Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **6** (2012), 82–94. doi:10.2298/AADM120108002I

GENERALIZED LUCAS CUBES

Aleksandar Ilić, Sandi Klavžar, Yoomi Rho

Let f be a binary string and $d \geq 1$. Then the generalized Lucas cube $Q_d(\overleftarrow{f}\,)$ is introduced as the graph obtained from the d-cube Q_d by removing all vertices that have a circulation containing f as a substring. The question for which f and d, the generalized Lucas cube $Q_d(\overleftarrow{f}\,)$ is an isometric subgraph of the d-cube Q_d is solved for all binary strings of length at most five. Several isometrically embeddable and non-embeddable infinite series where f is of arbitrary length are given. Some structural properties of generalized Lucas cubes are also presented.

1. INTRODUCTION

Hypercubes form one of the most applicable classes of graphs and offer challenging mathematical and computational problems. (Recall that the *d-cube* Q_d is the graph whose vertices are all binary strings of length d, two vertices are adjacent if they differ in exactly one position.) In this paper we continue the research initiated in [**7**] by studying graphs that are obtained from hypercubes by removing vertices that contain forbidden substrings.

The Fibonacci cube Γ_d is obtained from Q_d by removing the vertices that contain the substring 11. Fibonacci cubes found a couple of applications [**6, 10**] and have been extensively studied, see for instance [**2, 4, 11, 12, 14**]. For an up to date review on Fibonacci cubes see [**9**]. If f is an arbitrary binary string, we can more generally consider the subgraph $Q_d(f)$ of Q_d induced on vertices that do not contain f as a substring. These graphs were introduced in [**7**] and named generalized Fibonacci cubes.

Lucas cubes [**1, 2, 3, 5, 8, 11, 13**] form a class of graphs closely related to Fibonacci cubes. In fact, they are defined just as Fibonacci cubes with the

2010 Mathematics Subject Classification. 05C12, 05C75.
Keywords and Phrases. Hypercube, Lucas cube, generalized Lucas cube, isometric embedding.

additional request that 1 cannot occur simultaneously in the first and the last position. Hence the Lucas cubes are graphs in which the substring 11 is forbidden in a circular manner.

In this paper we introduce generalized Lucas cubes in the way analogous (in view of the above remark) to the generalized Fibonacci cubes. Although these families of graphs are similar, they have several different properties. The definition of the generalized Lucas cubes is more symmetric due to the circular consideration of the strings. However, it turns out rather surprisingly that generalized Fibonacci cubes embed isometrically into the corresponding cubes much more frequently than the generalized Lucas cubes. Moreover, among the generalized Lucas cubes non-connected graphs occur (very rarely though) and isometric embeddability of them into Q_d may depend on the congruence of d by some modulo.

The paper is organized as follows. In the rest of this section we give necessary definitions. In Section 2 we present our main results – classification of generalized Lucas cubes that embed isometrically into the corresponding cubes for all forbidden strings of length at most five. Then, in Section 3, some structural properties of generalized cubes are given. The final section contains proofs of the results from Section 2.

The *Lucas cube* Λ_d, $d \geq 1$, is the subgraph of Q_d induced by the binary strings $b_1 b_2 \ldots b_d$ such that $b_i b_{i+1} \ldots b_d b_1 \ldots b_{i-1}$ contains no two consecutive 1's for all $1 \leq i \leq d$. For each $1 \leq i \leq d$, call $b_i b_{i+1} \ldots b_d b_1 \ldots b_{i-1}$ the *i-th circulation* of $b_1 b_2 \ldots b_d$. For $d \geq 1$ and a binary string f, the *generalized Lucas cubes* $Q_d(\overleftarrow{f})$ is the graph obtained from Q_d by removing all vertices $b_1 b_2 \ldots b_d$ which have a circulation containing f as a substring. Hence, $\Lambda_d = Q_d(\overleftarrow{11})$.

The *distance* $d_G(u, v)$ between vertices u and v of a graph G is the usual shortest path distance. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. We will write $H \hookrightarrow G$ to denote that H is an isometric subgraph of G and $H \not\hookrightarrow G$ that this is not the case. For instance, $\Lambda_d \hookrightarrow Q_d$, see [**8**]. The *interval* $I_G(u, v)$ between vertices u and v of a graph G is the set of vertices of G that lie on some shortest u, v-path.

For a binary string b, let $\overline{b}$ be the binary complement of b and let $b^R = b_d b_{d-1} \ldots b_1$ be the reverse of b. For binary strings b and c of equal length, let $b + c$ denote their sum computed bitwise modulo 2. For $d \geq 1$ and $1 \leq i \leq d$, let e_i be the binary string of length d with 1 in the i-th position and 0 elsewhere. A non-extendable sequence of contiguous equal digits in a string b is a *block* of b. Let $b = uvw$ be a binary string obtained by a concatenation of u, v, and w, where u and w are allowed to be the empty string. Then we say that v is a *factor* of b.

2. (NON-)EMBEDDABILITY OF SHORT AND REGULAR STRINGS

In this section we give a solution to the question:

$$Q_d(\overleftarrow{f}) \hookrightarrow Q_d?$$

for all strings f of length at most 5. Along the way several additional (non-) embeddability results are obtained for strings of arbitrary length.

We begin with a collection of observations and leave their straightforward proofs to the reader.

Lemma 1.

(i) *Let f be a binary string of length r and $1 \le d < r$. Then $Q_d(\overleftarrow{f}) = Q_d$ and hence $Q_d(\overleftarrow{f}) \hookrightarrow Q_d$.*

(ii) *Let f be a binary string and $d \ge 1$. Then $Q_d(\overleftarrow{f})$ is isomorphic to $Q_d(\overleftarrow{\overline{f}})$.*

(iii) *Let f be a nonempty binary string and $d \ge 1$. Then $Q_d(\overleftarrow{f})$ is isomorphic to $Q_d(\overleftarrow{f^R})$.*

(iv) *Let $f = f_1 f_2 \ldots f_r$ and $d \ge r$. Then at least one of the circulations of a string b contains f as a factor if and only if b^2 does so. In other terms, $b \in Q_d(\overleftarrow{f})$ if and only if $b^2 \in Q_{2d}(f)$.*

Because of Lemma 1(i), let us call a dimension *trivial* provided that it is smaller than the length of the forbidden string considered, that is, $d < r$.

Forbidden strings with one block yield isometric embeddings:

Proposition 2. *Let $s \ge 1$. Then for any $d \ge 1$, $Q_d(\overleftarrow{1^s}) \hookrightarrow Q_d$.*

The situation with two blocks is more involved:

Theorem 3. *For any nontrivial dimension d, the following statements hold.*

(i) *Let $r \ge 1$. Then $Q_d(\overleftarrow{1^r 0}) \not\hookrightarrow Q_d$.*

(ii) *$Q_d(\overleftarrow{1^2 0^2}) \hookrightarrow Q_d$ if and only if $d = 5$.*

(iii) *Let $r \ge 3$. Then $Q_d(\overleftarrow{1^r 0^2}) \hookrightarrow Q_d$ if and only if $d = r + 3$, or $d = r + 4$.*

(iv) *Let $r, s \ge 3$. Then $Q_d(\overleftarrow{1^r 0^s}) \hookrightarrow Q_d$ if and only if $r + s + 1 \le d \le 2r + 2s - 3$.*

When f has three blocks, we have nonisometry for all nontrivial dimensions:

Proposition 4. *Let $r, s, t \ge 1$. Then $Q_d(\overleftarrow{1^r 0^s 1^t}) \not\hookrightarrow Q_d$ for any nontrivial dimension.*

To cover the cases where f is of length 4 and 5, we also need to consider f's with more than three blocks. We have the following related results:

Proposition 5. *Let $s \ge 1$. Then $Q_d(\overleftarrow{(10)^s 1}) \not\hookrightarrow Q_d$ for $d = 2s+1$ and for any $d \ge 4s$.*

Proposition 6. *Let $r, s \ge 1$. Then $Q_d(\overleftarrow{(10)^r 1 (10)^s}) \not\hookrightarrow Q_d$ for $d = 2r + 2s + 1$ and for any $d \ge 2r + 2s + 3$.*

Proposition 7. *Let $r_1, \ldots, r_n, s_1, \ldots, s_n \ge 1$ and let d be a nontrivial dimension. Then $Q_d(\overleftarrow{1^{r_1} 0^{s_1} \ldots 1^{r_n} 0^{s_n}}) \not\hookrightarrow Q_d$ for $d \equiv 0 \pmod{r_1 + \cdots + r_n + s_1 + \cdots + s_n - 1}$.*

The next result shows a striking difference between the isometry of generalized Lucas cubes and generalized Fibonacci cubes.

Theorem 8. *Let $s \geq 2$. Then for any nontrivial dimension d, $Q_d(\overleftarrow{(10)^s}) \hookrightarrow Q_d$ if and only if $d \not\equiv 0 \pmod{2s-1}$.*

In order to complete our classification, a single forbidden factor remains that is not covered by the above results: 11010. To take care of it we prove:

Proposition 9. *For every $d \geq 5$, it holds $Q_d(\overleftarrow{11010}) \not\hookrightarrow Q_d$.*

Now everything is ready for the main results of this paper which are presented in Table 1. In view of Lemma 1(ii) and (iii) we present only nonisomorphic generalized Lucas cubes, and in view of Lemma 1(i) we state results for nontrivial dimensions only. Only strings written in bold yield isometric embeddings, while strings labeled with $*$ indicate disconnected graphs (cf. Proposition 10). We note that in all the cases when $Q_d(f)$ is isometric in Q_d, our proofs imply that $Q_d(f)$ is connected.

length	forbidden factor
1	**1** (Proposition 2)
2	**11** (Proposition 2) 10* (Theorem 3(i))
3	**111** (Proposition 2) 110* (Theorem 3(i)) 101* (Proposition 4)
4	**1111** (Proposition 2) 1110* (Theorem 3(i)) **1100** ($d = 5$, Theorem 3(ii)) 1100 ($d = 4$ or $d \geq 6$, Theorem 3(ii)) **1010** ($d \leq 5$ or $d \not\equiv 0 \pmod 3$, Theorem 8) 1010 ($d \neq 3$ and $d \equiv 0 \pmod 3$, Proposition 7) 1101*, 1001 (Proposition 4)
5	**11111** (Proposition 2) 11110* (Theorem 3(i)) **11100** ($d = 6, 7$, Theorem 3(iii)) 11100 ($d = 5$ or $d \geq 8$, Theorem 3(iii)) 11001, 11101*, 11011*, 10001 (Proposition 4) **10110** (computer check for $d = 6$) 10110 ($d = 5$ or $d \geq 7$, Proposition 6) **10101** ($d = 6, 7$, by computer check) 10101 ($d = 5$ or $d \geq 8$, Proposition 5) 11010 (Proposition 9)

Table 1. Classification of embeddability of generalized Lucas cubes with forbidden factors of length at most 5

3. ON THE STRUCTURE OF GENERALIZED LUCAS CUBES

Note that $Q_d(\overline{0})$ contains only the vertex 1^d. We have already mentioned that generalized Lucas cubes need not be connected, see $Q_d(\overline{110})$ on Figure 1.

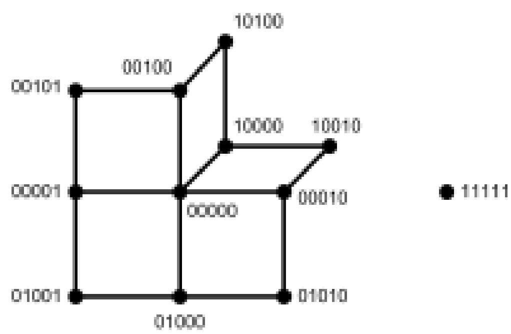

Figure 1. Generalized Lucas cube $Q_5(\overline{110})$.

More generally:

Proposition 10. *Generalized Lucas cubes* $Q_d(\overline{1^r01^s})$, $r+s \geq 1$, $d \geq r+s+1$, *are composed of an isolated vertex* 1^d *and a connected component.*

Proof. The vertex 1^d is isolated, since all its neighbors have circulations containing 1^r01^s. We will prove that for any other vertex there is a path to the vertex 0^d, which proves that the generalized Lucas cube $Q_d(\overline{1^r01^s})$ has exactly two connected components. Let b be an arbitrary vertex of $Q_d(\overline{1^r01^s})$, different from 1^d. Since b contains at least one bit 0, let i be the position of the first bit which is 0. Assume that the bits on positions $i, i+1, \ldots, j$ are all equal to 0, while the bit on the position $j+1$ is 1. We can change $(j+1)$-th or $(i-1)$-th bit from 1 to 0 by considering the positions by modulo d, and it can be easily shown that a vertex obtained in this way belongs to $Q_d(\overline{1^r01^s})$. In this way, we can increase the size of this block of zeros and finally reach the vertex 0^d. □

Note that $Q_d(\overline{10})$ contains only vertices 1^d and 0^d, so that the connected component of $Q_d(\overline{10})$ from Proposition 10 consists of only the vertex 0^d.

By Proposition 10, $H_d = Q_d(\overline{110})$ is disconnected with one isolated vertex 1^d. If some vertex of H_d contains two consecutive bits 1, then it can be easily proved that this vertex must be equal to 1^d. Therefore, for the vertices of the other connected component factor 11 is forbidden in a circular manner. Hence $V(H_d) = V(\Lambda_d) \cup \{1^d\}$. Using the results from [**8**], the following formulas hold:

(1) $$|V(H_d)| = |V(\Lambda_d)| + 1 = L_d + 1,$$

(2) $$|E(H_d)| = |E(\Lambda_d)| = dF_{n-1},$$

(3) $$|S(H_d)| = |S(\Lambda_d)|,$$

where L_d are the Lucas numbers defined with $L_0 = 2$, $L_1 = 1$, $L_d = L_{d-1} + L_{d-2}$ for $d \geq 2$; F_d are the Fibonacci numbers defined with $F_0 = 0$, $F_1 = 1$, $F_d = F_{d-1} + F_{d-2}$ for $d \geq 2$; and $S(H_d)$ is the set of 4-cycles of H_d.

Among the strings of length 3, 111 is the only one that gives isometric embeddings. For the Lucas cubes $Q_d(\overleftarrow{111})$ recurrent formulas similar to those for the generalized Fibonacci cubes $Q_d(111)$ [**7**] hold:

Proposition 11. *For $d \geq 0$, let $G_d = Q_d(\overleftarrow{111})$. Then*

(i) $|V(G_d)| = |V(G_{d-1})| + |V(G_{d-2})| + |V(G_{d-3})|, d \geq 6$,
and $|V(G_i)| = 1, 2, 4, 7, 11, 21$, *for* $i = 0, 1, \dots, 5$, *respectively;*

(ii) $|E(G_d)| = |E(G_{d-1})| + |E(G_{d-2})| + |E(G_{d-3})| + |V(G_{d-2})| + 2|V(G_{d-3})|, d \geq 6$,
and $|E(G_i)| = 0, 1, 4, 9, 16, 40$, *for* $i = 0, 1, \dots, 5$, *respectively;*

(iii) $|S(G_d)| = |S(G_{d-1})| + |S(G_{d-2})| + |S(G_{d-3})| + |E(G_{d-2})| + 2|E(G_{d-3})| + |V(G_{d-3})|$,
for $d \geq 6$, *and* $|S(G_i)| = 0, 0, 1, 3, 6, 25$, *for* $i = 0, 1, \dots, 5$, *respectively.*

Proof. The initial conditions can be easily checked. Consider the generalized Fibonacci cube $G'_d = Q_d(111)$. To get the generalized Lucas cube G_d from G'_d, we need to exclude the following sets of vertices $A = \{110v01 \mid v \in Q_{d-5}(111)\}$, $B = \{10v011 \mid v \in Q_{d-5}(111)\}$ and $C = \{110v011 \mid v \in Q_{d-6}(111)\}$, since the vertices of these sets have circulations containing 111. It follows that

$$|V(G_d)| = |V(G'_d)| - 2|V(G'_{d-5})| - |V(G'_{d-6})|\,.$$

Since $|V(G'_d)| = |V(G'_{d-1})| + |V(G'_{d-2})| + |V(G'_{d-3})|$ holds for $d \geq 3$ (see [**7**]), we get (i). For the number of edges, we need to exclude the edges from G'_d where at least one vertex belongs to $S = A \cup B \cup C$.

We claim that there are exactly $6|V(G'_{d-5})| + 2|V(G'_{d-6})|$ edges that connect one vertex from S and one vertex from $V(G_d)$. Consider the vertex $110\dots01$ from A. It has neighbors $010\dots01$, $100\dots01$, and $110\dots00$ (and $110\dots011$, but this vertex is from C, and we will count these edges in the second part). Therefore, the number of edges with one vertex in A or B and the other one in the Lucas cube is equal to $6|V(G'_{d-5})|$. Next consider the vertex $110\dots011$ from C. It has neighbors $010\dots011$ and $110\dots010$ (the neighbors $100\dots011$ and $110\dots001$ are from the set B and A, respectively). Therefore, the number of edges with one vertex in C and the other one in the Lucas cube is equal to $2|V(G'_{d-6})|$ and the claim is proved.

Moreover, there are $2|E(G'_{d-5})| + |E(G'_{d-6})| + 2|V(G'_{d-6})|$ edges that connect two vertices from S (the factor $2|V(G'_{d-6})|$ corresponds to the edges connecting the set C with sets A and B). Finally, this gives

$$|E(G_d)| = |E(G'_d)| - 2|E(G'_{d-5})| - |E(G'_{d-6})| - 6|V(G'_{d-5})| - 4|V(G'_{d-6})|.$$

Using a recurrent formula for $|E(G'_d)|$ we get (ii). Similarly we derive the formula for the number of squares.

4. PROOFS

Proof of Proposition 2. For $s = 1$ we have $Q_d(\overleftarrow{1}) = K_1$ and there is nothing to be proved. Let $s \geq 2$ and consider arbitrary vertices b and c of $Q_d(\overleftarrow{1^s})$ with $d_{Q_d}(b,c) = r \geq 1$. We need to show that $d_{Q_d(\overleftarrow{1^s})}(b,c) = r$ as well.

Let $b_j = 1$ and $c_j = 0$ for $j = i_1, \ldots, i_p$ and $b_j = 0$ and $c_j = 1$ for $j = i_{p+1}, \ldots, i_r$. Recalling that $+$ stands for the sum computed bitwise modulo 2 we then infer that

$$P : b \to (b + e_{i_1}) \to (b + e_{i_1} + e_{i_2}) \to \ldots \to (b + e_{i_1} + e_{i_2} + \ldots + e_{i_r})$$

is a b,c-path in Q_d of length r. The path P starts at b; then visits the vertices obtained from b by changing one by one the 1 bits of b such that the bits of c in the same positions are 0; then P traverses the vertices obtained from b by changing one by one the 0 bits of b whose corresponding bits of c are 1; in this way P finally reaches c. Now, if a circulation of some vertex of P would contain 1^s as a factor, 1^s would also be a factor of a circulation of c. Since this is not the case we conclude that P lies entirely in $Q_d(\overleftarrow{1^s})$. $\square$

Note that the proof of Proposition 2 implies that $Q_d(\overleftarrow{1^s})$ is connected.

Proof of Theorem 3. (i) $Q_d(\overleftarrow{1^r0})$ is not connected by Proposition 10.

(ii) We checked by computer that $Q_5(\overleftarrow{1^20^2}) \hookrightarrow Q_5$ holds. Dimension 6 will be covered by (the proof of) Proposition 7. Assume $d = 7$. Select vertices $b = 1^21010^2$ and $c = 1^20100^2$. Note that $b, c \in Q_d(\overleftarrow{1^20^2})$ and that they differ in three bits, that is, $d_{Q_d}(b,c) = 3$. The only neighbors of b in $I_{Q_d}(b,c)$ are 1^20010^2, 1^21110^2, and 1^21000^2, but none of them belongs to $Q_d(\overleftarrow{1^20^2})$. This implies that $d_{Q_d(\overleftarrow{1^20^2})}(b,c) > 3$. Attaching an appropriate number of 1's to the front of b and c, the above construction applies to $d > 7$ which proves (ii).

We next show that the smallest nontrivial dimension in (iii) and in (iv) does not give isometry. So to show that $Q_{r+s}(\overleftarrow{1^r0^s}) \not\hookrightarrow Q_{r+s}$, select vertices $b = 11^{r-1}10^{s-1}$ and $c = 01^{r-1}00^{s-1}$. Their common neighbors in $I_{Q_{r+s}}(b,c)$ are $01^{r-1}10^{s-1}$ and $11^{r-1}00^{s-1}$ but none of them belongs to $Q_{r+s}(\overleftarrow{1^r0^s})$.

(iii) The case $d = r + 2$ has already been treated above, hence assume that $d \geq r + 3$. For $d = r + 5$, consider the vertices $b = 1^r10100$ and $c = 1^r01100$ from $Q_d(\overleftarrow{1^r00})$, that are at distance 2. For higher dimensions, attach an appropriate number of 1's to the front of b and c.

Assume that $r + 3 \leq d \leq r + 4$. Let b and c be vertices of $Q_d(\overleftarrow{1^r00})$ with $p = d_{Q_d}(b,c)$. We proceed by induction on p and need to prove that $d_{Q_d(\overleftarrow{1^r00})}(b,c) = p$ as well. If $p = 1$, that is, $d_{Q_d}(b,c) = 1$, then by definition, b is adjacent to c in $Q_d(\overleftarrow{1^r00})$. Let $p \geq 2$ and let b and c differ on the position i. We may without loss of generality assume that $b_i = 1$ and $c_i = 0$. Let $b' = b + e_i$. If $b' \in Q_d(\overleftarrow{1^r00})$ then we are done by the induction assumption because $d_{Q_d(\overleftarrow{1^r00})}(b',c) = p-1$. Hence assume that $b' \notin Q_d(\overleftarrow{1^r00})$. Then b has 1^r as a prefix. Furthermore, we can assume that b starts with 1^r10 while c starts with 1^r01, i.e. $i = r+1$. Notice that the $(r+3)$-th bit in b must be 1, because otherwise b has a circulation that contains 1^r00 as a factor. Since $d \le r+4$, the vertex $b'' = b + e_{i+1}$ has at most one 0, and thus belongs to $Q_d(\overleftarrow{1^r00})$. Since $d_{Q_d}(b'',c) = p-1$, by induction assumption $d_{Q_d(\overleftarrow{1^r00})}(b'',c) = p-1$ also holds, hence we conclude that $d_{Q_d(\overleftarrow{1^r00})}(b,c) = p$.

(iv) The case $d = r+s$ has already been treated above. Let $d = 2r+2s-2$. Then vertices $b = 1^r0^{s-2}101^{r-2}0^s$ and $c = 1^r0^{s-2}011^{r-2}0^s$ do the job, as well as the vertices with 1's attached in front do for higher dimensions.

Assume that $r+s+1 \le d \le 2r+2s-3$. Let b and c be vertices of $Q_d(\overleftarrow{1^r0^s})$ with $p = d_{Q_d}(b,c)$. We proceed by induction on p, the base case $p=1$ is clearly true. Let $p \ge 2$ and let b and c differ on position i. We may without loss of generality assume that $b_i = 1$ and $c_i = 0$. Let $b' = b + e_i$. If $b' \notin Q_d(\overleftarrow{1^r0^s})$, then b has 1^r as a prefix. Furthermore, we can assume that b starts with $1^r0^x10^{s-1-x}$ while c starts with 1^r0^x0, i.e. $i = r+x+1$. Notice that c cannot have 1^r0^s as prefix, and therefore the vertex c must start with $1^r0^x00^y1$, where $x+1+y < s$. Let $b'' = b + e_{r+x+y+2}$.

If $x = 0$, the vertex b has to start with $1^r10^{s-1}1$, while we have that b'' starts with $1^r10^y10^{s-2-y}1$, and this prefix clearly does not have a block of size s with consecutive zeros. In order to b'' have a circulation that contains 1^r0^s as a factor, we need to have $s-2-y = 0$, and $1^{r-2}0^s$ appears after position $r+s+1$. But then $d \ge (r+s+1)+(r+s-2) = 2r+2s-1$, which is impossible. Therefore, in this case b'' belongs to $Q_d(\overleftarrow{1^r0^s})$. If $s > x+y+2$, the vertex b'' starts with $1^r0^x10^y10^{s-x-y-2}$, and similarly $b'' \in Q_d(\overleftarrow{1^r0^s})$.

Finally, let $s = x+y+2$ and $x > 0$. The vertices b and c start with $1^r0^x10^{s-2-x}0$ and $1^r0^x00^{s-2-x}1$, respectively. In order to b'' have a circulation that contains 1^r0^s as a factor, we need to have $s-2-x = 0$, and $1^{r-2}0^s$ appears after position $r+s$. But then $d \ge (r+s)+(r+s-2) = 2r+2s-2$, which is impossible. Since $d_{Q_d}(b'',c) = p-1$, this completes the inductive proof and the proof of Theorem 3.

Proof of Proposition 4. When $d = r+s+t$, note that $Q_d(\overleftarrow{1^r0^s1^t}) = Q_d(\overleftarrow{1^{r+t}0^s})$ and hence nonisometry follows from Theorem 3.

Suppose $d = r+s+t+1$. Then vertices $b = 1^r10^{s-1}11^t$ and $c = 1^r00^{s-1}01^t$ from $Q_d(\overleftarrow{1^r0^s1^t})$ demonstrate nonisometry. For higher dimensions proceed as in similar cases above.

Proof of Proposition 5. The case $s = 1$ has already been treated in Proposition 4. Assume in the rest that $s \ge 2$. For $d = 2s+1$ select $b = (10)100(10)^{s-2}$, $c = (10)111(10)^{s-2}$ and argue as above that $b,c \in Q_d(\overleftarrow{(10)^s1})$ but their common neighbors $(10)110(10)^{s-2}$ and $(10)101(10)^{s-2}$ are not vertices of $Q_d(\overleftarrow{(10)^s1})$.

For $d = 4s$ consider $b = (10)^{s-1}100(10)^{s-1}1$ and $c = (10)^{s-1}111(10)^{s-1}1$. The common neighbors $(10)^{s-1}110(10)^{s-1}1 = (10)^{s-1}1(10)^s1$ and $(10)^{s-1}101(10)^{s-1}1 =$

$(10)^s1(10)^{s-1}1$ do not belong to $Q_d(\overleftarrow{(10)^s1})$. For $d > 4s$ attach an appropriate number of 1's to the front of b and c.

Proof of Proposition 6. Let $d = 2r+2s+1$. Then $Q_d(\overleftarrow{(10)^r1(10)^s}) = Q_d(\overleftarrow{(10)^{r+s}1})$ and the assertion follows from Proposition 5.

Let $d = 2r + 2s + 3$. Select $b = (10)^r100(10)^s$ and $c = (10)^r111(10)^s$ to reach the desired conclusion. Again, for any $d > 2r+2s+3$ attach the appropriate number of 1's to the front of b and c.

Proof of Proposition 7. Let $f = 1^{r_1}0^{s_1}\dots1^{r_n}0^{s_n}$. Select vertices

$$b = (11^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n-1})^k = (1^{r_1}0^{s_1}\dots1^{r_n}0^{s_n-1})^k$$

and

$$c = (01^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n-1})^k = 0(1^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n})^{k-1}1^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n-1}\,.$$

Note that b and c differ in k bits. The only neighbors of b in $I_{Q_d}(b,c)$ are

$$(11^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n-1})^x(01^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n-1})(11^{r_1-1}0^{s_1}\dots1^{r_n}0^{s_n-1})^{k-x-1}$$

for some $0 \le x \le k-1$. But none of them belongs to $Q_d(\overleftarrow{f})$.

We still need to show that $b, c \in Q_d(\overleftarrow{f})$. In our earlier proofs this was straightforward, but now the task is more complicated.

Suppose by way of contradiction that $b \notin Q_d(\overleftarrow{f})$. Considering the length of f and the structure of b, we infer in the spirit of Lemma 1(iv) that a circulation of $(1^{r_1}0^{s_1}\dots1^{r_n}0^{s_n-1})^2$ and hence

$$g = 1^{r_{n-l+1}}0^{s_{n-l+1}}\dots1^{r_n}0^{s_n-1}1^{r_1}0^{s_1}\dots1^{r_{n-l}}0^{s_{n-l}}1^{r_{n-l+1}}0^{s_{n-l+1}}\,,$$

where $\ell \ge 1$, contains $f = 1^{r_1}0^{s_1}\dots1^{r_n}0^{s_n}$ as a factor. Note that the indices of f and g coincide except that g could have more 1's at the front and more 0's at the end. Note also that we could define g by ending it with $1^{r_{n-l}}0^{s_{n-l}}$ if $s_n > 1$. We distinguish two cases.

Case 1. $s_n > 1$. Then there is an index ℓ, $1 \le \ell < n$, such that

$$s_i = \begin{cases} s_{i+\ell-n}; & n-\ell+1 \le i < n, \\ s_\ell + 1; & i = n, \\ s_{i+\ell}; & 1 \le i < n-\ell, \end{cases}$$

and $s_{n-\ell} \ge s_n$. Let $n = \tilde{n}\ell + u$ for some $0 \le u < \ell$.

If $u = 0$, then $s_n - 1 = s_\ell = s_{2\ell} = \cdots = s_{(\tilde{n}-1)\ell} \ge s_n$, a contradiction. Assume $u > 0$. Let g be the greatest common divisor of ℓ and u. Set $\ell = \ell_1 g$ and $u = u_1 g$. Then

$$\begin{aligned} s_n - 1 &= s_{(\tilde{n})\ell=n-u} \\ &= s_{n-u+\ell-n} \quad (\text{as } n-\ell+1 \le n-u < n) \\ &= s_{n-u+2\ell-n} = \cdots = s_{n-u+\tilde{n}\ell-n=n-2u} (= s_{n+\ell-2u} \text{ if } n+\ell-2u < n) \\ &= \cdots = s_{n-\ell_1 u=n-u_1\ell} = s_{n-(u_1-1)\ell} = \cdots = s_{n-l} \ge s_n, \end{aligned}$$

and we get a contradiction.

Case 2. $s_n = 1$. Then there is an index ℓ, $1 \le \ell < n$, such that

$$r_i = \begin{cases} r_{i+\ell-n}; & n-\ell+2 \le i \le n-1, \\ r_\ell - r_1; & i = n, \\ r_{i+\ell-1}; & 2 \le i \le n-\ell+1, \end{cases}$$

and $r_n \ge r_1$. Let $n = \tilde{n}(\ell-1)+u$ for some $0 \le u < \ell-1$.

Let $r = r_n + r_1$. If $u = 0$, then

$$\begin{aligned} r &= r_\ell = r_{\ell+(\ell-1)} = \cdots = r_{(\tilde{n}-1)(\ell-1)+1=n-\ell+2} \\ &= r_{n-\ell+2+\ell-n=2} = r_{n-\ell+3} = r_3 = \cdots = r_{n-1} = r_{\ell-1} = r_{n-\ell+1} = r_n, \end{aligned}$$

a contradiction.

Assume $u > 0$. Then $r = r_\ell = r_{\tilde{n}(\ell-1)+1=n-u+1}$. If $u = 1$, then $r = r_n$, a contradiction. Assume $u \ge 2$. Let g be the greatest common divisor of $\ell-1$ and $u-1$. Set $\ell-1 = \ell_1 g$ and $u-1 = u_1 g$. Then

$$r = r_{n-u+1+\ell-n=\ell-u+1=\ell-1-u+2} = r_{\tilde{n}(\ell-1)-u+2=n-2u+2} \; (= r_{n-2u+2+\ell-1} \text{ if } n-2u+2 \le n-\ell).$$

Note that $n-2u+2 \ne n-\ell+1$ as otherwise $r = r_n$, a contradiction. Now

$$\begin{aligned} r &= r_{n-2u+2+(\ell-1)+l-n=(\ell-1)-2u+3+(\ell-1)} = r_{n-u-2u+3=n-3u+3} \\ &= \cdots = r_{n-\ell_1(u-1)=n-(\ell-1)u_1} = r_{n-(\ell-1)(u_1-1)} = \cdots = r_{n-(\ell-1)} = r_n, \end{aligned}$$

and we get a contradiction. This proves that $b \in Q_d(\overleftarrow{f})$.

Suppose $c \notin Q_d(\overleftarrow{f})$. Let $\tilde{c} = (1^{r_1-1}0^{s_1}\dots 1^{r_n}0^{s_n})^k$. Then $\tilde{c}$ is a circulation of c and hence $\tilde{c} \notin Q_d(\overleftarrow{f})$ either. Therefore $\tilde{c}^R = (0^{s_n}1^{r_n}\dots 0^{s_1}1^{r_1-1})^k$ contains $f^R = 0^{s_n}1^{r_n}\dots 0^{s_1}1^{r_1}$ as a factor. By a similar argument which we used for b we get contradictions and conclude that $c \in Q_d(\overleftarrow{f})$.

Proof of Theorem 8. By Proposition 7, $Q_d(\overleftarrow{(10)^s}) \not\hookrightarrow Q_d$ if $d \equiv 0 \pmod{2s-1}$. It is enough to show that $Q_d(\overleftarrow{(10)^s}) \hookrightarrow Q_d$ if $d \not\equiv 0 \pmod{2s-1}$, hence assume in the rest of the proof that $d \not\equiv 0 \pmod{2s-1}$.

Let b and c be vertices of $Q_d(\overleftarrow{(10)^s})$ with $p = d_{Q_d}(b,c)$. We proceed by induction on p and note that it is clear for $p = 1$. Let $p \ge 2$ and let $i < j$ be indices of bits such that $b_i \ne c_i$, $b_j \ne c_j$, and $b_r = c_r$ for all $i < r < j$. Also let

$$k(i) = \begin{cases} \max\{r \mid 0 < r < i \text{ s.t. } b_r \ne c_r\}; & \text{if the set is not empty,} \\ \max\{r-d \mid r \ge j \text{ s.t. } b_r \ne c_r\}; & \text{otherwise}, \end{cases}$$

and

$$\ell(i) = \begin{cases} k(i); & k(i) \ge 1, \\ k(i)+d; & k(i) \le 0. \end{cases}$$

Let $b' = b + e_i$. If $j - i \geq 2s$ and $i - k(i) \geq 2s$, then b' belongs to $Q_d(\overleftarrow{(10)^s})$ for otherwise $c \notin Q_d(\overleftarrow{(10)^s})$. Since b' differs from c in $p-1$ bits, induction assumption implies that there exists a b', c-path in $Q_d(\overleftarrow{(10)^s})$ of length $p-1$ and hence there exists a b, c-path in $Q_d(\overleftarrow{(10)^s})$ of length p.

Assume $j - i \leq 2s - 1$. Without loss of generality let $b_i = 1$ and $c_i = 0$. If any of b', $c' = c + e_i$, $b + e_j$, and $c + e_j$ lies in $Q_d(\overleftarrow{(10)^s})$, we can proceed by induction. Hence assume that none of them belongs to $Q_d(\overleftarrow{(10)^s})$. Then b'_i is preceded by 1 and c'_i is followed by 0. We distinguish two cases. In the arguments we give, we will consider c_{i-1}. If, however, $i - 1 = 0$, then instead of c_{i-1} consider c_d.

Case 1. $b_j = 0$ and $c_j = 1$. As $c + e_j \notin Q_d(\overleftarrow{(10)^s})$, c_j is preceded by 1.

If $c_{i-1} = 1$, then as $c' \notin Q_d(\overleftarrow{(10)^s})$, c'_i is followed by $0(10)^{s-1}$. If $c_{i-1} = 0$, then $k(i) = i - 1$ and $c + e_{l(i)}$ is on a shortest b, c-path in Q_d. The only possibility that $c + e_{l(i)}$ would not belong to $Q_d(\overleftarrow{(10)^s})$ is that $c_{l(i)}$ is preceded by 0 and in this case the only possibility that c' does not belong to $Q_d(\overleftarrow{(10)^s})$ is that c'_i is followed by $0(10)^{s-1}$. Therefore in any case, $c + e_{l(i)}$ which is on a shortest b, c-path in Q_d belongs to $Q_d(\overleftarrow{(10)^s})$ or c'_i is followed by $0(10)^{s-1}$. In the former case we can proceed by induction and in the latter case $j - i \geq 2s + 1$ as $c_{j-1} = c_j = 1$, and we get a contradiction.

Case 2. $b_j = 1$ and $c_j = 0$. As $c + e_j \notin Q_d(\overleftarrow{(10)^s})$, c_j is followed by 0. Similarly as in Case 1, $c + e_{l(i)}$ which is on a shortest b, c-path in Q_d would belong to $Q_d(\overleftarrow{(10)^s})$ or c'_i is followed by $0(10)^{s-1}$. In the former case we can proceed by induction and in the latter case $j - i \geq 2s - 1$ as $c_j = c_{j+1} = 0$. Assume the latter case. Then $j - i = 2s - 1$ and b'_i is followed by $0(10)^{s-1}$ as c'_i is. As $b' \notin Q_d(\overleftarrow{(10)^s})$, b'_i is preceded by $(10)^{s-1}1$ and as $c + e_j \notin Q_d(\overleftarrow{(10)^s})$, c_j is followed by $0(10)^{s-1}$. If $i - k(i) \geq 2s$, then c would not belong to $Q_d(\overleftarrow{(10)^s})$ as b' does not either and we get a contradiction. Therefore $i - k(i) \leq 2s - 1$. If $b_{l(i)} = 0$, then $b + e_{l(i)}$ which is on a shortest b, c-path in Q_d belongs to $Q_d(\overleftarrow{(10)^s})$ and we can proceed by induction. If $b_{l(i)} = 1$, then replacing i and j by $k(i)$ and i, respectively leads to $i - k(i) = 2s - 1$, $b_{l(i)}$ is preceded by $(10)^{s-1}1$ and $c_{l(i)}$ is followed by $0(10)^{s-1}$.

By Case 1 and Case 2, we conclude that if for some index t, $b_t = b_{t+2s-1} = 1$, $c_t = c_{t+2s-1} = 0$ and $b_r = c_r$ for all $t < r < t + 2s - 1$, then two cases follow. In the first case, at least one neighbor of b or c on a shortest b, c-path in Q_d belongs to $Q_d(\overleftarrow{(10)^s})$ and we can proceed by induction. In the second case, $k(t) = t - (2s - 1)$, $b_{l(t)} = 1$, $c_{l(t)} = 0$, $b_{l(t)}$ is preceded by $(10)^{s-1}1$ and $c_{l(t)}$ is followed by $0(10)^{s-1}$. As $d \not\equiv 0 \pmod{2s-1}$, in repeating this procedure there is an index t_0 such that $k(t_0) \neq t_0 - (2s - 1)$. Thus, at least one neighbor of b or c on a shortest b, c-path in Q_d belongs to $Q_d(\overleftarrow{(10)^s})$ and hence $d_{Q_d(\overleftarrow{(10)^s})}(b, c) = p$ by induction. $\square$

Proof of Proposition 9. We will distinguish three cases based on the residue of d modulo 3.

Case 1. $d = 3k$. Consider two vertices $b = (001)^k$ and $c = (101)^k$ of $Q_d(\overleftarrow{11010})$. They differ in k bits and changing any of these bits in b from 0 to 1 yields a circulation which has 11010 as a factor. It follows that $d_{Q_d(\overleftarrow{11010})}(b,c) > k$.

Case 2. $d = 3k + 1$. Consider $b = (001)^{k-1}0101$ and $c = (101)^{k-1}1101$ of $Q_d(\overleftarrow{11010})$. They differ in k bits and changing any of these bits in b from 0 to 1 yields a circulation which has 11010 as a factor. It follows that $d_{Q_d(\overleftarrow{11010})}(b,c) > k$.

Case 3. $d = 3k + 2$. For $d = 5$, consider 00101 and 11101 of $Q_d(\overleftarrow{11010})$, while for $d > 5$, vertices $(001)^{k-2}00101$ and $(101)^{k-2}11101$ do the job.

Acknowledgements. This work was supported by Research Grants 174010 and 174033 of Serbian Ministry of Education and Science, and by the Research Grant P1-0297 of Ministry of Higher Education, Science and Technology Slovenia. The work was partially done during a visit of A.I. at the University of Ljubljana supported by the grant P1-0297 of Ministry of Higher Education, Science and Technology Slovenia and during a sabbatical visit of Y.R. at the Faculty of Mathematics and Physics, University of Ljubljana.

REFERENCES

1. A. Castro, S. Klavžar, M. Mollard, Y. Rho: *On the domination number and the 2-packing number of Fibonacci cubes and Lucas cubes.* Comput. Math. Appl., **61** (9) (2011), 2655–2660.
2. A. Castro, M. Mollard: *The eccentricity sequences of Fibonacci and Lucas cubes.* Discrete Math., **312** (5) (2012), 1025–1037.
3. E. Dedó, D. Torri, N. Zagaglia Salvi: *The observability of the Fibonacci and the Lucas cubes.* Discrete Math., **255** (1-3) (2002), 55–63.
4. J. A. Ellis-Monaghan, D. A. Pike, Y. Zou: *Decycling of Fibonacci cubes.* Australas. J. Combin., **35** (2006), 31–40.
5. P. Gregor: *Recursive fault-tolerance of Fibonacci cube in hypercubes.* Discrete Math., **306** (13) (2006), 1327–1341.
6. W.-J. Hsu: *Fibonacci cubes – A new interconnection technology.* IEEE Trans. Parallel Distrib. Syst., **4** (1) (1993), 3–12.
7. A. Ilić, S. Klavžar, Y. Rho: *Generalized Fibonacci cubes.* Discrete Math., **312** (1) (2012), 2–11.
8. S. Klavžar: *On median nature and enumerative properties of Fibonacci-like cubes.* Discrete Math., **299** (1–3) (2005), 145–153.
9. S. Klavžar: *Structure of Fibonacci cubes: a survey.* To appear in J. Comb. Optim., doi: 10.1007/s10878-011-9433-z, 2012.
10. S. Klavžar, P. Žigert: *Fibonacci cubes are the resonance graphs of Fibonaccenes.* Fibonacci Quart., **43** (3) (2005), 269–276.

11. S. Klavžar, M. Mollard, M. Petkovšek: *The degree sequence of Fibonacci and Lucas cubes.* Discrete Math., **311** (14) (2011), 1310–1322.

12. E. Munarini, N. Salvi Zagaglia: *Structural and enumerative properties of the Fibonacci cubes.* Discrete Math., **255** (1–3) (2002), 317–324.

13. E. Munarini, C. Perelli Cippo, N. Zagaglia Salvi: *On the Lucas cubes.* Fibonacci Quart., **39** (1) (2001), 12–21.

14. A. Taranenko, A. Vesel: *Fast recognition of Fibonacci cubes.* Algorithmica, **49** (2) (2007), 81–93.

(Received August 14, 2011)
(Revised December 14, 2011)

Faculty of Sciences and Mathematics,
University of Niš
Serbia
E-mail: aleksandari@gmail.com

Faculty of Mathematics and Physics,
University of Ljubljana
Slovenia
and
Faculty of Natural Sciences and Mathematics,
University of Maribor
Slovenia
E-mail: sandi.klavzar@fmf.uni-lj.si

Department of Mathematics,
University of Incheon
Korea
e-mail: rho@incheon.ac.kr