

# Dominating sequences in grid-like and toroidal graphs

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## Abstract

A longest sequence  $S$  of distinct vertices of a graph  $G$  such that each vertex of  $S$  dominates some vertex that is not dominated by its preceding vertices, is called a Grundy dominating sequence; the length of  $S$  is the Grundy domination number of  $G$ . In this paper we study the Grundy domination number in the four standard graph products: the Cartesian, the lexicographic, the direct, and the strong product. For each of the products we present a lower bound for the Grundy domination number which turns out to be exact for the lexicographic product and is conjectured to be exact for the strong product. In most of the cases exact Grundy domination numbers are determined for products of paths and/or cycles.

**Keywords:** Grundy domination; graph product; edge clique cover; isoperimetric inequality

## 1 Introduction

If  $G$  is a graph, then a sequence  $S = (v_1, \dots, v_k)$  of distinct vertices of  $G$  is called a *legal (closed neighborhood) sequence* if, for each  $i$ ,

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset.$$

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In words, for each  $i$  the vertex  $v_i$  dominates at least one vertex not dominated by  $v_1, \dots, v_{i-1}$ . If the set of vertices from a legal sequence  $S$  forms a dominating set of  $G$ , then  $S$  is called a *dominating sequence* of  $G$ . Clearly, the length of a dominating sequence is bounded from below by the domination number  $\gamma(G)$  of a graph  $G$ . The maximum length of a dominating sequence in  $G$  is called the *Grundy domination number* of  $G$  and denoted by  $\gamma_{gr}(G)$ . The corresponding sequence is called a *Grundy dominating sequence* of  $G$  or  $\gamma_{gr}$ -sequence of  $G$ .

These concepts were introduced in [5], where the Grundy domination number was established for graphs of some well-known classes (such as split graphs and cographs), and a linear time algorithm to determine this number in an arbitrary tree was presented. It was also shown that the decision version of the problem is NP-complete, even when restricted to chordal graphs. Motivated by the question, when a Grundy dominating sequence produces a minimum dominating set (and hence any legal sequence is of the same length), the classes of graphs with domination numbers up to 3 having this property have been characterized. The concept was further studied in [4], where exact formulas for Grundy domination numbers of Sierpiński graphs were proven, and a linear algorithm for determining these numbers in arbitrary interval graphs was presented.

It is natural to study graph invariants on graph products, especially on the four standard ones: the Cartesian, the lexicographic, the direct and the strong product [9]. Among different reasons for the popularity of graph products, we emphasize the fact that several intriguing questions, related to products of graphs, gave new insights and new developments in studies of the involved invariants. Let us mention the still open, famous Vizing's conjecture on the domination number of the Cartesian product of graphs, which was posed in the 1960's [18], and initiated the introduction of several new concepts and methods, cf. [3]. The domination number of the Cartesian product of paths was completely determined only in 2011 [8], after a long period in which various attempts produced different approaches to graph domination problems. Cartesian products of paths by cycles were investigated in [15] and products of cycles in [11].

With respect to other standard products, an emphasis was given on  $r$ -perfect codes. Recall that a set  $S$  of vertices of a graph  $G$  is an  $r$ -perfect code if vertices from  $S$  are pairwise at distance at least  $2r + 1$  and every vertex of  $G$  is at distance at most at  $r$  from some vertex from  $S$ . It is easy to observe that if a graph admits a 1-perfect code, then its size is the domination number of the graph. In [12, 19] perfect codes in direct products of cycles were characterized, while in [1] and [17] perfect codes were investigated in strong and lexicographic products, respectively. In particular, the strong product of graphs has a perfect  $r$ -code if and only if each factor has a perfect  $r$ -code. For a uniform treatment of domination (and other invariants) on graph products see [13].

In this paper we focus on dominating sequences in the four standard graph products. While we could find some general results or bounds for Grundy domination numbers in all four graph products, we nevertheless focused mainly on the products of paths and/or cycles. (As noted above, obtaining the exact formulas for the standard domination number of these relatively simple classes of graphs has turned out to be rather non-trivial.) For each of the four products we consider the Grundy domination number of

*grids* (the products of paths), *cylinders* (the products of paths by cycles), and *tori* (the products of cycles) and proceed as follows. In the next section we introduce concepts and terminology needed, and present two general upper bounds on the Grundy domination number which will be applied later and could be of independent interest. In particular, the Grundy domination number is bounded by the edge clique number. In the subsequent four sections we respectively consider the Grundy domination number of the four standard graph products.

In Section 3 we first prove that the Grundy domination number of two graphs is bounded from below by the product of the Grundy domination number of one factor and the order of the other factor. We then show that this bound is attained in all grids, cylinders and tori, with the sole exception of the Cartesian product of two cycles of the same odd length. In addition, certain isoperimetric inequalities enable us to determine the Grundy domination numbers of certain multiple Cartesian products. In the case of the lexicographic product (see Section 4), we give a formula for the Grundy domination number of the product of two arbitrary graphs as a function of all dominating sequences of the first factor. This yields exact values of the Grundy domination numbers of products of paths and/or cycles. In the main part of Section 5 we consider lower bounds on the Grundy domination number of direct products of graphs. We also give an upper bound for the products of two paths. These results yield an exact value in the case when a shortest path is of even order. The main theme of the section on the strong product (Section 6) is the following conjecture posed for arbitrary graphs  $G$  and  $H$ :

$$\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H).$$

While  $\gamma_{gr}(G \boxtimes H) \geq \gamma_{gr}(G)\gamma_{gr}(H)$  is easily seen to be true, we prove the reversed inequality if one of the factors is a caterpillar.

## 2 Preliminaries

If  $S = (v_1, \dots, v_k)$  is a sequence of distinct vertices of a graph  $G$ , then the corresponding set  $\{v_1, \dots, v_k\}$  will be denoted by  $\widehat{S}$ . The initial segment  $(v_1, \dots, v_i)$  of  $S$  will be denoted by  $S_i$ . Each vertex  $u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$  is called a *private neighbor* of  $v_i$  with respect to  $\{v_1, \dots, v_i\}$ . We will also use a more suggestive term by saying that  $v_i$  *footprints* the vertices from  $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$ , and that  $v_i$  is the *footprinter* of any  $u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$ . For a dominating sequence  $S$  any vertex in  $V(G)$  has a unique footprinter in  $\widehat{S}$ . Thus the function  $f_S : V(G) \rightarrow \widehat{S}$  that maps each vertex to its footprinter is well defined.

Arising from the total domination number of graph, a related invariant was introduced recently in [6], which is defined on graphs  $G$  with no isolated vertices. A sequence  $S = (v_1, \dots, v_k)$  of distinct vertices of  $G$ , is called a *legal open neighborhood sequence* if  $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ , holds for every  $i \in \{2, \dots, k\}$ . If, in addition,  $\widehat{S}$  is a total dominating set of  $G$ , then we call  $S$  a *total dominating sequence* of  $G$ . (The meaning of *footprinting* in the context of total dominating sequences is analogous as above, and should be clear.) The maximum length of a total dominating sequence in  $G$  is called the *Grundy total*

domination number of  $G$  and is denoted by  $\gamma_{gr}^t(G)$ . The corresponding sequence is called a *Grundy total dominating sequence* of  $G$ .

Recall that for all of the standard graph products, the vertex set of the product of graphs  $G$  and  $H$  is equal to  $V(G) \times V(H)$ , while their edge-sets are as follows. In the *lexicographic product*  $G \circ H$  (also denoted by  $G[H]$ ), vertices  $(g_1, h_1)$  and  $(g_2, h_2)$  are adjacent if either  $g_1g_2 \in E(G)$  or  $(g_1 = g_2 \text{ and } h_1h_2 \in E(H))$ . In the *strong product*  $G \boxtimes H$  vertices  $(g_1, h_1)$  and  $(g_2, h_2)$  are adjacent whenever  $(g_1g_2 \in E(G) \text{ and } h_1 = h_2)$  or  $(g_1 = g_2 \text{ and } h_1h_2 \in E(H))$  or  $(g_1g_2 \in E(G) \text{ and } h_1h_2 \in E(H))$ . In the *direct product*  $G \times H$  vertices  $(g_1, h_1)$  and  $(g_2, h_2)$  are adjacent when  $g_1g_2 \in E(G)$  and  $h_1h_2 \in E(H)$ . Finally, in the *Cartesian product*  $G \square H$  vertices  $(g_1, h_1)$  and  $(g_2, h_2)$  are adjacent when  $(g_1g_2 \in E(G) \text{ and } h_1 = h_2)$  or  $(g_1 = g_2 \text{ and } h_1h_2 \in E(H))$ . Hence in general we have  $E(G \square H) \subseteq E(G \boxtimes H) \subseteq E(G \circ H)$ , and  $E(G \times H) \subseteq E(G \boxtimes H)$ ,  $E(G \times H) \cap E(G \square H) = \emptyset$ , while  $E(G \times H) \cup E(G \square H) = E(G \boxtimes H)$ . With the exception of the lexicographic product, the standard graph products are commutative. For this and other properties of the standard products see [9].

Let  $G$  and  $H$  be graphs and  $*$  be one of the four graph products under consideration. For a vertex  $h \in V(H)$ , we call the set  $G^h = \{(g, h) \in V(G * H) : g \in V(G)\}$  a *G-layer* or a *row* of  $G * H$ . By abuse of notation we will also consider  $G^h$  as the corresponding induced subgraph. Clearly  $G^h$  is isomorphic to  $G$  unless  $*$  is the direct product in which case it is an edgeless graph of order  $|V(G)|$ . For  $g \in V(G)$ , the *H-layer* or the *column*  ${}^gH$  is defined as  ${}^gH = \{(g, h) \in V(G * H) : h \in V(H)\}$ . We may again consider  ${}^gH$  as an induced subgraph when appropriate. The map  $p_G : V(G * H) \rightarrow V(G)$ ,  $p_G(g, h) = g$ , is the *projection* onto  $G$  and  $p_H : V(G * H) \rightarrow V(H)$ ,  $p_H(g, h) = h$ , is the *projection* onto  $H$ . We say that  $G * H$  is *nontrivial* if both factors are graphs on at least two vertices.

We now give two upper bounds on the Grundy domination number of arbitrary graphs. If  $G$  is a graph and  $S \subseteq V(G)$ , then the *boundary*  $\partial S$  of  $S$  is defined with

$$\partial S = \{u \in V(G) \setminus S : u \text{ has a neighbor in } S\}.$$

The first upper bounds reads as follows.

**Lemma 1.** *For a graph  $G$  the inequality  $\gamma_{gr}(G) \leq |V| - k$  holds if and only if every Grundy dominating sequence has an initial segment  $S$  such that  $|\partial S| \geq k$ . In particular, if there exists an  $m \leq |V(G)|$  such that for any  $m$ -subset  $A$  of  $V(G)$  we have  $|\partial A| \geq k$ , then  $\gamma_{gr}(G) \leq |V| - k$  holds.*

*Proof.* If  $\gamma_{gr}(G) \leq |V| - k$  holds, then for the set  $S$  of all the elements of any Grundy dominating sequence we must have  $|\partial S| \geq k$ . The other direction follows from the fact that the size of the boundary of the initial segments is non-decreasing and obviously for the set  $S$  of all the elements of any Grundy dominating sequence we have  $|S| + |\partial S| = |V(G)|$ .  $\square$

We follow with the next general upper bound, which will be used in Section 22 in the context of strong products of graphs. Let  $G$  be a graph. If  $\mathcal{C} = \{Q_1, \dots, Q_r\}$  is a set of cliques of  $G$  such that any edge of  $G$  is contained in a clique of  $\mathcal{C}$ , then  $\mathcal{C}$  is called an *edge*

clique cover of  $G$ . The size of a smallest edge clique cover of  $G$  is the *edge clique cover number* of  $G$  and denoted by  $\theta_e(G)$ . It is well-known that  $\theta_e(G)$  is equal to the intersection number of  $G$ , cf. [7, 14].

**Proposition 2.** *If  $G$  is a graph without isolated vertices, then  $\gamma_{gr}(G) \leq \theta_e(G)$ .*

*Proof.* Let  $\mathcal{Q}$  be a minimum edge clique cover of  $G$  and let  $S = (d_1, \dots, d_p)$  be a Grundy dominating sequence in  $G$ , where  $p = \gamma_{gr}(G)$ . We claim that at each term of the sequence  $S$  at least one clique of  $\mathcal{Q}$  becomes completely dominated. Since  $d_i$  is a legal choice, at least one vertex from  $N[d_i]$  is not dominated by  $\widehat{S_{i-1}}$ . If  $d_i$  footprints itself, let  $x$  be an arbitrary neighbor of  $d_i$ . (Note that  $x$  exists because  $G$  has no isolated vertices.) And if  $d_i$  footprints some other vertex, let  $x$  be this vertex. In either of the cases the edge  $d_i x$  lies in a clique  $Q \in \mathcal{Q}$  that is not yet completely dominated. It follows that after  $d_i$  is selected, all the vertices of  $Q$  are dominated. Consequently,  $\mathcal{Q}$  contains at least  $p$  cliques, that is  $\theta_e(G) = |\mathcal{Q}| \geq p = \gamma_{gr}(G)$ .  $\square$

The bound in Proposition 2 is sharp, as demonstrated by complete graphs  $K_n$  and paths  $P_n$ . Moreover, within the class of trees, the above bound is sharp precisely in caterpillars (i.e., the trees in which there is no vertex having more than two non-leaf neighbors). Indeed, note that in bipartite graphs  $\theta_e(G)$  equals the number of edges, and, as mentioned in [5], in any caterpillar  $G$  we have  $\gamma_{gr}(G) = |V(G)| - 1$ . On the other hand, [5, Lemma 2.4] implies that if there exists a vertex in a tree with more than two non-leaf neighbors, then  $\gamma_{gr}(G) < |V(G)| - 1$ . Hence, caterpillars are really the only trees that enjoy the equality in the bound from Proposition 2.

### 3 Cartesian product

In this section we determine the Grundy domination number of grids, cylinders and tori (with respect to the Cartesian product) and use isoperimetric inequalities to determine the Grundy domination number of some products of several paths and cycles. We begin with the following general lower bound.

**Proposition 3.** *For any two graphs  $G$  and  $H$ ,*

$$\gamma_{gr}(G \square H) \geq \max\{\gamma_{gr}(G)|V(H)|, \gamma_{gr}(H)|V(G)|\}.$$

*Proof.* Set  $V(H) = \{h_1, \dots, h_k\}$ . Let  $m = \gamma_{gr}(G)$  and let  $(d_1, \dots, d_m)$  be a Grundy dominating sequence in  $G$ . Observe that

$$((d_1, h_1), \dots, (d_1, h_k), (d_2, h_1), \dots, (d_2, h_k), \dots, (d_m, h_1), \dots, (d_m, h_k))$$

is a dominating sequence in  $G \square H$ . Indeed,  $(d_i, h_j)$  footprints  $(g_i, h_j)$ , where  $g_i$  is footprinted by  $d_i$  in  $G$ . Hence  $\gamma_{gr}(G \square H) \geq mk = \gamma_{gr}(G)|V(H)|$ . By reversing the roles of  $G$  and  $H$  the statement follows.  $\square$

It is natural to ask whether in the inequality in Proposition 3 always equality holds. To see that this need not be the case consider the Cartesian product of complete graphs. Indeed, let  $V(K_n) = \{a_1, \dots, a_n\}$ ,  $V(K_m) = \{b_1, \dots, b_m\}$  and  $m, n \geq 3$ . Then

$$((a_1, b_1), (a_2, b_1), \dots, (a_{n-1}, b_1), (a_1, b_2), (a_1, b_3), \dots, (a_1, b_m))$$

is a dominating sequence of length  $n + m - 2$ , which in turn implies that

$$\max\{\gamma_{gr}(K_n)|V(K_m)|, \gamma_{gr}(K_m)|V(K_n)|\} = \max\{m, n\} < n + m - 2 \leq \gamma_{gr}(K_n \square K_m).$$

This example actually shows that the left-hand side of the inequality of Proposition 3 can be arbitrary larger than the right-hand side of it.

**Theorem 4.** *Let  $P_k$  be the path and  $C_k$  be the cycle on  $k$  vertices. Then we have*

- (a)  $\gamma_{gr}(P_k \square P_l) = k(l - 1)$ , if  $2 \leq k \leq l$ ;
- (b)  $\gamma_{gr}(P_k \square C_l) = \max\{l(k - 1), k(l - 2)\}$ , if  $2 \leq k$  and  $3 \leq l$ ;
- (c)  $\gamma_{gr}(C_k \square C_l) = k(l - 2)$ , if  $3 \leq k \leq l$  and  $(k, l) \neq (2t + 1, 2t + 1)$  for some  $1 \leq t$ ;
- (d)  $\gamma_{gr}(C_k \square C_k) = k(k - 2) + 1$  if  $k$  is odd.

*Proof.* The fact that the left-hand side in (a), (b) and (c) is at least the right-hand side is a consequence of Proposition 3 and the facts that  $\gamma_{gr}(P_l) = l - 1$  and  $\gamma_{gr}(C_k) = k - 2$ .

To prove the same for (d) we construct a dominating sequence of length  $k(k - 2) + 1$  in  $C_k \square C_k$  for odd  $k$ . We do it in two steps.

Suppose that  $k = 2t + 1$ . Let  $V(C_k \square C_k) = \{(x, y) \in \mathbb{Z}^2 : |x|, |y| \leq t\}$  and  $E(C_k \square C_k) = \{(x, y), (x', y')\} : |x - x'| + |y - y'| = 1, \text{ or } x = x' \text{ and } y = -y' = t, \text{ or } y = y' \text{ and } x = -x' = t\}$ .

Set

$$A_1 := \{(x, y) \in V(C_k \square C_k) : |x - 1/2| + |y| \leq t - 1/2\}$$

with the following ordering: for  $(x, y), (x', y') \in A_1$ ,  $(x, y) \prec_1 (x', y')$  if and only if

- $|y| < |y'|$ , or
- $|y| = |y'|$  and  $y' < 0 < y$ , or
- $y = y'$  and  $x < x'$ .

It is obvious that  $\prec_1$  is a total (and so a well-)ordering of  $A_1$ . We choose the vertices of  $A_1$  into our dominating sequence in the order  $\prec_1$ . As  $(x, y) \in A_1$  will dominate the undominated vertex  $(x, y + 1)$  if  $y \geq 0$  and  $(x, y - 1)$  if  $y < 0$ , each element of  $A_1$  will be a legal choice in this order.

In the next step we consider the set

$$A_2 := \{(x, y) \in V(C_k \square C_k) : |x| < t \text{ and } |x - 1/2| + |y| > t - 1/2\}$$

with the following ordering. For  $(x, y), (x', y') \in A_2$ ,  $(x, y) \prec_2 (x', y')$  if and only if

- $|x| < |x'|$ , or
- $|x| = |x'|$  and  $x > 0 > x'$ , or
- $x = x'$  and  $y < y'$ .

One can easily check that  $\prec_2$  is a total (and hence a well-)ordering of  $A_2$ . We will choose the elements of the dominating sequence after the elements of  $A_1$  in the order  $\prec_2$ . Each element can be chosen into a legal sequence, since  $(x, y) \in A_2$  dominates the so far undominated element  $(x + 1, y)$  if  $x > 0$  and  $(x - 1, y)$  if  $x \leq 0$ .

We are done since the length of this dominating sequence is  $k(k-2)+1$ . A dominating sequence of  $C_5 \square C_5$  is constructed in Figure 1.

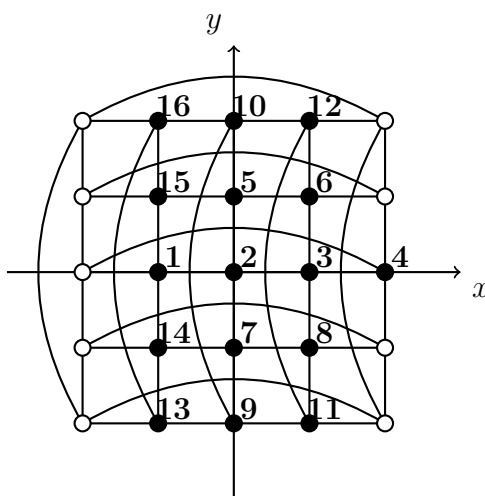


Figure 1: A dominating sequence of  $C_5 \square C_5$

Now we prove that the left-hand side in **(a)**, **(b)**, **(c)**, and **(d)** is at most the right-hand side. We will use Lemma 1.

To prove **(a)** let  $S = ((a_1, b_1), (a_2, b_2), \dots, (a_m, b_m))$  be a Grundy dominating sequence of  $P_k \square P_l$ . Let  $i$  be the minimum index such that either there is a column containing  $l$  vertices of  $\widehat{S}_i$  or a row containing  $k$  vertices of  $\widehat{S}_i$ . If there is no such  $i$ , then  $m \leq k(l-1)$  and we are done. Note that the two cases cannot happen simultaneously at any step  $i$  since otherwise all the neighbors of  $(a_i, b_i)$  and  $(a_i, b_i)$  itself would already be dominated by  $\widehat{S}_{i-1}$  and so  $S$  would not be a legal sequence. So without loss of generality we can assume that there is horizontal segment of length  $k$ . Since in each column of  $P_k \square P_l$  there is at least one and at most  $l-1$  elements of the set  $\widehat{S}_i$ , we have that  $|\partial \widehat{S}_i| \geq k$ , which—by Lemma 1—proves **(a)**.

To prove **(b)** let  $S = ((a_1, b_1), (a_2, b_2), \dots, (a_m, b_m))$  be a Grundy dominating sequence of  $P_k \square C_l$ . Let  $i$  be the minimum index such that either there is a column containing  $l-1$  vertices of  $\widehat{S}_i$  or a row containing  $k$  vertices of  $\widehat{S}_i$ . There exists such an  $i$  as otherwise  $m \leq \max\{l(k-1), k(l-2)\}$  and we are done. We observe again that both cases cannot

happen at the same time since otherwise all the neighbors of  $(a_i, b_i)$  and  $(a_i, b_i)$  itself would already be dominated. Suppose first that there is a column containing  $l - 1$  vertices of  $\widehat{S}_i$ . Then in each row there is an element of the boundary of  $\widehat{S}_i$ . By Lemma 1, this means that the length of the Grundy dominating sequence is at most  $l(k - 1)$ . Now suppose that there is column containing  $k$  vertices of  $\widehat{S}_i$ . Then—since there is an element in each column and there is no column containing  $l - 1$  vertices of  $\widehat{S}_i$ —we have that the cardinality of the boundary of  $\widehat{S}_i$  is at least  $2k$ . By Lemma 1, the length of the Grundy dominating sequence is at most  $k(l - 2)$  and we are done with **(b)**.

Now we prove **(c)**.

CASE I.  $k + 1 \leq l$ .

Let  $S = (a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)$  be a Grundy dominating sequence of  $C_k \square C_l$ . Let  $i_1$  be the minimum index such that either there is a column containing  $l - 1$  vertices of  $\widehat{S}_i$  or there is a row containing  $k - 1$  vertices of  $\widehat{S}_i$ . Note that it cannot happen that both of these hold as then  $(a_{i_1}, b_{i_1})$  would not dominate any new vertex.

SUBCASE IA. There exists a column containing  $l - 1$  vertices of  $\widehat{S}_{i_1}$ .

Then  $|\partial\widehat{S}_{i_1}| \geq 2(l - 1) \geq 2k$  and we are done by Lemma 1.

SUBCASE IB1. There exists a row  $P_k^v$  containing  $k - 1$  vertices of  $\widehat{S}_{i_1}$ , and before the appearance of another such row and before the first row containing  $k$  vertices appears, there exists a column  ${}^uP_l$  containing  $l - 1$  vertices of  $\widehat{S}_{i_2}$  for some  $i_2 > i_1$ .

Then the boundary of  $\widehat{S}_{i_2}$  contains  ${}^uP_l \setminus \widehat{S}_{i_2}$ ,  $P_k^v \setminus \widehat{S}_{i_2}$  and 2 vertices from every other row. Therefore  $|\partial\widehat{S}_{i_2}| \geq 1 + 1 + 2(l - 2) \geq 2k$  holds and we are done by Lemma 1.

SUBCASE IB2. There exists a row  $P_k^v$  containing  $k - 1$  vertices of  $\widehat{S}_{i_1}$ , and before the appearance of another such row and before a column  ${}^uP_l$  containing  $l - 1$  vertices appears, the vertex in  $P_k^v \setminus \widehat{S}_{i_1}$  becomes the  $i_2$ nd vertex of  $S$  for some  $i_2 > i_1$ .

Then the boundary of  $\widehat{S}_{i_2}$  contains two vertices from every column and therefore  $|\partial\widehat{S}_{i_2}| \geq 2k$  and we are done by Lemma 1.

SUBCASE IB3. There exists a row  $P_k^v$  containing  $k - 1$  vertices of  $\widehat{S}_{i_1}$ , and before the appearance of a column  ${}^uP_l$  containing  $l - 1$  vertices and before the vertex in  $P_k^v \setminus \widehat{S}_{i_1}$  appears in  $S$ , there exists another row  $P_k^w$  containing  $k - 1$  vertices from  $\widehat{S}_{i_2}$  for some  $i_2 > i_1$ .

Then either every column contains a vertex from  $\widehat{S}_{i_2}$  and then  $|\partial\widehat{S}_{i_2}| \geq 2k$  or the vertices of  $P_k^v \setminus \widehat{S}_{i_2}$  and  $P_k^w \setminus \widehat{S}_{i_2}$  are in the same column. Then these two vertices are in the boundary of  $\widehat{S}_{i_2}$  and two vertices from every other column belongs to  $\partial\widehat{S}_{i_2}$  and thus  $|\partial\widehat{S}_{i_2}| \geq 2 + 2(k - 1) = 2k$ . We are done by Lemma 1.

CASE II.  $k = l$  even.

Let  $k = 2t$ ,  $V(C_k \square C_k) = \{(x, y) \in \mathbb{Z}^2 : -t + 1 \leq x, y \leq t\}$  and  $E(C_k \square C_k) = \{((x, y), (x', y')) : |x - x'| + |y - y'| = 1, \text{ or } x = x' \text{ and } |y - y'| = k, \text{ or } y = y' \text{ and } |x - x'| = k\}$ . Let  $S = (a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)$  be a Grundy dominating sequence of  $C_k \square C_k$  and let  $i$  be the minimum index such that either a row or a column contains  $k - 1$  vertices from  $\widehat{S}_i$ . As in the previous case, there cannot be both a row and a column that contain



$k - 1$  vertices of  $\widehat{S}_i$  as  $N[(a_i, b_i)]$  would already be dominated by  $\widehat{S}_{i-1}$ . We can assume without loss of generality that  $\widehat{S}_i$  contains  $\{(0, y) : |y| \leq t - 1\}$ . As then every row contains at most  $k - 2$  vertices of  $\widehat{S}_i$ , we obtain that  $\partial\widehat{S}_i$  contains 2 vertices in each row except the row  $R_t = \{(x, t) : -t + 1 \leq x \leq t\}$ . The row  $\{(x, y) : -t + 1 \leq x \leq t\}$  will be denoted by  $R_y$ . As  $(0, t) \in \partial\widehat{S}_i$ , we have  $|\partial\widehat{S}_i| \geq 2k - 1$ . If  $|\partial\widehat{S}_i| \geq 2k$  holds, then we are done by Lemma 1. Otherwise, we must have  $|\partial\widehat{S}_i| = 2k - 1$  and  $\partial\widehat{S}_i \cap R_t = \{(0, t)\}$ . This implies  $R_{-t+1} \cap \widehat{S}_i = \{(0, -t + 1)\}$  and  $R_{t-1} \cap \widehat{S}_i = \{(0, t - 1)\}$ . Note that if for some row  $|R_u \cap \widehat{S}_i| < k - 2$ , then  $R_u \cap \widehat{S}_i$  must be an interval as otherwise  $|R_u \cap \partial\widehat{S}_i| \geq 3$  and thus  $|\partial\widehat{S}_i| \geq 2k$  would hold. Also, if for some row  $|R_u \cap \widehat{S}_i| < k - 2$  and  $R_u \cap \widehat{S}_i = \{(x, u) : \alpha \leq x \leq \beta\}$ , then  $R_{u-1} \cap \widehat{S}_i \subseteq \{(x, u - 1) : \alpha - 1 \leq x \leq \beta + 1\}$  and  $R_{u+1} \cap \widehat{S}_i \subseteq \{(x, u + 1) : \alpha - 1 \leq x \leq \beta + 1\}$  hold as otherwise  $|R_u \cap \partial\widehat{S}_i| \geq 3$  and thus  $|\partial\widehat{S}_i| \geq 2k$  would hold.

From the above it follows that  $\widehat{S}_i \subseteq T := \{(x, y) : |x| + |y| \leq t - 1\}$  holds, see Figure 2. However the number of vertices in  $T$  is less than  $k(k - 2)$ , so there must exist a minimal  $j$  such that  $\widehat{S}_j$  contains a vertex outside of  $T$ . An easy case analysis shows that  $|\partial\widehat{S}_j| \geq 2k$  holds and therefore we are done by Lemma 1.

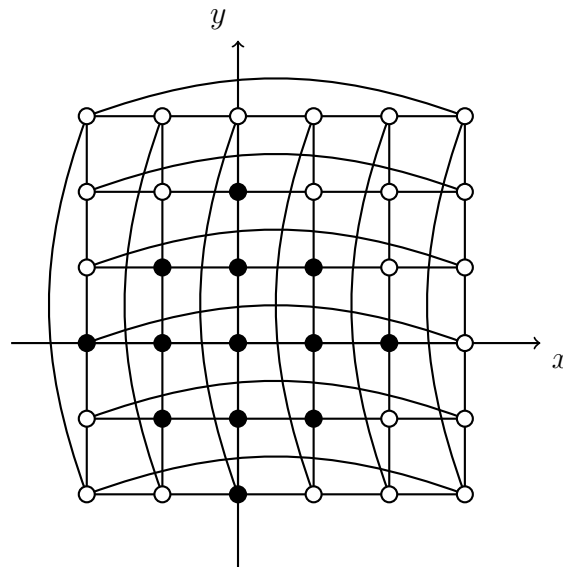


Figure 2: A set  $T$  in  $C_6 \square C_6$

Finally, let us prove the upper bound in **(d)**. Let  $S = (a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)$  be a Grundy dominating sequence of  $C_k \square C_k$  for odd  $k$ . Let  $i$  be the minimum index such that there is either a row or a column containing  $k - 1$  vertices from  $\widehat{S}_i$ . We observe again that both cannot happen at step  $i$  since otherwise all the neighbors of  $(a_i, b_i)$  and  $(a_i, b_i)$  itself would already be dominated. So we can assume without loss of generality that there exists a column containing  $k - 1$  vertices from  $\widehat{S}_i$ . Then in each row we have at least two elements in the boundary of  $\widehat{S}_i$ , except one, where we have at least one element. By this

we have  $|\partial\widehat{S}_i| \geq 2k - 1$ , and the upper bound of **(d)** follows by Lemma 1.  $\square$

In the remainder of this section we show how Lemma 1 and isoperimetric inequalities can be applied to prove results on the Grundy domination number of products of several paths and products of several even cycles. For a graph  $G$ , a vertex  $v \in V(G)$  and natural number  $r$  let us denote by  $B(G, v, r)$  those points in  $G$ , whose distance from  $v$  is at most  $r$ . We will need the following two isoperimetric theorems that are consequences of the cited results for graphs of special vertex cardinalities.

**Theorem 5.** (Bollobás, Leader, [2, Theorem 8]) *Suppose that  $H \subset C_{2k_1} \square \cdots \square C_{2k_n}$  and  $|V(H)| = |B(C_{2k_1} \square \cdots \square C_{2k_n}, v, r)|$  for some  $v \in V(C_{2k_1} \square \cdots \square C_{2k_n})$  and a natural number  $r$ . Then*

$$|\partial H| \geq |\partial B(C_{2k_1} \square \cdots \square C_{2k_n}, v, r)|.$$

**Theorem 6.** (Riordan, [16, Theorem 1.1]) *Suppose that  $H \subset P_{k_1} \square \cdots \square P_{k_n}$  and*

$$|V(H)| = |B(P_{k_1} \square \cdots \square P_{k_n}, v, r)|$$

*for some vertex  $v \in V(P_{k_1} \square \cdots \square P_{k_n})$  of minimum degree and some natural number  $r$ . Then*

$$|\partial H| \geq |\partial B(P_{k_1} \square \cdots \square P_{k_n}, v, r)|.$$

**Proposition 7.** *If  $k_1 \leq \cdots \leq k_n$  and  $\sum_{i=1}^{n-1} k_i + 2 \leq k_n$ , then we have*

$$\gamma_{gr}(C_{2k_1} \square \cdots \square C_{2k_n}) = 2^n k_1 \cdots k_{n-1} (k_n - 1).$$

*If  $k_1 \leq \cdots \leq k_n$  and  $\sum_{i=1}^{n-1} k_i + 1 \leq k_n$ , then we have*

$$\gamma_{gr}(P_{k_1} \square \cdots \square P_{k_n}) = k_1 \cdots k_{n-1} (k_n - 1).$$

*Proof.* In both statements, the lower bound on the Grundy domination number follows from Proposition 3.

The upper bound in the first statement follows from Lemma 1 and Theorem 5, while the upper bound of the second statement follows from Lemma 1 and Theorem 6.  $\square$

## 4 Lexicographic product

In this section we give an expression for the Grundy domination number of the lexicographic product of graphs in terms of corresponding invariants of the factors. From this result we in particular did use explicit formulas for the Grundy domination number of grids, cylinders and tori (with respect to the lexicographic product).

**Proposition 8.** *Let  $G$  and  $H$  be graphs. Then*

$$\gamma_{gr}(G \circ H) \geq \max\{\alpha(G)\gamma_{gr}(H), \gamma_{gr}(G)\}.$$

*Proof.* Let  $\ell = \alpha(G)$ ,  $n = \gamma_{gr}(H)$ , let  $\{d_1, \dots, d_\ell\}$  be a maximum independent set in  $G$ , and  $(d'_1, \dots, d'_n)$  be a Grundy dominating sequence in  $H$ . Observe that

$$((d_1, d'_1), \dots, (d_1, d'_n), (d_2, d'_1), \dots, (d_2, d'_n), \dots, (d_\ell, d'_1), \dots, (d_\ell, d'_n))$$

is a dominating sequence in  $G \circ H$ . Hence  $\gamma_{gr}(G \circ H) \geq \ell n = \alpha(G)\gamma_{gr}(H)$ .

The bound  $\gamma_{gr}(G \circ H) \geq \gamma_{gr}(G)$  is trivial. □

The lower bound from Proposition 8 is sharp. For instance, if  $H$  is a complete graph then clearly  $\gamma_{gr}(G \circ H) = \gamma_{gr}(G)$ . It is easy to see that  $\gamma_{gr}(G)\gamma_{gr}(H)$  is an upper bound for  $\gamma_{gr}(G \circ H)$ . Thus for any graph  $G$ , in which  $\gamma_{gr}(G) = \alpha(G)$ , we have  $\gamma_{gr}(G \circ H) = \alpha(G)\gamma_{gr}(H)$ . On the other hand, there are graphs in which this bound is not sharp. For instance, let  $T$  be the tree from Fig. 3, and let  $H$  be any graph, which is not complete, thus  $\gamma_{gr}(H) \geq 2$ . Note that the filled vertices in the figure present a maximum independent set of  $T$ , and so  $\alpha(T) = 5$ . Hence the bound in Proposition 8 is  $\alpha(T)\gamma_{gr}(H) = 5\gamma_{gr}(H)$ , but  $\gamma_{gr}(T \circ H) \geq 5\gamma_{gr}(H) + 1$ . Indeed, let  $S$  be the sequence that starts in the layer  ${}^uH$ , by legally picking  $\gamma_{gr}(H)$  vertices, then choosing a vertex from  ${}^vH$ , and then choosing  $\gamma_{gr}(H)$  vertices in each of the layers that correspond to the remaining four leaves of  $T$ . This yields a legal sequence of the desired length.

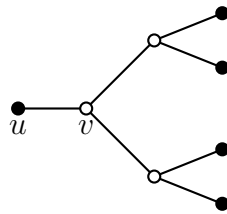


Figure 3: The tree  $T$ .

Next, we present an exact formula for  $\gamma_{gr}(G \circ H)$ . Given a dominating sequence  $D = (d_1, \dots, d_k)$  in a graph  $G$ , let  $a(D)$  denote the cardinality of the set of vertices  $d_i$  from  $D$ , which are not adjacent to any vertex from  $\{d_1, \dots, d_{i-1}\}$ .

**Theorem 9.** *Let  $G$  and  $H$  be graphs. Then*

$$\gamma_{gr}(G \circ H) = \max\{a(D)(\gamma_{gr}(H) - 1) + |\widehat{D}|; D \text{ is a dominating sequence of } G\}.$$

*Proof.* Let  $D = (d_1, \dots, d_m)$  be any dominating sequence of  $G$ , and let  $(d'_1, \dots, d'_k)$  be a Grundy dominating sequence of  $H$ . Then one can find a dominating sequence  $S$  in  $G \circ H$  of length  $a(D)(\gamma_{gr}(H) - 1) + |\widehat{D}|$  as follows. Let  $S$  be the sequence that corresponds to  $D$ , only those vertices  $d_i \in \widehat{D}$  which are not adjacent to any vertex from  $\{d_1, \dots, d_{i-1}\}$  are repeated  $\gamma_{gr}(H)$  times in a row, so that the corresponding subsequence is of the form  $((d_i, d'_1), \dots, (d_i, d'_k))$ . On the other hand, the vertices  $d_i \in \widehat{D}$  which are adjacent to some vertex from  $\{d_1, \dots, d_{i-1}\}$  are projected only once, from any vertex  $(d_i, h) \in {}^{d_i}H$ . It is easy to see that in either case the vertices in  $S$  are legally chosen. In the first case this is true because no vertex of  ${}^{d_i}H$  is dominated at the point when  $(d_i, d'_1)$  is chosen,

thus  $((d_i, d'_1), \dots, (d_i, d'_k))$  is a legal subsequence. In the second case this is true because  $D$  is a legal sequence in  $G$ , and so when  $d_i$  is chosen, which is at that point already dominated, this implies that there exists another vertex  $t \in V(G)$  that  $d_i$  footprints. Hence, when  $(d_i, h)$  is chosen in  $S$ , it footprints vertices from  ${}^tH$ . Note that the length of  $S$  is  $a(D)(\gamma_{gr}(H) - 1) + |\widehat{D}|$ , and that  $D$  is an arbitrary legal sequence of  $G$ . This implies that  $\gamma_{gr}(G \circ H) \geq \max\{a(D)(\gamma_{gr}(H) - 1) + |\widehat{D}|\}$ .

For the converse, let  $S$  be an arbitrary dominating sequence in  $G \circ H$ . Let  $s_i = (x, y)$  be a vertex from  $S$ , where  $x \in V(G), y \in V(H)$ . Note that when  $s_i$  is added to  $S$ , all vertices from the layers  ${}^hH$ , where  $h \in N(x)$ , are dominated. In particular, at most one vertex from each of the layers  ${}^hH$  can be in  $S$  after  $s_i$ .

Now, consider the sequence  $D$  of vertices from  $G$ , defined as follows: for each  $s_i = (x, y) \in S$  add  $x$  to  $D$  if  $s_i$  is the first vertex from  $S$  which is in  ${}^xH$ . First note that  $D$  is a legal sequence of  $G$ . Indeed, when  $x$  is added to  $D$ ,  $(x, y)$  footprints some vertex, either from  ${}^xH$  or from  ${}^zH$ , where  $z \in N(x)$ . If  $(x, y)$  footprints a vertex  $(x, h)$ , this implies that no vertex  $s_j \in S$ , where  $j < i$ , is in  $N[x] \times V(H)$ . Thus in this case  $x$  footprints itself with respect to  $D$ . In the case when  $(x, y)$  footprints a vertex from a layer  ${}^gH$ , where  $g \in N(x)$ ,  $x$  footprints  $g$  with respect to  $D$ . In either case  $x$  is a legal choice, hence  $D$  is a legal sequence in  $G$ . It is clear that  $\widehat{D}$  is a dominating set, since  $\widehat{D} = p_G(\widehat{S})$  and  $\widehat{S}$  is a dominating set of  $G \circ H$ .

Let  $A(D)$  be the set of all vertices  $d_k$  from  $D$ , such that  $d_k$  is not adjacent to any vertex from  $\{d_1, \dots, d_{k-1}\}$ . (Note that  $|A(D)| = a(D)$  by definition.) By the way  $D$  is constructed, if  $d_k \notin A(D)$ , then at the point when  $s_i = (d_k, y)$  is added to  $S$ , all vertices of the layer  ${}^{d_k}H$  are already dominated. Hence, at most one vertex from  ${}^{d_k}H$  can lie in  $S$ . On the other hand, clearly at most  $\gamma_{gr}(H)$  vertices from layers  ${}^{d_k}H$ , where  $d_k \in A(D)$ , can lie in  $S$ . We infer that  $|\widehat{S}| \leq (|\widehat{D}| - a(D)) + a(D)\gamma_{gr}(H)$ , from which the desired inequality follows.  $\square$

**Corollary 10.** *Let  $H$  be an arbitrary graph that is not a complete graph. Then*

$$\gamma_{gr}(P_k \circ H) = \begin{cases} \frac{k}{2} \cdot \gamma_{gr}(H) + 1, & k \text{ is even, } k \neq 2 \\ \lceil \frac{k}{2} \rceil \cdot \gamma_{gr}(H), & k \text{ is odd.} \end{cases}$$

*Proof.* Let  $P_k = v_1, v_2, \dots, v_k$ . Using Theorem 9, we would like to find a dominating sequence  $D = (d_1, \dots, d_t)$  of  $P_k$  with the largest value  $a(D)\gamma_{gr}(H) + (|\widehat{D}| - a(D))$ . From the proof of Theorem 9 it follows that the vertices  $d_i \in \widehat{D}$  that are not adjacent to vertices  $\widehat{D}_{i-1}$  contribute  $\gamma_{gr}(H) \geq 2$  to the Grundy dominating sequence of  $P_k \circ H$ , but the other vertices of  $\widehat{D}$  contribute just 1 to the Grundy dominating sequence of  $P_k \circ H$ . Therefore  $D$  is optimal if and only if  $a(D)$  is as big as possible. Thus  $a(D) = \lceil \frac{k}{2} \rceil$ . In the case when  $k$  is odd,  $D = (v_1, v_3, \dots, v_k)$  and this is the only case with  $a(D) = \frac{k+1}{2}$ . Thus  $\gamma_{gr}(P_k \circ H) = \lceil \frac{k}{2} \rceil \cdot \gamma_{gr}(H)$ . If  $k$  is even, then  $a(D) = \frac{k}{2}$  implies  $|\widehat{D}| \leq \frac{k}{2} + 1$ . The set  $D$  can be chosen in such a way that the equality holds, for example  $D = (v_1, v_2, v_4, v_6, \dots, v_k)$  contains  $\frac{k}{2} + 1$  vertices. Thus  $\gamma_{gr}(P_k \circ H) = \frac{k}{2} \cdot \gamma_{gr}(H) + 1$ .  $\square$

**Corollary 11.** *Let  $k, l > 2$ . Then*

$$\gamma_{gr}(P_k \circ P_l) = \begin{cases} \frac{k}{2} \cdot (l-1) + 1, & k \text{ is even} \\ \lceil \frac{k}{2} \rceil \cdot (l-1), & k \text{ is odd.} \end{cases}$$

**Corollary 12.** *Let  $k, l > 2$ . Then*

$$\gamma_{gr}(P_k \circ C_l) = \begin{cases} \frac{k}{2} \cdot (l-2) + 1, & k \text{ is even} \\ \lceil \frac{k}{2} \rceil \cdot (l-2), & k \text{ is odd.} \end{cases}$$

Using Theorem 9 and the same ideas as in the proof of Corollary 10, we obtain the following result.

**Corollary 13.** *Let  $H$  be an arbitrary graph that is not a complete graph, and let  $k > 3$ . Then*

$$\gamma_{gr}(C_k \circ H) = \begin{cases} \frac{k}{2} \cdot \gamma_{gr}(H), & k \text{ is even} \\ \lceil \frac{k}{2} \rceil \cdot \gamma_{gr}(H) + 1, & k \text{ is odd.} \end{cases}$$

**Corollary 14.** *Let  $k, l > 3$ . Then*

$$\gamma_{gr}(C_k \circ C_l) = \begin{cases} \frac{k}{2} \cdot (l-2), & k \text{ is even} \\ \lceil \frac{k}{2} \rceil \cdot (l-2) + 1, & k \text{ is odd.} \end{cases}$$

## 5 Direct product

Quite often the direct product is the most difficult one among the standard products when investigating its invariants and similar problems. This phenomenon is also true in the case of the Grundy domination number. In this section we give a general lower bound on the Grundy domination number and specialize it to products of paths and cycles. In the case of the product of two paths where a shorter path is of even order we also give an exact value. We begin with the following general lower bound where  $a(D)$  is the same function as in Section 4.

**Proposition 15.** *Let  $G$  and  $H$  be graphs. Then*

$$\gamma_{gr}(G \times H) \geq \max \left\{ \begin{array}{l} \max\{a(D)|V(H)| + \gamma_{gr}^t(H)(|\widehat{D}| - a(D)); D \text{ is a dominating sequence of } G\} \\ \max\{a(D)|V(G)| + \gamma_{gr}^t(G)(|\widehat{D}| - a(D)); D \text{ is a dominating sequence of } H\} \end{array} \right\}$$

*Proof.* We first note that it suffices to present a construction that yields a dominating sequence of length

$$\max\{a(D)|V(H)| + \gamma_{gr}^t(H)(|\widehat{D}| - a(D)); D \text{ is a dominating sequence of } G\}.$$

Let  $D$  be a dominating sequence of  $G$ . Now we construct a sequence  $S$  in  $G \times H$  that corresponds to  $D$  in the following way. For a vertex  $d_i \in \widehat{D}$ , which is not adjacent to any

vertex from  $\{d_1, \dots, d_{i-1}\}$ , all vertices of  $d_i H$  are added to  $S$  (this is legal because  $d_i H$  is an independent set, and  $(d_i, h)$  is not adjacent to any  $(d_j, h')$  for  $j < i$ ). On the other hand, if  $d_i$  is adjacent to some  $d_j$ ,  $j < i$ , then  $d_i$  footprints a vertex  $g$  with respect to  $D$  in  $G$ . Given a Grundy total dominating sequence  $T = (t_1, \dots, t_r)$  in  $H$ , where  $r = \gamma_{gr}^t(H)$ , we add to  $S$  the sequence  $(d_i, t_1), \dots, (d_i, t_r)$  (indeed this is legal because  $(d_i, t_j)$  footprints  $(g, t'_j)$ , where  $t'_j$  is a neighbor of  $t_j$  footprinted by  $t_j$  with respect to  $T$ ). The length of  $S$  is  $a(D)|V(H)| + \gamma_{gr}^t(H)(|\widehat{D}| - a(D))$ , as desired.  $\square$

In the next few results we will use the following notations  $V(P_k) = [k]$ ,  $E(P_k) = \{\{i, i+1\} : 1 \leq i \leq k-1\}$  and  $V(C_k) = [k]$ ,  $E(C_k) = \{\{i, i+1\} : 1 \leq i \leq k-1\} \cup \{1, k\}$ .

Proposition 15 yields the following lower bounds.

**Corollary 16.** *If  $k \geq 2$  and  $l \geq 4$ , then*

$$\gamma_{gr}(P_k \times C_l) \geq \begin{cases} \max\{kl - 2k - l + 6, kl - 2k\}, & k, l \text{ are even} \\ kl - k - l + 3, & k, l \text{ are odd.} \\ \max\{kl - 2k, kl - k - l + 3\}, & k \text{ is even, } l \text{ is odd} \\ kl - 2k - l + 6, & k \text{ is odd, } l \text{ is even} \end{cases}$$

*Proof.* The proof of all four cases is similar, the only difference is in total Grundy domination number of paths and cycles with respect to the parity of the length. That is

$$\gamma_{gr}^t(P_k) = \begin{cases} k, & k \text{ is even} \\ k - 1, & k \text{ is odd} \end{cases}$$

and

$$\gamma_{gr}^t(C_l) = \begin{cases} l - 2, & l \text{ is even} \\ l - 1, & l \text{ is odd.} \end{cases}$$

Let  $D = (1, \dots, k-2, k)$  be a dominating sequence of  $P_k$  and  $D' = (1, \dots, l-2)$  a dominating sequence of  $C_l$ . Then  $a(D) = 2$  and  $a(D') = 1$ . First let  $k$  and  $l$  be even. Since  $\gamma_{gr}^t(C_l) = l - 2$ , Proposition 15 implies that  $\gamma_{gr}(P_k \times C_l) \geq a(D)|V(C_l)| + \gamma_{gr}^t(C_l)(|\widehat{D}| - a(D)) = 2l + (k-3)(l-2) = kl - 2k - l + 6$ . Since  $\gamma_{gr}^t(P_k) = k$  it follows from Proposition 15 that  $\gamma_{gr}(P_k \times C_l) \geq a(D')|V(P_k)| + \gamma_{gr}^t(P_k)(|\widehat{D}'| - a(D')) = k + (l-3)k = kl - 2k$ .

If  $k$  is even and  $l$  is odd, then  $\gamma_{gr}(P_k \times C_l) \geq a(D)|V(C_l)| + \gamma_{gr}^t(C_l)(|\widehat{D}| - a(D)) = 2l + (k-3)(l-1) = kl - k - l + 3$  and  $\gamma_{gr}(P_k \times C_l) \geq a(D')|V(P_k)| + \gamma_{gr}^t(P_k)(|\widehat{D}'| - a(D')) = k + k(l-3) = kl - 2k$ .

If  $k$  is odd and  $l$  is even, then  $\gamma_{gr}(P_k \times C_l) \geq a(D)|V(C_l)| + \gamma_{gr}^t(C_l)(|\widehat{D}| - a(D)) = 2l + (k-3)(l-2) = kl - 2k - l + 6$ .

Finally let  $k$  and  $l$  be odd. Then  $\gamma_{gr}(P_k \times C_l) \geq a(D)|V(C_l)| + \gamma_{gr}^t(C_l)(|\widehat{D}| - a(D)) = 2l + (k-3)(l-1) = kl - k - l + 3$ .  $\square$

The proof of the next result is omitted, as it follows from the Grundy total domination number of a cycle and a dominating sequence of a cycle  $C_n$  of length  $n - 2$ .

**Corollary 17.** *If  $l \geq k \geq 4$ , then*

$$\gamma_{gr}(C_k \times C_l) \geq \begin{cases} kl - 2k - 2l + 6, & k, l \text{ are even} \\ kl - 2k - l + 3, & k \text{ is odd.} \\ kl - k - 2l + 3, & k \text{ is even, } l \text{ is odd.} \end{cases}$$

**Corollary 18.** *If  $2 \leq k \leq l$ , then*

$$\gamma_{gr}(P_k \times P_l) \geq \begin{cases} kl - k, & k \text{ is even} \\ kl - k - l + 3, & k, l \text{ are odd.} \\ \max\{kl - l, kl - k - l + 3\}, & k \text{ is odd, } l \text{ is even.} \end{cases}$$

*Proof.* Let  $D = (1, \dots, k-2, k)$  be a dominating sequence of  $P_k$  and  $D' = (1, \dots, l-2, l)$  a dominating sequence of  $P_l$ . Then  $a(D) = a(D') = 2$ . If  $k$  is even, then it follows from Proposition 15 that  $\gamma_{gr}(P_k \times P_l) \geq a(D')|V(P_k)| + \gamma_{gr}^t(P_k)(|D'| - a(D')) = 2k + (l-3)k = kl - k$ . If both  $k$  and  $l$  are odd, then  $\gamma_{gr}(P_k \times P_l) \geq 2l + (k-3)(l-1) = kl - k - l + 3$ . If  $k$  is odd and  $l$  is even, then  $\gamma_{gr}(P_k \times P_l) \geq \max\{2l + (k-3)l, 2k + (l-3)(k-1)\} = \max\{kl - l, kl - k - l + 3\}$ .  $\square$

**Proposition 19.** *If  $2 \leq k \leq l$ , then  $\gamma_{gr}(P_k \times P_l) \leq kl - k$ .*

*Proof.* The proof is very similar to that of the upper bound in Theorem 4 (a). The graph  $P_k \times P_l$  has two components  $C_{odd} = \{(a, b) : a+b \text{ is odd}\}$  and  $C_{even} = \{(a, b) : a+b \text{ is even}\}$ . We will prove that the length of a dominating sequence is  $\lfloor k/2 \rfloor$  less than the number of vertices in one of them and  $\lceil k/2 \rceil$  less than the number of vertices in the other. Let us consider the component  $C_{odd}$ . Let  $L_{odd,+}$  denote the set of lines  $\{(x, y) : x + y = c\}$  where  $c$  is an odd integer with  $\lfloor k/2 + 1 \rfloor \leq c \leq \lfloor 3k/2 + 1 \rfloor$ , and let  $L_{odd,-}$  denote the set of lines  $\{(x, y) : x - y = c\}$  where  $c$  is an odd integer with  $1 - \lfloor k/2 \rfloor \leq c \leq \lfloor k/2 \rfloor$ . Finally, let  $S = ((a_1, b_1), (a_2, b_2), \dots, (a_m, b_m))$  be a dominating sequence of  $C_{odd}$  and let  $i$  be the smallest index for which either every line in  $L_{odd,-}$  contains a vertex from  $\widehat{S}_i$  or every line in  $L_{odd,+}$  contains a vertex from  $\widehat{S}_i$  (or both). Such an index exists as if not then the vertex  $\ell_- \cap \ell_+$  with  $\ell_- \in L_{odd,-}, \ell_+ \in L_{odd,+}$  is not dominated by  $\widehat{S}$  if none of  $\ell_-$  and  $\ell_+$  contain a vertex from  $\widehat{S}$ . Observe that  $|\partial \widehat{S}_i| \geq \min\{|L_{odd,-}|, |L_{odd,+}|\}$  holds as if every line in  $L_{odd,-}$  contains a vertex from  $\widehat{S}_i$ , then every line in  $L_{odd,+}$  contains a vertex from  $\partial \widehat{S}_i$  and vice versa.  $\square$

Corollary 18 and Proposition 19 give the following exact result.

**Corollary 20.** *Let  $2 \leq k \leq l$  and let  $k$  be even. Then  $\gamma_{gr}(P_k \times P_l) = kl - k$ .*

## 6 Strong product

In this section we first observe that  $\gamma_{gr}(G \boxtimes H) \geq \gamma_{gr}(G)\gamma_{gr}(H)$  holds for any graphs  $G$  and  $H$ , and conjecture that it always holds with equality. Among other results proved here we confirm the conjecture for strong products of caterpillars with arbitrary graphs.

**Proposition 21.** For any graphs  $G$  and  $H$ ,

$$\gamma_{gr}(G \boxtimes H) \geq \gamma_{gr}(G)\gamma_{gr}(H).$$

*Proof.* Let  $p = \gamma_{gr}(G)$ ,  $q = \gamma_{gr}(H)$ , let  $(d_1, \dots, d_p)$  be a Grundy dominating sequence in  $G$ , and  $(d'_1, \dots, d'_q)$  be a Grundy dominating sequence in  $H$ . Let  $d_i$  footprint  $u_i$ ,  $i \in [p]$  and let  $d'_j$  footprint  $u'_j$ ,  $j \in [q]$ . Consider the following sequence

$$S = ((d_1, d'_1), \dots, (d_1, d'_q), (d_2, d'_1), \dots, (d_2, d'_q), \dots, (d_p, d'_1), \dots, (d_p, d'_q))$$

of vertices of  $G \boxtimes H$ . It is clear that  $\widehat{S}$  is a dominating set. Moreover a chosen vertex  $(d_i, d'_j)$  is legal since it footprints  $(u_i, u'_j)$ . Hence  $\gamma_{gr}(G \boxtimes H) \geq pq = \gamma_{gr}(G)\gamma_{gr}(H)$ .  $\square$

We conjecture that the lower bound of Proposition 21 is always tight:

**Conjecture 22.** For any graphs  $G$  and  $H$ ,  $\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H)$ .

Recall that the edge cover number  $\theta_e(G)$  presents an upper bound for the  $\gamma_{gr}(G)$  in any graph  $G$ . We next show that this parameter behaves nicely in strong products of triangle-free graphs.

**Proposition 23.** If  $G$  and  $H$  are triangle-free graphs, then  $\theta_e(G \boxtimes H) = |E(G)| \cdot |E(H)|$ .

*Proof.* Let  $e = gg'$  and  $f = hh'$  be arbitrary edges in  $G$  and  $H$ , respectively. Then the edge  $(g, h)(g'h')$  of  $G \boxtimes H$  lies in a unique maximal clique  $Q_{e,f}$  induced by the four vertices in  $\{g, g'\} \times \{h, h'\}$ . It follows that  $\theta_e(G \boxtimes H) \geq |E(G)| \cdot |E(H)| = \theta_e(G)\theta_e(H)$ . On the other hand the set of cliques  $\{Q_{e,f} : e \in E(G), f \in E(H)\}$  forms an edge clique cover of  $G \boxtimes H$ , so that  $\theta_e(G \boxtimes H) \leq |E(G)| \cdot |E(H)|$ .  $\square$

**Corollary 24.** If  $k, l \geq 2$ , then  $\gamma_{gr}(P_k \boxtimes P_l) = (k - 1)(l - 1)$ .

*Proof.* The lower bound follows from Proposition 21 and the upper bound from Proposition 2 and Proposition 23.  $\square$

We approach Conjecture 22 with the following upper bound on the Grundy domination number of the strong product of graphs.

**Proposition 25.** Let  $G$  and  $H$  be arbitrary graphs. Then

$$\gamma_{gr}(G \boxtimes H) \leq \min\{|V(G)|\gamma_{gr}(H), \gamma_{gr}(G)|V(H)|\}.$$

*Proof.* Let  $D$  be a (Grundy) dominating sequence of  $G \boxtimes H$ . Consider a layer  $G^h$ ,  $h \in V(H)$ , and let  $D^h$  be the subsequence of  $D$  that consists only of the vertices in  $G^h$ . We claim that the corresponding sequence  $p_G(D^h)$  is a legal sequence in  $G$ . Indeed, let  $(g, h) \in D^h$  and let  $(g', h')$  be a vertex footprinted by  $(g, h)$  with respect to the sequence  $D$ . It is clear that  $g$  footprints  $g'$  with respect to the sequence  $p_G(D^h)$ . This implies that in each  $G$ -layer there are at most  $\gamma_{gr}(G)$  vertices from  $D$ . Therefore,  $\gamma_{gr}(G \boxtimes H) \leq \gamma_{gr}(G)|V(H)|$ . By reversing the roles of  $G$  and  $H$  the claimed inequality follows.  $\square$



Recall that a vertex is called simplicial if its neighborhood induces a complete graph.

**Proposition 26.** *Let  $G$  and  $H$  be arbitrary graphs. If  $v$  is a simplicial vertex in  $G$ , then*

$$\gamma_{gr}(G \boxtimes H) \leq \gamma_{gr}(H) + \gamma_{gr}((G - v) \boxtimes H).$$

*Proof.* Consider a Grundy dominating sequence  $D$  of  $G \boxtimes H$  such that  $D$  contains maximum number of vertices from  ${}^vH$ . Let  $D^1$  be the subsequence of  $D$  that consists only of the vertices in  ${}^vH$  and let  $D^2$  be the complementary subsequence of  $D$ . Analogously to the proof of Proposition 25, one can show that  $D^1$  is a dominating sequence in  ${}^vH$ . Consequently,  $|\widehat{D^1}| \leq \gamma_{gr}(H)$ .

Now assume that a vertex  $(g, h)$ , which is not from  ${}^vH$ , footprints a vertex  $(v, h')$  in  ${}^vH$ . Since  $v$  is simplicial,  $N_G[v] \subseteq N_G[g]$ . Further, either  $h = h'$  or  $(v, h)$  and  $(v, h')$  must be adjacent. Therefore, if  $(g, h)$  is replaced by  $(v, h)$  in  $D$ , we obtain a Grundy dominating sequence again, and this one contains more vertices from  ${}^vH$  than  $D$  did. This contradicts the choice of  $D$  and proves that no vertex from  $\widehat{D^2}$  footprints a vertex outside  $(G - v) \boxtimes H$ . We may conclude that  $D^2$  is a dominating sequence in  $(G - v) \boxtimes H$ , and the desired inequality

$$\gamma_{gr}(G \boxtimes H) = |\widehat{D^1}| + |\widehat{D^2}| \leq \gamma_{gr}(H) + \gamma_{gr}((G - v) \boxtimes H)$$

holds. □

We say that a graph  $G$  satisfies Conjecture 22 if for every graph  $H$ ,  $\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H)$  holds. As a consequence of Proposition 26, we obtain the following result related to the conjecture.

**Corollary 27.** *Every caterpillar satisfies Conjecture 22.*

*Proof.* Let  $G$  be a caterpillar and let  $H$  be an arbitrary graph. If  $G$  is of order 1 or 2,  $\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H) = \gamma_{gr}(H)$  can be shown directly. Then, we proceed by induction on  $n$ . As we mentioned at the end of Section 2, it is shown in [5] that any nontrivial caterpillar  $G$  of order  $n$  satisfies  $\gamma_{gr}(G) = n - 1$ . If  $n \geq 3$ , delete an arbitrary leaf  $v$  from  $G$ . By the induction hypothesis we have  $\gamma_{gr}((G - v) \boxtimes H) = (n - 2)\gamma_{gr}(H)$ . Then, Proposition 26 gives

$$\gamma_{gr}(G \boxtimes H) \leq \gamma_{gr}(H) + (n - 2)\gamma_{gr}(H) = (n - 1)\gamma_{gr}(H) = \gamma_{gr}(G)\gamma_{gr}(H).$$

Together with Proposition 21 this establishes our statement. □

We immediately derive the exact value of Grundy domination number of cylinders and an upper bound for tori.

**Corollary 28.** *If  $k \geq 2$  and  $l \geq 3$ , then  $\gamma_{gr}(P_k \boxtimes C_l) = (k - 1)(l - 2)$ .*

**Corollary 29.** *If  $3 \leq k \leq l$ , then  $\gamma_{gr}(C_k \boxtimes C_l) \leq (k - 2)(l - 1)$ .*

Note that the conjectured value for a torus is  $\gamma_{gr}(C_k \boxtimes C_l) = (k-2)(l-2)$ . We further remark that, by Corollary 27, strong products of any number of caterpillars also satisfy Conjecture 22. Particularly, the following exact results can be derived:

$$\begin{aligned}\gamma_{gr}(P_{k_1} \boxtimes \cdots \boxtimes P_{k_n}) &= (k_1 - 1) \cdots (k_n - 1) \\ \gamma_{gr}(P_{k_1} \boxtimes \cdots \boxtimes P_{k_n} \boxtimes C_l) &= (k_1 - 1) \cdots (k_n - 1)(l - 2).\end{aligned}$$

Finally, we consider a graph operation related to the conjecture. We say that  $G'$  is obtained from  $G$  by substituting a vertex  $v \in V(G)$  with  $K_\ell$  if  $v$  is replaced with the complete graph on  $\ell$  vertices such that each of these  $\ell$  new vertices is made adjacent to the entire  $N_G(v)$ .

**Proposition 30.** *If  $G$  satisfies Conjecture 22 and  $G'$  is obtained from  $G$  by substituting a vertex  $v \in V(G)$  with a complete graph, then  $G'$  also satisfies Conjecture 22.*

*Proof.* Clearly, a graph isomorphic to  $G'$  can be obtained from  $G$  by successively adding (true) twin vertices to  $v$ . It was observed in [4] that if  $x$  has a twin in a graph  $F$  then  $\gamma_{gr}(F - x) = \gamma_{gr}(F)$ . Further, if  $x$  has a twin in  $F$ , then for every graph  $H$  and for every  $h \in V(H)$ , the vertex  $(x, h)$  also has a twin in  $F \boxtimes H$ . These imply  $\gamma_{gr}((F - x) \boxtimes H) = \gamma_{gr}(F \boxtimes H)$ . Using these statements in opposite direction and successively, we obtain the following. If  $G$  satisfies Conjecture 22 then

$$\gamma_{gr}(G' \boxtimes H) = \gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H) = \gamma_{gr}(G')\gamma_{gr}(H)$$

holds for every graph  $H$ . □

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