_1 = \langle(0,1,0,0)\rangle$, $H_2 = \langle(0,0,a_2,0)\rangle$ and $H_3 = \langle(0,0,0,a_3)\rangle$. It is obvious that $H_i \cap H_j = \{(0,0,0,0)\}$ for $i,j \in [4]$, $i \neq j$. As $\mathcal{G}$ is abelian it follows that $H_0H_1H_2H_3 = \{h_0+h_1+h_2+h_3 \mid h_i \in H_i \text{ for } i \in [4]\}$ is a subgroup of order $\frac{n_0n_1n_2n_3}{9}$ in $\mathcal{G}$. Let us use the notation $r = \frac{n_0n_1n_2n_3}{9}$ and $P = H_0H_1H_2H_3 = \{p_1,p_2,...,p_r\}$, where we may without loss of generality assume that $p_1 = (0,0,0,0)$. Let $\beta_k$ and $\gamma_k$ be any elements in $Z_{n_k} \setminus \{0\}$, with $\beta_k \neq \gamma_k$ for $k \in \{0,1\}$, and let $\beta_j$ and $\gamma_j$ be any elements in $Z_{n_j} \setminus \{0\}$, with $\beta_j \neq \gamma_j$; $\beta_j,\gamma_j \notin \langle a_j\rangle$; and if $a_j \neq 0$ then $\beta_j \not\equiv \gamma_j \bmod a_j$, $j \in \{2,3\}$. By standard group theory arguments,

$$P,(0,\beta_1,\beta_2,\beta_3)+P,(0,\gamma_1,\gamma_2,\gamma_3)+P,\\ (\beta_0,0,\beta_2,\gamma_3)+P,(\beta_0,\gamma_1,0,\beta_3)+P,(\beta_0,\beta_1,\gamma_2,0)+P,\\ (\gamma_0,0,\gamma_2,\beta_3)+P,(\gamma_0,\beta_1,0,\gamma_3)+P,(\gamma_0,\gamma_1,\beta_2,0)+P$$

is a partition of $\mathcal{G}$ into left cosets of $P$. If we denote

$$D = \{(0,\beta_1,\beta_2,\beta_3),(0,\gamma_1,\gamma_2,\gamma_3),(\beta_0,0,\beta_2,\gamma_3),(\beta_0,\gamma_1,0,\beta_3),\\ (\beta_0,\beta_1,\gamma_2,0),(\gamma_0,0,\gamma_2,\beta_3),(\gamma_0,\beta_1,0,\gamma_3),(\gamma_0,\gamma_1,\beta_2,0)\},$$

then no element from $D$ belongs to the subgroup $P$ due to the construction of $D$. Moreover, there exists no element $z \in P$ such that $x+z = y$ for some pairwise different elements $x,y \in D$. Otherwise, $z = (z_0,z_1,z_2,z_3)$ is such that

$z_2 \in \{\pm\beta_2, \pm\gamma_2, \pm(\beta_2-\gamma_2)\}$ or $z_3 \in \{\pm\beta_3, \pm\gamma_3, \pm(\beta_3-\gamma_3)\}$. All these possibilities lead to a contradiction with the conditions of choosing $\beta_j$ and $\gamma_j$, $j \in \{2,3\}$. Hence our statement about the partition of $\mathcal{G}$ into left cosets of $P$ holds. For instance, in the above construction we could have chosen $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 1$ and $\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 2$.

Now we will construct an idomatic partition of $G$ into $T_1$-sets. For every $p_i \in P$, $1 \le i \le r$, we introduce

$$\begin{aligned} C_i \;\; = \;\; & \{p_i, p_i + (0,\beta_1,\beta_2,\beta_3), p_i + (0,\gamma_1,\gamma_2,\gamma_3), \\ & p_i + (\beta_0, 0, \beta_2, \gamma_3), p_i + (\beta_0,\gamma_1,0,\beta_3), p_i + (\beta_0,\beta_1,\gamma_2,0), \\ & p_i + (\gamma_0,0,\gamma_2,\beta_3), p_i + (\gamma_0,\beta_1,0,\gamma_3), p_i + (\gamma_0,\gamma_1,\beta_2,0)\}, \end{aligned}$$

where

$$\begin{aligned} & p_i + x = (p_{i_0}, p_{i_1}, p_{i_2}, p_{i_3}) + (x_0, x_1, x_2, x_3) \\ & = (p_{i_0} + x_0 \bmod n_0, p_{i_1} + x_1 \bmod n_1, p_{i_2} + x_2 \bmod n_2, p_{i_3} + x_3 \bmod n_3). \end{aligned}$$

It is clear that for arbitrary pairwise different $x, y \in D \cup \{(0,0,0,0)\}$ and $i \in \{1,2,...,r\}$ the vertices $p_i+x$ and $p_i+y$ are non-adjacent because of non-adjacency of $x$ and $y$. That is, all $C_i$ are independent. By Theorem 3.2, all $C_i$ are $T_1$-sets and by our statement above we get $\bigcup_{i=1}^r C_i = G$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Hence $C_1, C_2, ..., C_r$ is an idomatic partition of the graph $G$ into $T_1$-sets. □

We add that the arguments of the proof of Theorem 3.3 are in part parallel to those from [16].

# 5 Concluding remarks

The proof of Theorem 3.2 is quite technical and lengthy. Hence it seems that one needs another approach to deal with direct products of more than four factors. In this respect, we pose the following conjecture.

**Conjecture 5.1** *Let $I$ be a $T_1$-set of $\times_{i=1}^k K_{n_i}$, where $k \ge 5$. Then $|I| = (k-1)^2$.*

For instance, we can show that the following subset of the direct product of six factors

| | | | | |
|---|---|---|---|---|
| (0,0,0,0,0,0) | (1,0,1,2,3,4) | (2,0,4,3,2,1) | (3,0,2,4,1,3) | (4,0,3,1,4,2) |
| (0,1,1,1,1,1) | (1,4,0,1,2,3) | (2,1,0,4,3,2) | (3,3,0,2,4,1) | (4,2,0,3,1,4) |
| (0,2,2,2,2,2) | (1,3,4,0,1,2) | (2,2,1,0,4,3) | (3,1,3,0,2,4) | (4,4,2,0,3,1) |
| (0,3,3,3,3,3) | (1,2,3,4,0,1) | (2,3,2,1,0,4) | (3,4,1,3,0,2) | (4,1,4,2,0,3) |
| (0,4,4,4,4,4) | (1,1,2,3,4,0) | (2,4,3,2,1,0) | (3,2,4,1,3,0) | (4,3,1,4,2,0) |

is a dominating $T_1$-set. In fact, it has a structure similar to the one for four factors from Theorem 3.2 so perhaps (at least for even $k$) such sets are the only dominating $T_1$-sets.

The $T_{1,2}$-sets and $T_{1,2,3}$-sets seem more involved than the $T_1$-sets. Hence we pose:

**Problem 5.2** *Characterize $T_{1,2}$-sets and $T_{1,2,3}$-sets of the direct product of four complete graphs.*

We have already noted that fall colorings coincide with idomatic partitions. The *fall chromatic number* $\chi_f(G)$ of a graph $G$ is defined as the minimum order of a fall coloring of $G$. Clearly, $\chi_f(G) \geq \chi(G)$. Since Hedetniemi's conjecture holds for complete graphs we have $\chi(\times_{i=1}^k K_{n_i}) = \min\{n_i \mid 1 \leq i \leq k\}$. Let $n_\ell = \min\{n_i \mid 1 \leq i \leq k\}$. In Section 2 we have noticed (for four factors) that

$$I_j = [n_1] \times \cdots \times [n_{\ell-1}] \times \{j\} \times [n_{\ell+1}] \times \cdots \times [n_k]\,,$$

where $j \in [n_\ell]$, is an independent dominating set. Consequently, $\{I_j \mid j \in [n_\ell]\}$ is a fall coloring. Hence $\chi_f(\times_{i=1}^k K_{n_i}) \leq n_\ell = \chi(\times_{i=1}^k K_{n_i}) \leq \chi_f(\times_{i=1}^k K_{n_i})$ and so

$$\chi_f(\times_{i=1}^k K_{n_i}) = \chi(\times_{i=1}^k K_{n_i})\,.$$

An alternative argument for the above conclusion could use the independence number of the direct product of complete graphs, see [17, Corollary 1].

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