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HYPERCUBES AS DIRECT PRODUCTS*

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Abstract. Let G be a connected bipartite graph. An involution α of G that preserves the bipartition of G is called bipartite. Let G^α be the graph obtained from G by adding to G the natural perfect matching induced by α. We show that the k-cube Q_k is isomorphic to the direct product $G \times H$ if and only if G is isomorphic to Q_{k-1}^α for some bipartite involution α of Q_{k-1} and $H = K_2$.

Key words. direct product, hypercube, automorphism, involution

AMS subject classifications. 05C70, 05C25

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1. Introduction. This paper is concerned with hypercubes and the direct product of graphs. The main result is the characterization of all graphs G for which $G \times K_2$ is a hypercube and the proof of the fact that there are no other factorizations of the hypercube with respect to the direct product.

The paper was motivated by the problem of representing median graphs—that is, retracts of hypercubes—as direct products [2]. In this context the first question pertains to the possibility of decomposing the hypercube itself. The original proof of the result was unwieldy and long but could be considerably simplified by the application of ideas connected with the density of subgraphs of sparse graphs, together with the concept of the Cartesian skeleton [10, 11], which was introduced for the investigation of the direct product.

The paper illustrates the importance and applicability of Graham's density lemma and adds to the numerous interesting properties of the hypercube, which plays a prominent role in many areas of mathematics and computer science; see, e.g., the papers [8, 16, 20] on networks, routings, and flows, respectively. It may also shed some light on the decomposition of bipartite graphs with respect to the direct product.

The direct product, together with the Cartesian, the strong, and the lexicographic product, is one of the four standard graph products [11]. It is the natural product in the category of graphs [7] and harbors intriguing and challenging problems. Foremost of all is Hedetniemi's conjecture, which asserts that the chromatic number of the direct product is the minimum of the chromatic numbers of its factors. It is the big open problem in the area and has led to many different approaches and new concepts; cf. surveys [17, 21]. More generally, the direct product is a widely used tool in the area of graph colorings; see, for instance, [6, 22, 23]. It is also replete with interesting

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ideas and concepts relating to other areas of graph theory, for example to matching theory [1, 9] and stability in graphs [3, 13].

This product has been introduced and studied from several points of view and is known under many different names, for instance as the cardinal product, the Kronecker product, and the categorical product. Moreover, it is universal in the sense that every graph is an induced subgraph of a suitable direct product of complete graphs [15].

In 1971 McKenzie [14] proved that finite, nonbipartite, connected graphs have unique prime factor decomposition (UPFD) with respect to the direct product in the class of undirected graphs with loops. Many years later, in 1998, this result was extended in [10] by showing that the UPFD can be found in polynomial time. For disconnected graphs or bipartite graphs the prime factorizations need not be unique. It is also not unique for finite nonbipartite graphs in the class of simple graphs without loops.

Despite the extensive and deep investigations of the direct product, factorizations of bipartite graphs have rarely been investigated. If a bipartite graph is a direct product of two graphs, one factor must be bipartite, but not the other. (The direct product of two connected bipartite graphs consists of two connected (bipartite) components [19].) This also holds for the hypercube, and we cannot directly apply the above results to our problem. Nevertheless, the concept of the Cartesian skeleton that proved useful in the nonbipartite case can be fruitfully applied here, too. In the nonbipartite case the Cartesian skeleton is connected, but not in the bipartite one, and this accounts for the nonuniqueness of the factorizations.

In the remainder of the section we fix terminology and notation. All graphs considered here are undirected, finite graphs that may contain loops.

The *direct product* $G \times H$ of two graphs G and H is defined on the Cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. Its edge set is the set of all pairs of vertices $(a, x), (b, y) \in V(G) \times V(H)$, where $ab \in E(G)$ and $xy \in E(H)$. It is commutative and associative, and the one-vertex graph with a loop is a unit.

The *Cartesian product* $G \Box H$ has the same vertex set as the direct product. Its edge set consists of all pairs (a, x), (b, y) with $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. It is also commutative and associative. Its unit is K_1.

The subgraph of $G \Box H$ induced by the vertices (a, x), $x \in V(H)$, is called an *H-layer* of $G \Box H$ and denoted by $H^{(a,x)}$. Note that any H-layer is isomorphic to H. Analogously one defines G-layers. The d-dimensional *hypercube* or *d-cube* Q_d is the Cartesian product of d copies of the complete graph K_2 on two vertices. So $Q_1 = K_2$ and we also set $Q_0 = K_1$. Let $Q_{d-1} \Box K_2$ be an arbitrary factorization of Q_d. The edges between the two Q_{d-1}-layers are said to be of the same *color* or *parallel* in Q_d.

Let $V(Q_d) = X + Y$ be the bipartition of Q_d. Then the *halved cube* Q'_d is the graph with $V(Q'_d) = X$, where u is adjacent to v in Q'_d if u and v have a common neighbor in Q_d. A subgraph H of G is called *spanning* if $V(H) = V(G)$.

The concept of layers is defined analogously for the direct product. In the case of the direct product the layer $H^{(a,x)}$ is isomorphic to H only if a carries a loop (in G); otherwise the edge-set of $H^{(a,x)}$ is empty.

2. Graham's density lemma for hypercubes. At a first glance the hypercube looks simple, and from many points of view this is true. Nevertheless, it has a rich subgraph structure. For example, if one subdivides every edge of a given graph G on n vertices into a path of length two and adds a vertex that is adjacent to the original n vertices of G, then the resulting graph can be isometrically embedded into

Q_n; see [12]. This ambivalence between simplicity and structure definitely adds to its attractiveness.

As the number $|Q_k|$ of vertices of Q_k is 2^k and the number of edges $k2^k/2$, the *density* of Q_k; that is, the quotient $|E(Q_k)|/|Q_k|$ is $k/2$. This is rather small if one considers the fact that the complete graph on the same number of vertices as Q_k has density $(2^k - 1)/2$. More important, this sparseness is inherited by the subgraphs of the hypercube.

Before formulating the lemma, we note that the statement $|E(Q_k)| = \frac{1}{2}|Q_k| \cdot \log_2 |Q_k|$ is equivalent to the assertion that the density of Q_k is $k/2$.

We are now ready for Graham's density lemma from [5]. We include a proof (modelled after the proof given in [11]) because its main idea appears again in the proof of Lemma 2.

LEMMA 1 (density lemma). *Let G be a subgraph of a hypercube. Then*

$$|E(G)| \le \frac{|G|}{2} \log_2 |G|, \tag{2.1}$$

with equality holding if and only if G is a hypercube.

Proof. The proof is similar to that of [11, Proposition 1.24]. Let G be a subgraph of Q_k. We proceed by induction on k. The assertions of the lemma are true for $k = 1$ and 2. Suppose they are true for $k \ge 2$ and that G is a subgraph of $Q_{k+1} = Q_k \Box K_2$. If G meets only one Q_k-layer, then the assertion is true by the induction hypothesis. Thus both intersections G_1 and G_2 of G with the Q_k-layers are nonempty. Let the notation be chosen such that $x = |G_1| \ge |G_2| = y \ge 1$. Again by the induction hypothesis $|E(G_1)| \le \frac{x}{2} \log_2 x$ and $|E(G_2)| \le \frac{y}{2} \log_2 y$. Since every vertex of G_2 has at most one neighbor in G_1 the number z of edges between G_1 and G_2 is at most y. We thus have

$$|E(G)| \le \frac{x}{2} \log_2 x + z + \frac{y}{2} \log_2 y. \tag{2.2}$$

Since $z \le y$ and $\frac{1}{2}(x + y) \log_2(x + y) = \frac{1}{2}|G| \log_2 |G|$ it suffices to show that

$$\frac{x}{2} \log_2 x + y + \frac{y}{2} \log_2 y \le \frac{x + y}{2} \log_2(x + y) \tag{2.3}$$

and that equality holds in (2.1) if and only if G is a hypercube.

We show the validity of inequality (2.3) first. It is clearly true for $x = y$; in this case the equality sign holds. We now fix y and increase x. Comparing the partial derivatives with respect to x on both sides of (2.3) we arrive at the inequality

$$\frac{1}{2} \log_2 x + \frac{1}{2} \log_2 e < \frac{1}{2} \log_2(x + y) + \frac{1}{2} \log_2 e.$$

This means that the right side grows strictly faster than the left and in (2.3) equality only holds for $x = y$.

Now, suppose $|E(G)| = \frac{1}{2}|G| \log_2 |G|$. Then the equality sign must hold everywhere, $z = y$ and $x = y$. Also, $|E(G_1)|$ must be $\frac{x}{2} \log_2 x$, just as $|E(G_2)|$ must be $\frac{y}{2} \log_2 y$. By the induction hypothesis both G_1 and G_2 are hypercubes. Since $x = y$ they have the same dimension, and $z = y$ implies that G is the Cartesian product of a hypercube of dimension $\log_2 x$ by a K_2, with the layers G_1 and G_2.

This completes the proof, because equality clearly holds in (2.1) if G is a hypercube. $\square$

This result has been generalized by Squier, Torrence, and Vogt [18] to Cartesian products of complete graphs. They prove that subgraphs G of the k-fold Cartesian product of K_p have at most $\frac{1}{2}(p-1)|G|\log_p |G|$ edges, with equality holding if and only if G is a Cartesian power of K_p.

3. Factorizations of hypercubes. We continue with the investigation of the structure of the graphs G with $Q_k = G \times K_2$. It is easy to see that every hypercube Q_k of dimension $k > 0$ can be represented as a nontrivial direct product $G \times K_2$, where G is obtained from Q_{k-1} by addition of a loop to every vertex. This is a special case of the following lemma.

LEMMA 2. *Let $Q_k = G \times K_2$. Then G has a spanning hypercube.*

Proof. Let $V(K_2) = \{b, w\}$. For convenience we color b black, w white, and assign the same colors to the vertices of Q_k that are mapped into b and w, respectively. Moreover, for $x = (g, b)$ and $y = (g, w)$ we set $x' = y$ and $y' = x$.

We proceed by induction on the size of G. It suffices to show that Q_k has a factorization $Q_{k-1}\Box K_2$ such that both Q_{k-1}-layers are mapped injectively into G. Clearly the theorem is true for Q_1. In this case G is the graph on one vertex with a loop and Q_{k-1} in the decomposition $Q_{k-1}\Box K_2$ is $Q_0 = K_1$.

Suppose it is true for all Q_i with $1 \leq i < k$. Let $Q_k = G \times K_2$ be a given factorization. We consider all decompositions $Q\Box K_2$ of the given Q_k, where Q is a $(k-1)$-dimensional hypercube. Without loss of generality we can assume that Q is a subgraph of Q_k. In the rest of the proof let p_G denote the projection map onto G. If p_G projects Q injectively into G there is nothing to show. Furthermore, since the color classes in a regular bipartite graph have the same size, the numbers of black and white vertices of Q are equal.

Suppose there is a Q whose projection $p_G Q$ meets exactly half the vertices of G. By induction Q has a $(k-2)$-dimensional subcube H that is mapped injectively into $p_G Q$. Let H_b be the set of the black vertices of H and H_w be its set of white ones. Note that $H_b' \cup H_w'$ also spans a subcube of Q with dimension $k-2$. We denote it by H'; it is the other H-layer in the decomposition $H\Box K_2$.

Let $\overline{Q}$ be the second Q-layer of Q_k and F be the set of edges between Q and $\overline{Q}$; we color them blue. The blue edges induce matchings between H and $\overline{H}$ and between H' and $\overline{H'}$. With every edge uv from a vertex $u \in H$ to a vertex $v \in \overline{H}$ the pair $u'v'$ is an edge from H' to $\overline{H'}$. Hence $p_G\overline{H} = p_G\overline{H'}$. Since the union of these projections is $p_G\overline{Q}$ all three projections are equal. Thus $p_G(H \cup \overline{H}) = V(G)$ and $H \cup \overline{H}$ induces a hypercube of dimension $k-1$.

In the remaining case there is a nonempty part A of Q with $p_G A_b = p_G A_w$ and a nonempty part B that maps injectively into G. In other words, the sets $p_G B_b$ and $p_G B_w$ are disjoint and at least one of them is nonempty. Since Q has the same number of vertices as G this means that there is a further nonempty part C of Q_k with $p_G C_b = p_G C_w$. Of course this is only possible if $k \geq 3$, which we will assume henceforth. A simple calculation shows that $|A| = |C|$. The corresponding situation of this last case is schematically shown in Figure 1.

We wish to show now that A and B are hypercubes of dimension $k-2$. We introduce the notation $x = |A|$, $y = |B|$ and show first that the number of edges between A and B is at most $\min(x, y)$. By the definition of the direct product the number of edges between A and B is the same as the number of edges between A' (which is A) and B'.

For an estimate we consider $\overline{Q}$, the second layer of Q. It is spanned by the union of B' and C. This means that the number of edges between A and B'—they are part

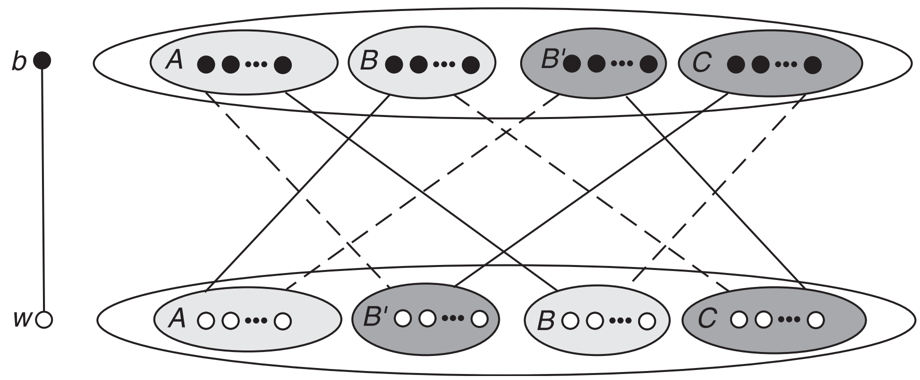

FIG. 1. *Situation from the proof;* $Q = [A \cup B]$ *is brighter and* $\overline{Q} = [B' \cup C]$ *darker.*

of the matching between Q and $\overline{Q}$—is at most $\min(x, y)$.

By the density lemma $\frac{1}{2}x \log_2 x + \min(x, y) + \frac{1}{2}y \log_2 y$ is an upper bound for the number of edges in Q, but the latter equals $\frac{1}{2}(x+y)\log_2(x+y)$ since Q is a hypercube. We thus arrive at the inequality

$$\frac{x}{2}\log_2 x + \min(x, y) + \frac{y}{2}\log_2 y \geq \frac{x+y}{2}\log_2(x+y).$$

As the proof of the density lemma shows this is only possible if both sides are equal, $x = y$, and both A and B are hypercubes.

Thus, A and B have the same size x and are hypercubes of dimension $k-2$. Moreover, there are exactly x edges between them and they form a matching. We color them red. By the matching between Q and $\overline{Q}$ they correspond to edges in $\overline{Q}$ that we also color red; cf. Figure 1, which schematically shows the matchings of red edges by unbroken lines. The edges of the matching between Q and $\overline{Q}$ we color blue. These edges have the same projections into G as the red ones and are indicated in the picture by broken lines.

By the induction hypothesis there is a color in A, call it green, whose removal decomposes A into two hypercubes that are projected injectively into G by p_G. Let us remove all edges from Q_k that are parallel to the green edges in A. The resulting graph consists of two hypercubes of dimension $k-1$. Let H^* be one of these components. We wish to show that H^* projects injectively into G. To see this, we consider $A^* = A \cap H^*$ and extend it to $B^* = B \cap H^*$ by the matching induced by the red edges and to $B'^* = B' \cap H^*$ by the matching induced by the blue ones. The matching to C^* can then be effected either from B^* by blue edges or B'^* by red ones; cf. Figure 2.

Note that $A_b \setminus A^*$ and A^*_w have the same projections into G. Since the red and blue edges also have the same projections into G one sees that $B'_b \setminus B'^*$ and B^*_b have the same projections too, from which we infer that B'^*_b and B^*_b have different ones. Continuing this way it is easily seen that H^* projects injectively into G. $\square$

An *involution* of a graph is an automorphism of order two. For a bipartite graph G with bipartition $V(G) = X + Y$ we call an involution α *bipartite* if $\alpha(X) = X$. For a bipartite involution α we let G^α denote the graph obtained from G by addition of the perfect matching $\{uv \mid u = \alpha(v), v \in V(G)\}$.

THEOREM 3. *The hypercube* Q_k *is representable as a product of the form* $G \times K_2$ *if and only if* G *is isomorphic to* Q^α_{k-1} *for some bipartite involution* α *of* Q_{k-1}.

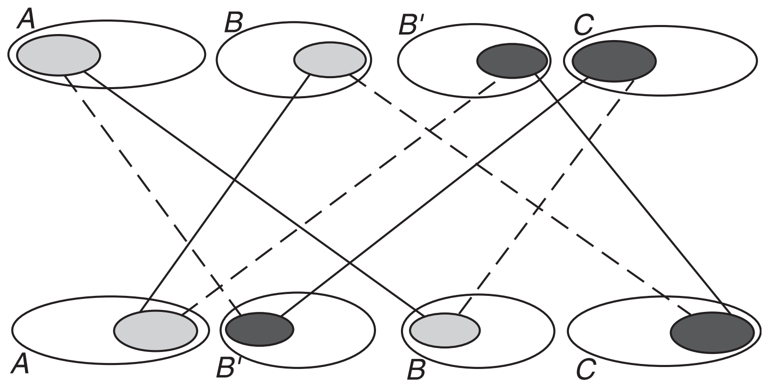

FIG. 2. *Situation from the proof; H^* is indicated brighter.*

Proof. Recall that the vertices of Q_k can also be represented as strings from $\{0,1\}^k$ and that all vertices with an even number of 1's form one of the bipartition sets of Q_k. Clearly, any two such vertices have even distance.

Suppose that $G \times K_2$ is a k-cube. By Lemma 2, G contains Q_{k-1} as a spanning subgraph; we denote it by S. Then $S \times K_2$ is a subgraph of $G \times K_2$ that consists of two disjoint hypercubes Q_{k-1}, say S_1 and S_2. As $G \times K_2$ is isomorphic to Q_k, each vertex x of S_1 is incident with an edge from $(G \times K_2) \setminus (S \times K_2)$ that connects x with a vertex y of S_2. Hence the distance in S between $p_G(x)$ and $p_G(y)$ must be even. Moreover, the edges between S_1 and S_2 induce an isomorphism between S_1 and S_2, so their projections to G induce an automorphism α of Q_{k-1} which maps each vertex v to a vertex $\alpha(v)$ with even distance from v. Also, the projections of the edges from $(G \times K_2) \setminus (S \times K_2)$ form a perfect matching of G. We conclude that α is a bipartite involution of G.

For the converse it suffices to show that every $Q_{k-1}^{\alpha} \times K_2$ is isomorphic to Q_k. ∎

If we are interested only in simple graphs G that factor Q_k with respect to the direct product, it suffices to restrict attention to fixed point free involutions α. We state this as a corollary.

COROLLARY 4. *The hypercube Q_k is representable as a direct product $G \times K_2$ of a simple graph G by K_2 if and only if G is isomorphic to Q_{k-1}^{α} for some fixed point free bipartite involution α of Q_{k-1}.*

4. The direct product representations of Q_k. To find all representations of Q_k as a direct product we first note that no two vertices of Q_k have the same set of neighbors. Such graphs are called *thin*; their prime factorizations with respect to the direct product are similar to the prime factorizations of graphs with respect to the Cartesian product. For any thin graph G one can show the existence of a *Cartesian skeleton* H. It is defined on the vertex set of G, is invariant under automorphisms of G, and, most important, to any decomposition $G_1 \times G_2$ of G corresponds a decomposition $H_1 \Box H_2$ of H such that the vertex-sets of the G_i-layers of G are the vertex-sets of the H_i-layers of H. In particular, this means that G is prime with respect to the direct product if its Cartesian skeleton H is prime with respect to the Cartesian product.

The Cartesian skeleton was introduced in [10] (see also [11]) to investigate the decomposition properties of graphs with respect to the direct product. It led to a polynomial algorithm for the prime factorization of nonbipartite connected graphs with respect to the direct product and to a new proof of the uniqueness of this decomposition for such graphs. It generalizes ideas of Feigenbaum and Schäffer [4], who presented a polynomial algorithm for the prime factorization of connected graphs with respect to the strong product and a new proof of its uniqueness.

For Q_k we cannot apply the result in full strength, because the Cartesian skeleton of bipartite graphs is disconnected, whereas it is connected in the nonbipartite case. However, we can use the results of [11, Lemmas 5.34 and 5.35], which hold for Cartesian skeletons in general.

In particular, we can apply the fact from [11, Lemma 5.35] that two vertices x and y are an edge of the Cartesian skeleton if the intersection $N(x,y) = N(x) \cap N(y)$ of the neighborhoods $N(x)$ of x and $N(y)$ of y is strictly maximal in the set $\mathcal{N}(x) = \{N(x,y) \mid N(x,y) \neq \emptyset\}$.

Proposition 5. *The Cartesian skeleton of Q_k consists of two (isomorphic) halved cubes H_1 and H_2.*

Proof. Any two vertices x and y with intersecting neighborhoods $N(x)$ and $N(y)$ have distance two; the intersection $N(x,y) = N(x) \cap N(y)$ has exactly two elements (for $k > 1$) and $N(x,y) = N(x,z)$ if and only if $x = y$. This implies that every $N(x,y)$ is strictly maximal in the set $\mathcal{N}(x) = \{N(x,y) \mid N(x,y) \neq \emptyset\}$. Therefore xy is an edge of the Cartesian skeleton H of Q_k if and only if $d(x,y) = 2$. Thus the Cartesian skeleton H of Q_k consists of the two halved cubes H_1 and H_2. □

Clearly H is disconnected because Q_k is bipartite. Nevertheless, every factorization of Q_k with respect to the direct product induces a factorization of H with respect to the Cartesian product. This means that Q_k cannot be a product of more factors with respect to the direct product than H with respect to the Cartesian one. We therefore decompose H first.

Either any two edges ab and ac of a halved cube are in a triangle abc or there are two triangles abd and adc (with the common edge bd). This implies that every halved cube is prime with respect to the Cartesian product. Thus, the only possible factorization of H with respect to the Cartesian product is $H_1 \square D_2$, where D_2 is the graph on two vertices without edges or loops.

For Q_k this implies that it can only be decomposed into a product $G \times K$ of two factors, where K is a graph on two vertices: where $V(H_1)$ projects onto one vertex of K and $V(H_2)$ onto the other. Since no pair of vertices in either H_1 or H_2 is adjacent in G, we infer that K cannot have loops.

Moreover, both G and K must be connected because Q_k is. We thus show the following proposition.

Proposition 6. *Every factorization of Q_k with respect to the direct product is of the form $G \times K_2$. All such graphs G are prime with respect to the direct product.*

Together with Theorem 3 we can summarize our findings in the following theorem.

Theorem 7. *Every decomposition of the hypercube Q_k into a direct product has exactly two factors. One factor is always K_2 and the other one any of the graphs Q_{k-1}^{α} for a bipartite involution α of Q_{k-1}.*

It would be interesting to enumerate the bipartite involutions of Q_k as well as the factorizations of Q_k with respect to the direct product. These questions are open.

We wish to conclude with the remark that nonunique factorizations can easily be found, also for factors different from K_2. For example, the direct product of a path P_n with a triangle is isomorphic to the product of P_n by a path of length two with loops added to the endpoints; cf. Figure 3 where an isomorphism is indicated for the case $n = 5$.

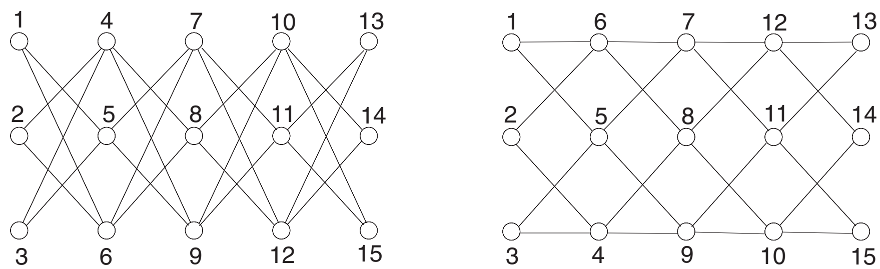

FIG. 3. *Isomorphic direct products.*

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