Redefining Fractal Cubic Networks and Determining Their Metric Dimension and Fault-Tolerant Metric Dimension

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Abstract

In network theory, distance parameters are crucial in analyzing structural aspects of the networks under investigation, including their symmetry, connectedness, and tendency to form clusters. To this end, the metric dimension and the fault-tolerant metric dimension are important distance invariants of networks. In this note we consider fractal cubic networks, a variant of hypercubes. We first correct their definition from the seminal paper [Engineering Science and Technology, an International Journal 18 (2015) 32–41]. After that we determine their metric dimension and fault-tolerant metric dimension, which is in striking contrast to the situation with hypercubes, where these invariants are intrinsically difficult.

Keywords: resolving set; metric dimension; fault-tolerant metric dimension; hypercube; fractal cubic network

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1 Introduction

A combinatorial concept called the metric dimension (**MD**) of a finite, simple graph was initially investigated in [9, 29] and later in different terms reconsidered in [4, 15]. It can be equivalently described as the problem of determining the fewest landmark nodes from which any two nodes can be distinguished in a network by only using distance measurements. Numerous applications of the metric dimension exist in a variety of branches of research and technology, says robot navigation [18], computational chemistry [4], and network discovery [3] all employ the MD to determine the bare minimum of landmark nodes necessary for their respective tasks. Moreover, **MD** evolves in areas such as coin weighing problems [30], pattern recognition [22], chemistry [15], robot navigation [18], geometric routing protocols [20], and mastermind strategic game [7].

In a practical example provided in [5], metric base elements were called censors. If one of the censors fails to function, we won't have enough data to effectively deal with the invader. This leads to a fairly new development in the field of the **MD**, to the idea of the fault-tolerant metric dimension (**FTMD**). This notion was developed in [13] to address the above-mentioned problems. If one of the censors stops functioning, a fault-tolerant resolving set (**FTRS**) will still return accurate results. Therefore, the **FTMD** has applications in all the fields where the **MD** is reported.

The NP-completeness of determining the **MD** was proved in general in [6] and, in particular, for bipartite class [21]. Later, authors proved it for directed graphs [25]. As a result, the problem has been studied on many classes of graphs, and indeed many such papers have been written. Rather than making a selection among them, let us refer to two quite topical and very competent review papers [19, 31].

In this article we are interested in the fractal family of graphs named fractal cubic network. First, we correct the explicit definition from the original article, and then we determine their **MD** and their **FTMD**. But before we get to these networks, we give the basic definitions we need and recall a known result.

We use established graph terminology, in particular the degree $\deg_G(x)$ of a vertex x of G, the distance $d_G(x, y)$ between x and y, and the diameter $\mathrm{diam}(G)$ have an established meaning. Let $R = \{r_1, \ldots, r_k\}$ be an ordered set of vertices. For $x \in V(G)$, the *metric representation* of x with reference to R is the k-vector

$$(d_G(x, r_1), \ldots, d_G(x, r_k)) .$$

R is a *resolving set* for G if vertices of G have pairwise different metric representations with reference to R. Alike, for every two different vertices $x, y \in V(G)$ there is a vertex $r \in R$ s.t. $d_G(x, r) \neq d_G(y, r)$. See Fig. 1(a). The minimum cardinality of a resolving set of G is the *metric dimension* $\dim(G)$ of G. Further, $F \subseteq V(G)$ is a *fault-tolerant resolving set* if for every $r \in F$, $F - \{r\}$ is a resolving set of G.

In other words, for every two vertices $x, y \in V(G)$ there exist $r_1, r_2 \in F$ s.t. $d_G(x, r_1) \neq d_G(y, r_1)$ and $d(x, r_2) \neq d(y, r_2)$. See Fig. 1(b). The minimum cardinality of a fault-tolerant resolving set of G is the *fault-tolerant metric dimension* $\dim'(G)$ of G. This concept has been also well studied, a selection of these studies is [2, 8, 11, 13, 23, 24, 26–28].

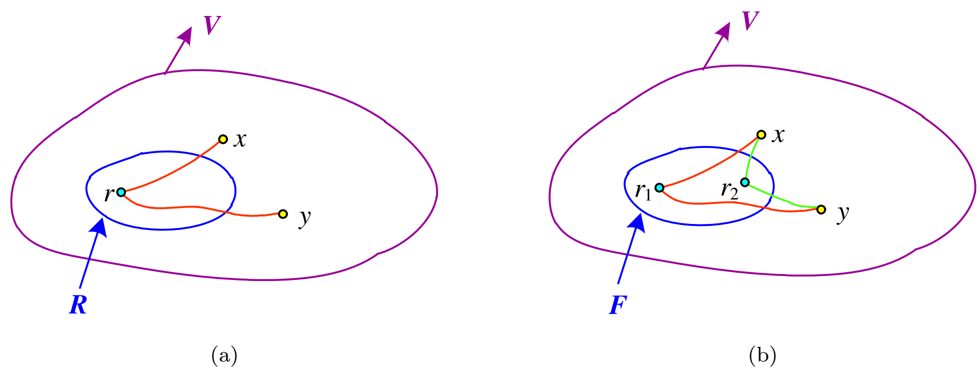


Figure 1: (a) Resolving set R; (b) Fault-tolerant resolving set F

2 Interconnection Networks and Fractal Cubic Networks

To connect a significant fraction of consistently replicated processor memory pairs, each of which is referred to as processors (nodes), multiprocessor interconnection networks are quite often needed. Message passing is typically utilised to provide complete synchronisation and transmission among the processors for programmed execution rather than shared memory. There has been noticeable interest in designing and implementing multiprocessor interconnection networks due to the accessibility of low-cost, powerful microprocessors and memory chips. The creation of CPUs, interconnection networks, and routing algorithms are three of the main study fields in supercomputers and massively parallel computing systems. The interconnection network, which contains links among betwixt millions of processors, is vital when developing a supercomputer.

An interconnection net is made up of multiple of processors, each of which has its own local sensing and cognitive links (edges) that allow data to be sent amongst processors (nodes). It can be represented as a graph G defined earlier, where two nodes p_i, p_j are directly connected by a communication link. The broadcasting time, bisection width, diameter, degree, and fault-tolerance are the characteristics used to assess the effectiveness of interconnection networks [1]. A common interconnection network topology, the hypercube has a number of distinguishing qualities, including regularity, symmetry, easy routing, high connectedness, and recursive structure. In the recent decades, hypercube have been extensively investigated on various properties, cf. [10]. There have been several proposed hypercube variations, including augmented cubes, folded hypercubes, exchanged hypercubes, crossed cubes, twisted cubes, and shuffle cubes to name a selection of them.

The investigation of the **MD** of hypercubes Q_d were initiated in [4]. The exact values are known only for small dimensions, while the best known upper bounds for dimensions at least 8 are due to [14] and read as follows:

$$\dim(Q_r) \leq \begin{cases} r-2; & r \in \{8,9\}, \\ r-3; & r \in \{10,11\}, \\ r-4; & r = 12, \\ r-5; & r \in \{13,14\}, \\ r-6; & r \in \{15,16\}, \\ r-7; & r = 17\,. \end{cases}$$

As an appealing recent contribution is this direction, it was proved in [17] that the **MD** and the edge **MD** differ by at most one on an arbitrary hypercube and that the **MD** and the mixed **MD** of hypercubes coincide.

Though there are several investigations cast path on variations of hypercube stated above, the problem metric dimension is investigated for none of the above variants of hypercube listed above. With this motivation, we investigate here the problem for fractal cubic network (FCN).

We first restate the definition of the FCN as given in [16]. For $r > 0$, the *Fractal Cubic Network* $FCN(r) = (V_1(r), E_1(r))$ has the vertex set

$$V_1(r) = 11 \parallel V_1(r-1) \bigcup 01 \parallel V_1(r-1) \bigcup 10 \parallel V_1(r-1) \bigcup 00 \parallel V_1(r-1),$$

and the edge set

$$E_1(r) = 11 \parallel E_1(r-1) \bigcup 01 \parallel E_1(r-1) \bigcup 10 \parallel E_1(r-1) \bigcup 00 \parallel E_1(r-1) \bigcup E^{'},$$

where $E^{'} = \{(e_i, e_j)|$, $e_i = \underbrace{\text{string}}_{m}\ ab\ \underbrace{\text{string}}_{2r-m}$, $e_j = \underbrace{\text{string}}_{m}\ cd\ \underbrace{\text{string}}_{2r-m}$ and $\sum((ab) \oplus (cd)) = 1$, ($\oplus$ is and xor operator) the label of e_i and e_j are same except ab of e_i and cd of $e_j\}$, where $m = 0, 2, 4, \ldots$.

In the above definition, $\parallel$ stands for the concatenation of strings, for instance, $11 \parallel V_1(r-1)$ is the set of all strings obtained by attaching 11 to the front of each of the strings in $V_1(r-1)$.

However, the corresponding figures and subsequent research show that the definition is inadequate because the set $E^{'}$ is too large. In fact, we can imagine $FCN(r)$ to be constructed from four disjoint copies of $FCN(r-1)$ by adding just four additional edges. Hence the corrected definition reads as follows.

$FCN(0)$ is the 4-cycle with vertices $00, 01, 10, 11$. In other words, $FCN(0) = Q_2$. If $r \geq 1$, then

$$V_1(r) = 11 \parallel V_1(r-1) \bigcup 01 \parallel V_1(r-1) \bigcup 10 \parallel V_1(r-1) \bigcup 00 \parallel V_1(r-1),$$

and

$$E_1(r) = 11 \parallel E_1(r-1) \bigcup 01 \parallel E_1(r-1) \bigcup 10 \parallel E_1(r-1) \bigcup 00 \parallel E_1(r-1)$$
$$\bigcup \{\{\mathbf{00}1100\ldots0, \mathbf{10}1100\ldots0\}, \{\mathbf{10}1100\ldots0, \mathbf{11}1100\ldots0\}\}$$
$$\bigcup \{\{\mathbf{11}1100\ldots0, \mathbf{01}1100\ldots0\}, \{\mathbf{01}1100\ldots0, \mathbf{00}1100\ldots0\}\}$$

See Fig. 2 where $FCN(0)$, $FCN(1)$, and $FCN(2)$ are drawn.

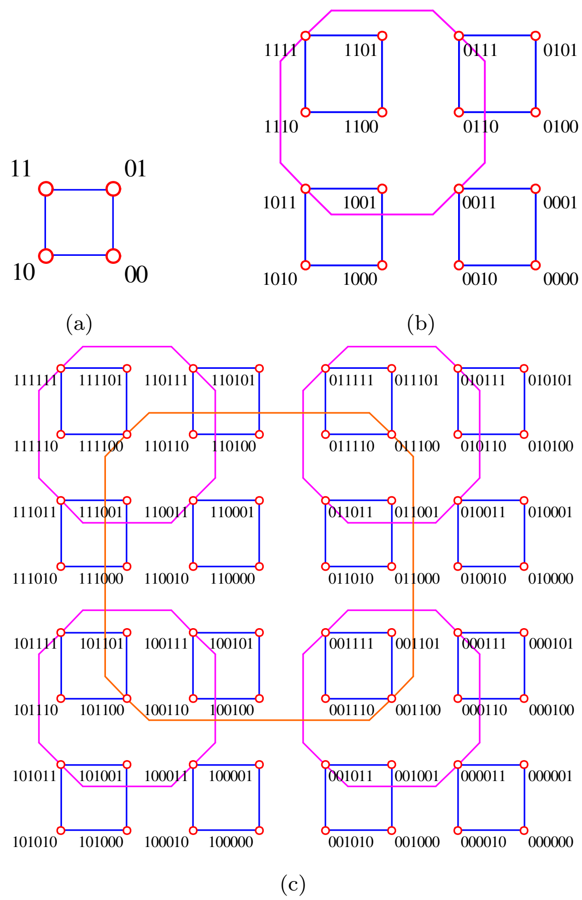


Figure 2: (a) $FCN(0)$; (b) $FCN(1)$; (c) $FCN(2)$

3 The (Fault-Tolerant) Metric Dimension of FCNs

The open neighborhood of $x \in V(G)$ is the set $N_G(x) = \{y \in V(G) : d_G(x,y) = 1\}$, and the set $N_G[x] = N_G(x) \cup \{x\}$ is the closed neighborhood of x. Two vertices $x, y \in V(G)$ are *twins* if $N_G(x) = N_G(y)$ or $N_G[x] = N_G[y]$. It is clear that if $N_G[x] = N_G[y]$, then $d_G(x,y) = 1$, and if

$N_G(x) = N_G(y)$, then $d_G(x,y) \neq 1$. Define the relation $\equiv$ on $V(G)$ by $x \equiv y$ if x and y are twins. It is an equivalence relation, let $\tau(G)$ be the set of twin equivalence classes. Then we infer that if R is a resolving set of G and $\tau \in \tau(G)$, then $|R \cap \tau| \geq |\tau| - 1$. This in turn implies the following fact first observed in [12]:

$$\dim(G) \geq \sum_{\tau \in \tau(G)} (|\tau| - 1). \tag{1}$$

Using the same argument we also see that

$$\dim'(G) \geq \sum_{\tau \in \tau(G)} |\tau|, \tag{2}$$

a result explicitly stated in [24, Lemma 2.1]. From these facts we can deduce the following:

Lemma 1. *Let G be a connected graph with twin sets τ_i, $i \in [k]$, such that $|\tau_i| = 2$. If $\dim(G) = k$, then $\dim'(G) = 2k$.*

Proof. By (2) and the assumption that $|\tau_i| = 2$ holds for $i \in [k]$, we have

$$\dim'(G) \geq \sum_{\tau \in \tau(G)} |\tau| = 2k.$$

On the other hand, since $\dim(G) = k$ and $|\tau_i| = 2$ for $i \in [k]$, we infer that each smallest resolving set contains exactly one vertex from each twin equivalence class. But then the set of all twins of G, which is of cardinality $2k$, is a resolving set. Moreover, it is also a fault-tolerant resolving set. We conclude that $\dim'(G) \leq 2k$ holds. □

Now we can state the announced result.

Theorem 2. *If $r > 0$, then $\dim(FCN(r)) = 2^{2r}$ and $\dim'(FCN(r)) = 2^{2r+1}$.*

Proof. By the construction of $FCN(r)$, $N_{FCN(r)}(x_{2r+2}x_{2r+1}\ldots x_3 01) = N_{FCN(r)}(x_{2r+2}x_{2r+1}\ldots x_3 10)$, where $x_{2r+2}x_{2r+1}\ldots x_3$ is an arbitrary binary sting of length $2r$. That is,

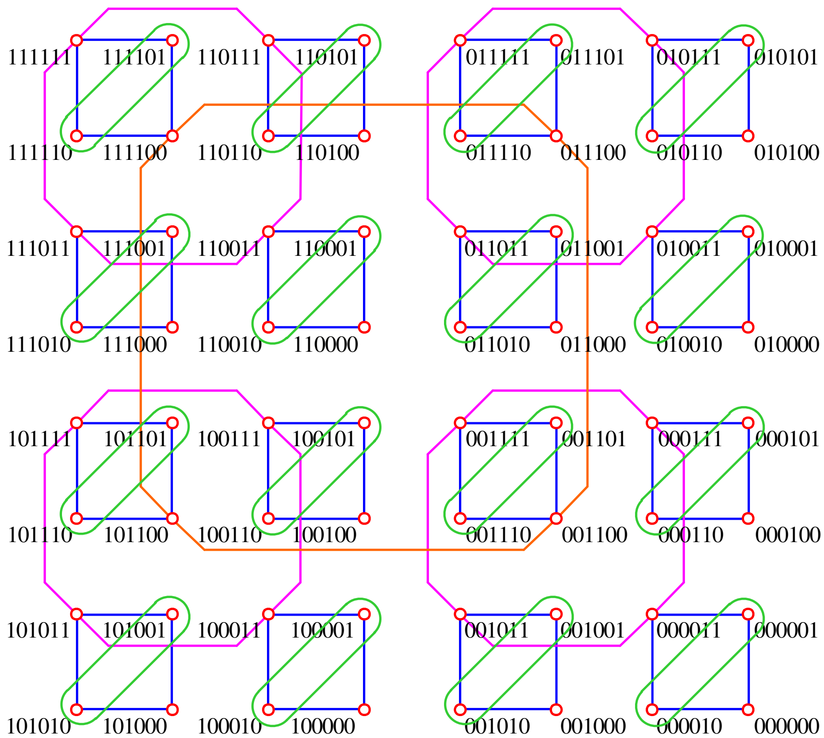
$$\{x_{2r+2}x_{2r+1}\ldots x_3 01, x_{2r+2}x_{2r+1}\ldots x_3 10\}$$

is a twin set, where $x_{2r+2}x_{2r+1}\ldots x_3 \in \{0,1\}^{2r}$. See Fig. 3 where 2^4 twin sets in $FCN(2)$ are marked.

Hence, since $FCN(r)$ contains 2^{2r} twin classes, each of cardinality 2, we infer from (1) that $\dim(FCN(r)) \geq 2^{2r}$. For the reverse inequality $\dim(FCN(r)) \leq 2^{2r}$, we set $R = \{x_{2r+2}x_{2r+1}\ldots x_3 01 : x_i \in \{0,1\}\}$ and claim that R is a resolving set. For this sake we need to show that x and y are resolved by R, where x and y are arbitrary vertices of $FCN(r)$. We distinguish several cases.

Case 1: $x, y \in \{x_{2r+2}x_{2r+1}\ldots x_3 00, x_{2r+2}x_{2r+1}\ldots x_3 10, x_{2r+2}x_{2r+1}\ldots x_3 11\}$.

Subcase 1.1: $x = x_{2r+2}x_{2r+1}\ldots x_3 00$ and $y = x_{2r+2}x_{2r+1}\ldots x_3 11$.
In this case, if $r \in R - \{x_{2r+2}x_{2r+1}\ldots x_3 01\}$, then $d_{FCN(r)}(r,x) \neq d_{FCN(r)}(r,y)$.



(a)

Figure 3: Twins in $FCN(2)$

Subcase 1.2: $x = x_{2r+2}x_{2r+1}\dots x_3 10$ and $y = x_{2r+2}x_{2r+1}\dots x_3 11$ or $x_{2r+2}x_{2r+1}\dots x_3 00$.
In this case consider a vertex $x_{2r+2}x_{2r+1}\dots x_3 01 \in R$. Then we have $d_{FCN(r)}(x_{2r+2}x_{2r+1}\dots x_3 01, x)$ $= d_{FCN(r)}(x_{2r+2}x_{2r+1}\dots x_3 01, y) + 1$.

Case 2: $x \in \{x_{2r+2}x_{2r+1}\dots x_3 00, x_{2r+2}x_{2r+1}\dots x_3 10, x_{2r+2}x_{2r+1}\dots x_3 11\}$ and
$y \notin \{x_{2r+2}x_{2r+1}\dots x_3 00, x_{2r+2}x_{2r+1}\dots x_3 10, x_{2r+2}x_{2r+1}\dots x_3 11\}$.

Subcase 2.1: $x \in N_{FCN(r)}(x_{2r+2}x_{2r+1}\dots x_3 01)$.
In this case consider the vertex $x_{2r+2}x_{2r+1}\dots x_3 01 \in R$. Then $d_{FCN(r)}(x_{2r+2}x_{2r+1}\dots x_3 01, x) = 1$ and $d_{FCN(r)}(x_{2r+2}x_{2r+1}\dots x_3 01, y) > 1$.

Subcase 2.2: $x \notin N_{FCN(r)}(x_{2r+2}x_{2r+1}\dots x_3 01)$.
If $N_{FCN(r)}(y) \cap R = \emptyset$, then $d_{FCN(r)}(x, x_{2r+2}x_{2r+1}\dots x_3 01) < d_{FCN(r)}(y, x_{2r+2}x_{2r+1}\dots x_3 01)$, and if $N_{FCN(r)}(y) \cap R \neq \emptyset$, then there exists an $r \in N_{FCN(r)}(y) \cap R$, s.t. $d_{FCN(r)}(x, r) > d_{FCN(r)}(y, r)$.

This completes the proof of $\dim(FCN(r)) \leq 2^{2r}$, hence we can conclude that $\dim(FCN(r)) = 2^{2r}$. The assertion that $\dim'(FCN(r)) = 2^{2r+1}$ now follows by Corollary 1. □

4 Conclusion

Covering the nodes of an undirected graph so that any node is uniquely recognized by examining the nodes covering it. As a combinatorial concept, the **MD** of a graph captures the fewest landmark nodes required to separate any two nodes in the graph using graph geodesic. Interconnection networks are crucial to parallel computing systems because they determine how well they perform on a wide scale. A crucial performance indicator in parallel computing systems is communication efficiency. And a crucial metric of communication effectiveness is the diameter of an interconnection network. Numerous interconnection networks have been suggested thus far. Hypercubes are popular among all connectivity networks due to their beneficial characteristics. There are several potential hypercube variations by modifying some linkages. The fractal cubic network is one among them. In this article, we have defined the fractal cubic network precisely and dealt with the problem of giving the nodes of a fractal cubic network their unique representation in the best possible way.

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