THE ALL-PATHS TRANSIT FUNCTION OF A GRAPH

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(Received October 29, 1998)

Abstract. A transit function R on a set V is a function $R\colon V \times V \to 2^V$ satisfying the axioms $u \in R(u,v)$, $R(u,v) = R(v,u)$ and $R(u,u) = \{u\}$, for all $u, v \in V$. The all-paths transit function of a connected graph is characterized by transit axioms.

Keywords: all-paths convexity, transit function, block graph

MSC 2000: 05C99, 05C75

1. Introduction

The geodesic interval function I and the geodesic convexity of a connected graph G are important tools for the study of the metric properties of G, cf. e.g. [1], [6], where a geodesic is a shortest path. An example of a class of graphs where these tools are indispensable, is that of median graphs. Such graphs are defined by the property that, for any triple of vertices, the intervals between the pairs of the triple intersect in exactly one vertex. This class of graphs is well studied, see [5], [6]. In the area of metric graph theory, the notions of the gate and the gated set also play an important role.

* This work was done while the first author was visiting the Econometric Institute of Erasmus University, Rotterdam, as a BOYSCAST fellow of the Department of Science and Technology (DST) of the Ministry of Science and Technology of India, March–September 1998. The financial support of the DST, New Delhi, and the hospitality of the Econometric Institute, Rotterdam, are greatly acknowledged.

** This work was supported by the Ministry of Science and Technology of Slovenia under the grant J1-0498-0101.

The definition of I is in terms of the distance function of G. Nebeský [9], [10] has given an axiomatic characterization of the geodesic interval function without any reference to metric notions.

Apart from the geodesic convexity, the convexity generated by the induced path function J is an interesting concept and various authors have studied it, see [2], [4]. The analogue of median graphs in the case of the function J was studied in [8]. There are also analogues available of the gate and the gated set with respect to J and other transit functions, see [7]. From the convexity point of view, the convexity generated by the all-paths function A has also been studied in [2], [11], where it was called the coarsest path interval.

Convexities defined by a function such as those above are called interval convexities, or interval spaces in e.g. [3], [12]. Because obvious properties of betweenness are only present in the case of I, but in general not in the case of J and A, we follow the terminology of [7] and call these functions transit functions.

Motivated by the characterization of the geodesic interval function by Nebeský [9], [10] using axioms on I only, we present in this paper an axiomatic characterization of the all-paths function A. The all-paths function has a nice structure, reflecting the block cut-vertex structure of the graph. Besides the characterization of A in terms of transit axioms only, we consider briefly the obvious analogues of the notions discussed above related to the functions I and J. Thus we follow the suggestion in [7] to study such problems for any transit function.

In Section 2 we introduce the concept of a transit function and in particular the all-paths transit function which is the central notion of our paper. In parallel to results for some other transit functions we list several basic properties of the all-paths function, for instance on betweenness, gatedness and convexity. In Section 3 we prove our main theorem which characterizes the all-paths transit function of a connected graph by transit axioms, i.e. by axioms on the function only. As it will turn out, three relatively simple axioms suffice for this purpose.

All graphs in this paper are finite, simple, loopless and connected. Recall that a subgraph H of a graph G is a *block* of G if either H is a bridge (and its endvertices) or it is a maximal 2-connected subgraph of G. A basic property of blocks that we use in the sequel is the following: let u, v, w be three distinct vertices of a block H, then there exists a path in H between u and v through w. A *block graph* is a connected graph, in which each block is a complete subgraph.

Let V be a (finite) set. A *transit function* on V is a function $R\colon V \times V \to 2^V$ satisfying the following axioms (for any $u, v \in V$):

(t1) $u \in R(u,v)$;

(t2) $R(u,v) = R(v,u)$;

(t3) $R(u,u) = \{u\}$.

If, moreover, $G = (V,E)$ is a graph with a vertex set V, then we say that R is a *transit function on* G. The term transit function was introduced in [7] as a unifying concept for the study of various notions such as convexity, betweenness, medians, etcetera. Examples of transit functions on graphs are provided by the geodesic interval function

$$I(u,v) = \{w \in V \mid w \text{ lies on a shortest } u,v\text{-path in } G\},$$

and the induced path function

$$J(u,v) = \{w \in V \mid w \text{ lies on an induced } u,v\text{-path in } G\}.$$

For more examples we refer to [7]. Here we consider the *all-paths function* A_G of a graph G defined as

$$A_G(u,v) = \{w \in V \mid w \text{ lies on some } u,v\text{-path in } G\}.$$

Clearly, the all-paths function is a well-defined transit function. If no confusion can arise, we write $A = A_G$.

Let R be a transit function on a set V. We say that a set $W \subseteq V$ is *R-convex* if $R(u,v) \subseteq W$ for all u, v in W. The family of R-convex sets forms an abstract convexity on V in the sense that it is closed under intersections and both the empty set $\emptyset$ and the whole set V are R-convex. If R is a transit function on a graph G, then the R-convexity is a convexity on G as well. The A-convexity was considered by Sampathkumar [11] and Duchet [2] who evaluated the usual convexity invariants, namely the Helly, Carathéodory and Radon numbers.

A transit function R on V is a *betweenness* if it satisfies the *betweenness axioms* (for any $u, v \in V$):

(b1) $x \in R(u,v)$, $x \neq v \Rightarrow v \notin R(u,x)$;

(b2) $x \in R(u,v) \Rightarrow R(u,x) \subseteq R(u,v)$.

For a subset W of V and a vertex $z \in V$, a vertex $x \in W$ is an *R-gate* for z in W if x lies in $R(z,w)$ for any w in W. The set W is called *R-gated*, if every vertex

$z \in V$ has a unique R-gate in W. Note that by (t3) every vertex w in W is its own unique R-gate.

Let $G = (V, E)$ be a connected graph. A *tree of blocks* in G is a connected subgraph such that, whenever it contains two distinct vertices u and v of some block of G, it contains the whole block. Let u and v be distinct vertices of G. Then it follows easily from the basic property of blocks mentioned in the introduction that the subgraph of G induced by $A(u, v)$ is the smallest tree of blocks containing u and v. We call this tree of blocks the *path of blocks* between u and v. If u and v are distinct vertices of the same block of G, then $A(u, v)$ induces the block of G containing u and v. Note that if G is itself a block, then the family of A-convex sets in G is just the trivial convexity consisting of the empty set $\emptyset$, all the singletons and V. Note also that every set $A(u, v)$ is A-convex and hence the A-convex hull of u and v is just $A(u, v)$.

For a connected graph G, we define *block closure* $\widetilde{G}$ of G as the graph with vertex V, where distinct u and v in V are adjacent in $\widetilde{G}$ if and only if u and v belong to the same block in G. Clearly $\widetilde{G}$ is a block graph, and $G = \widetilde{G}$ if and only if G is a block graph. The following fact is basic for the all-paths function of a graph.

Fact 1. *Let G be a connected graph with the block closure $\widetilde{G}$. Then $A_G = A_{\widetilde{G}}$.*

As mentioned in the introduction, the geodesic interval function I and the minimal path function J were a source of inspiration for this paper. Therefore, we list here some more facts on the all-paths function A, which are obvious analogues of prototype problems and results on I and J, cf. [7].

Let G be a connected graph with all-paths function A. The *underlying graph* $G(A)$ of A is the graph with a vertex set V, where distinct vertices u and v are adjacent if and only if $A(u, v) = \{u, v\}$.

Fact 2. *Let G be a connected graph with all-paths function A. Then $G = G(A)$ if and only if G is a tree.*

The all-paths function of a graph trivially satisfies axiom (b2) of betweenness, but in general it does not satisfy axiom (b1).

Fact 3. *Let G be a connected graph with all-paths function A. Then A satisfies the betweenness axiom* (b1) *if and only if G is a tree.*

Of course, in the case of a tree, we have $I = J = A$.

For a transit function R, we write $R(u, v, w) = R(u, v) \cap R(v, w) \cap R(w, u)$, cf. [7]. In the case of the geodesic interval function I, the sets $I(u, v, w)$ are usually empty. So the interesting cases here are when this set is nonempty for all u, v, w in V (the case of modular graphs), or when $|I(u, v, w)| = 1$ for all u, v, w in V (the case of median graphs). In the case of the induced path function J, the sets $J(u,v,w)$ are usually nonempty. The interesting case, when $|J(u,v,w)| \leqslant 1$ for all u,v,w in V, is discussed in [8]. The sets $A(u,v,w)$ are always nonempty for any connected graph G.

Fact 4. *Let $G=(V,E)$ be a connected graph with all-paths function A. Then $|A(u,v,w)|=1$ for all u,v,w in V, if and only if G is a tree.*

Note that, if G is not bipartite, then no edge in an isometric odd cycle in G is I-gated. So I-convex sets need not be I-gated. On the other hand, it is easy to see that I-gated sets are I-convex. Let W be an I-gated set in G. The gate x in W of a vertex z is necessarily the (unique) vertex in W nearest to z. Using this fact, one can easily deduce that the intersection of I-gated sets is again I-gated. Thus the family of I-gated sets of G forms an abstract convexity. In general, however, the family of R-gated sets of a transit function R on G need not form a convexity. For example, let G be the 5-cycle. Then any three consecutive vertices on the 5-cycle form a J-gated set, whereas no edge is J-gated. So on the 5-cycle, the J-gated sets do not form a convexity. It is straightforward to show that a subset of vertices of G is A-gated if and only if it induces a tree of blocks.

Fact 5. *Let $G=(V,E)$ be a connected graph with all-paths function A. Then a subset W of V is A-gated if and only if it is A-convex.*

3. A characterization of the A-function

For a transit function A defined on a set V we introduce the following additional axioms:

(a4) $w \in A(u,v) \Rightarrow A(w,v) \subseteq A(u,v)$;
(a5) $A(u,x) \cap A(x,v) = \{x\} \Rightarrow A(u,x) \cup A(x,v) = A(u,v)$;
(a6) $A(u,v) \subsetneqq A(u,w),\ u \neq v \Rightarrow \exists x \in A(u,v),\ x \neq u$ such that $A(u,x) \cap A(x,w) = \{x\}$.

Note that the axiom (a4) is just the betweenness axiom (b2). Throughout this section we assume that the transit function A satisfies also the axioms (a4), (a5) and (a6).

Before we prove our main result, we need a series of lemmas.

Lemma 1. *Let u,v,x be elements of a set V such that $A(u,x) \cap A(x,v) = \{x\}$. Then for any element w of V we have*

$$A(u,x) \cap A(x,w) = \{x\} \quad \text{or} \quad A(v,x) \cap A(x,w) = \{x\}.$$

P r o o f. Assume the contrary, and let u, v, w, x, y, z be elements of V with x distinct from y and z such that

$$A(u,x) \cap A(x,v) = \{x\}, \tag{1}$$

$$y \in A(u,x) \cap A(x,w), \tag{2}$$

$$z \in A(v,x) \cap A(x,w). \tag{3}$$

By (a5) we have

$$A(u,x) \cup A(x,v) = A(u,v). \tag{4}$$

By (1) and (2) we have $y \notin A(x,v)$ and, by (1) and (3), we have $z \notin A(u,x)$. If w were in $A(u,v)$, then by (4) w would be in $A(u,x)$ or $A(x,v)$, say in $A(u,x)$. But then by (a4) we would have $A(w,x) \cap A(x,v) \subseteq A(u,x) \cap A(x,v) = \{x\}$, which is in conflict with (3). So we have

$$w \notin A(u,x) \cup A(x,v) = A(u,v). \tag{5}$$

By (a4) we have $A(y,x) \cap A(x,z) \subseteq A(u,x) \cap A(x,v)$, so that by (1)

$$A(y,x) \cap A(x,z) = \{x\}, \tag{6}$$

whence, by (a5),

$$A(y,x) \cup A(x,z) = A(y,z).$$

By (a4), we have $A(x,y) \subseteq A(u,x) \cap A(x,w) \subseteq A(u,v)$, so, by (5), it follows that

$$w \notin A(x,y). \tag{7}$$

Then, by (a6) and (a5), there exists a y' distinct from x in $A(x,y)$ such that

$$A(x,y') \cap A(y',w) = \{y'\}, \tag{8}$$

$$A(x,y') \cup A(y',w) = A(x,w). \tag{9}$$

By (a4) we have

$$A(x,y') \subseteq A(x,y) \subseteq A(u,x).$$

Similarly, by (7), (a6) and (a5), there exists a z' distinct from x in $A(x,z)$ such that

$$A(x,z') \cap A(z',w) = \{z'\}, \tag{10}$$

$$A(x,z') \cup A(z',w) = A(x,w).$$

By (a4) we have

$$A(x, z') \subseteq A(x, z) \subseteq A(x, v).$$

By (a4), we have $A(y', x) \cap A(x, z') \subseteq A(y, x) \cap A(x, z)$, so by (6)

$$A(y', x) \cap A(x, z') = \{x\}. \tag{11}$$

Hence, by (a6), we have

$$A(y', z') = A(y', x) \cup A(x, z'). \tag{12}$$

Since y' is in $A(x, w)$, it follows from (11) and (10) that y' is in $A(z', w)$. Similarly, since z' is in $A(x, w)$, it follows from (11) and (9) that z' is in $A(y', w)$. Hence we have

$$y', z' \in A(y', w) \cap A(z', w).$$

Therefore, by (12) and (a4) we have

$$A(y', z') \subseteq A(y', w) \cap A(z', w). \tag{13}$$

By (11) and (13) we have

$$x, y' \in A(x, y') \cap A(y', w).$$

However, this is in conflict with (8). This impossibility concludes the proof of the lemma. □

Let A be a transit function defined on a set V. Then the *transit graph* G_A of A is defined as follows. It has V as the vertex set and uv is an edge of G_A if there is no $x \neq u, v$ such that $A(u, x) \cap A(x, v) = \{x\}$.

Lemma 2. *Let G_A be the transit graph of a transit function A on V. Then G_A is connected.*

P r o o f. It suffices to show that $A(u, v)$ contains a u, v-path for each u, v in V. Assume the contrary, and let u, v be such that $A(u, v)$ does not contain a u, v-path with $|A(u, v)|$ as small as possible. Clearly, u and v cannot be adjacent. So, by definition of G_A, there exists a vertex x distinct from u and v with $A(u, x) \cap A(x, v) = \{x\}$. By (a5), we have $A(u, v) = A(u, x) \cup A(x, v)$, in particular $x \in A(u, v)$. If v was in $A(u, x)$, then, by (a4), we would have $A(u, x) = A(u, v)$. This would imply that v is in $A(u, x) \cap A(x, v)$ too, which is impossible. So v is not in $A(u, x)$ and, similarly, u is not in $A(x, v)$. Hence we have $|A(u, x)| < |A(u, v)|$ and $|A(x, v)| < |A(u, v)|$. By minimality of $|A(u, v)|$ there exists a u, x-path in $A(u, x)$ and an x, v-path in $A(x, v)$. Combining these paths, we obtain a u, v-path in $A(u, x) \cup A(x, v) = A(u, v)$, contrary to our assumption. □

Lemma 3. *Let G_A be the transit graph of a transit function A on V. Let u, v, x be distinct vertices of V with $A(u,x) \cap A(x,v) = \{x\}$. Then x is a cut-vertex in G_A and u and v are in different components of $G_A - x$.*

Proof. Let

$$W_u = \{w \mid |A(u,x) \cap A(x,w)| \geqslant 2\}$$

and

$$W_v = \{w \mid |A(v,x) \cap A(x,w)| \geqslant 2\}.$$

Note that u lies in W_u and v lies in W_v. It follows from Lemma 1 that $W_u \cap W_v = \emptyset$. Let $Z = V - [\{x\} \cup W_u \cup W_v]$. By definition, there is no edge between W_u and $W_v \cup Z$. Hence by Lemma 2 x is a cut-vertex in G_A, and u and v are in different components of $G_A - x$. □

Lemma 4. *Let G_A be the transit graph of a transit function A on V. Then G_A is a block graph.*

Proof. Let u, v be non-adjacent vertices of G_A. By the definition of G_A and by Lemma 2, there exists a cut-vertex x of G_A such that u and v are in different components of $G_A - x$. Hence every block of G_A is complete. □

We are now ready for our main result. Loosely speaking the next theorem amounts to the following equation: $A_{G_A} = A$.

Theorem 5. *Let A be a transit function on a set V satisfying* (a4), (a5) *and* (a6), *and let G_A be the transit graph of A. Then A is the all-paths transit function of G_A.*

Proof. Let A_0 denote the all-paths transit function of G_A.

First, let u, v be adjacent vertices of G_A.

Then $A_0(u,v)$ is the block B of G_A containing uv. Let w be a vertex in B distinct from u and v. Because of Lemma 4, w is adjacent to u and v, thus we have $|A(u,v) \cap A(u,w)| \geqslant 2$. Let x be a vertex in $A(u,v) \cap A(u,w)$ distinct from u. By (a4), we have $A(u,x) \subseteq A(u,v) \cap A(u,w)$. Since u and v are adjacent, it follows from (a6) and Lemma 3 that $A(u,x) = A(u,v)$. Similarly, it follows from the adjacency of w and u that $A(u,x) = A(u,w)$. So $A(u,w) = A(u,v)$, whence w is in $A(u,v)$.

Conversely, let z be a vertex in $A(u,v)$ distinct from u and v. By (a4), we have $A(u,z) \subseteq A(u,v)$. Since u and v are adjacent, it follows from (a6) and Lemma 3 that $A(u,z) = A(u,v)$. Suppose there is a vertex x distinct from u and z such that $A(u,x) \cap A(x,z) = \{x\}$. From (a5) it follows that x is in $A(u,z) = A(u,v)$. As in the case of z, we deduce that $A(u,x) = A(u,v)$. So $A(u,x) = A(u,z)$. But this would imply that $A(u,x) \cap A(x,z)$ would contain z as well, contrary to our assumption on x. Hence we conclude that z is adjacent to u and, similarly, z is adjacent to v. Therefore, z is in B. Thus we have shown that $A(u,v) = B = A_0(u,v)$.

Second, let u, v be non-adjacent vertices of G_A. Let $u = u_0 \to u_1 \to \ldots \to u_k \to u_{k+1} = v$ be a shortest u,v-path in G_A. Since G_A is a block graph, u_i is a cut-vertex in G_A for $i = 1, \ldots, k$. Hence by the first step of the proof, we have $A(u_i, u_{i+1}) = A_0(u_i, u_{i+1})$ for $i = 0, \ldots, k$. Moreover, $A(u_i, u_{i+1}) \cap A(u_{i+1}, u_{i+2}) = \{u_{i+1}\}$ for $i = 0, \ldots, k-1$, so that, by (a5), we have

$$A(u,v) = A(u,u_1) \cup A(u_1,u_2) \cup \ldots \cup A(u_k,v).$$

From these facts we deduce that $A(u,v) = A_0(u,v)$ is precisely the path of blocks between u and v. □

Let G be a graph and let R be a transit function on G. A graph G is called *R-monotone* if the sets $R(u,v)$ are R-convex for all u, v in G, see [7]. We note that graphs for which each geodesic interval $I(u,v)$ is I-convex were introduced in [6] as *interval monotone graphs*. A characterization of this class of graphs is still an open problem. On the other hand, we have already observed that any connected graph is A-monotone.

Let R be a transit function on V. We say that R is *monotone* if the following holds for any u, v in V: if $x, y \in R(u,v)$, then $R(x,y) \subseteq R(u,v)$, cf. [7]. Then we have the following immediate corollary of Theorem 5.

Corollary 6. *Let A be a transit function on a set V satisfying* (a4), (a5) *and* (a6). *Then A is monotone.*

References

[1] *P. Duchet*: Convexity in combinatorial structures. Rend. Circ. Mat. Palermo (2) Suppl. *14* (1987), 261–293.

[2] *P. Duchet*: Convex sets in graphs II. Minimal path convexity. J. Combin. Theory Ser. B *44* (1988), 307–316.

[3] *J. Calder*: Some elementary properties of interval convexities. J. London Math. Soc. *3* (1971), 422–428.

[4] *M. Farber and R. E. Jamison*: Convexity in graphs and hypergraphs. SIAM J. Algebraic Discrete Methods *7* (1986), 433–444.

[5] *S. Klavžar and H. M. Mulder*: Median graphs: characterizations, location theory and related structures. J. Combin. Math. Combin. Comput. *30* (1999), 103–127.

[6] *H. M. Mulder*: The Interval Function of a Graph. Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.

[7] *H. M. Mulder*: Transit functions on graphs. In preparation.

[8] *M. A. Morgana and H. M. Mulder*: The induced path convexity, betweenness, and svelte graphs. Discrete Math. To appear.

[9] *L. Nebeský*: A characterization of the interval function of a connected graph. Czechoslovak Math. J. *44(119)* (1994), 173–178.

[10] *L. Nebeský*: Characterizing the interval function of a connected graph. Math. Bohem. *123(2)* (1998), 137–144.

[11] *E. Sampathkumar*: Convex sets in graphs. Indian J. Pure Appl. Math. *15* (1984), 1065–1071.

[12] *M. L. J. van de Vel*: Theory of Convex Structures. North Holland, Amsterdam, 1993.

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