

# Characterizing almost-median graphs II

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## Abstract

Let  $\mathcal{M}$ ,  $\mathcal{A}$ ,  $\mathcal{S}$ , and  $\mathcal{P}$ , be the sets of median graphs, almost-median graphs, semi-median graphs and partial cubes, respectively. Then  $\mathcal{M} \subset \mathcal{A} \subset \mathcal{S} \subset \mathcal{P}$ . It is proved that a partial cube is almost-median if and only if it contains no convex cycle of length greater than four. This extends the result of Brešar [2] who proved that the same property characterizes almost-median graphs within the class of semi-median graphs.

**Key words:** partial cube; median graph; convex cycle; almost-median graph

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## 1 Introduction

Median graphs, one of the central classes in metric graph theory, can be characterized as partial cubes for which all sets  $U_{uv}$  are convex. (See Section 2 for the definition of  $U_{uv}$ .) Motivated by this fact, almost-median graphs were introduced in [7] as those partial cubes for which the sets  $U_{uv}$  are isometric. Another reason for this definition was to better understand the structure of partial cubes and to find faster recognition

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algorithms for partial cubes. Along these lines, it was shown in [3] that prism-free almost-median graphs and some related classes of graphs can be recognized in time  $O(m \log n)$ , where  $n$  is the number of vertices and  $m$  the number of edges of a given graph. An algorithm of the same time complexity has been developed in [8] for an additional class of almost-median graphs that in particular includes all planar almost-median graphs. The fastest recognition algorithm for the general partial cube is due to Eppstein [6]. It is also known that median graphs are the almost-median graphs that contain no convex vertex-deleted  $Q_3$ , see [7]. For the position of almost-median graphs in the hierarchy of classes of partial cubes see [4].

Several characterizations of almost-median graphs are known. They are precisely the graphs that can be obtained from a single vertex by a sequence of isometric expansions, where in each expansion covering sets induce almost-median graphs [3]. In addition, they can be characterized among partial cubes with the so-called almost-quadrangle property [10].

Our main motivation is the following characterization due to Brešar [2]: A graph  $G$  is an almost-median graph if and only if  $G$  is a semi-median graph that contains no convex cycle of length greater than four. Using a closer inspection of convex cycles in partial cubes we prove in this note that an absence of long convex cycles characterizes almost-median among all partial cubes.

At least two related results involving forbidden convex subgraphs should be mentioned here. Polat [12] proved that nontrivial netlike partial cubes (the class of graphs introduced in [11]) that contain at most one convex cycle of length greater than four can be characterized with the so-called prism-retractable property. On the other hand, Bandelt and Chepoi [1] proved that graphs of acyclic cubical complexes are precisely median graphs not containing any convex bipartite wheels.

We proceed as follows. In the next section the necessary definitions are collected, while Section 3 contains a key lemma about convex cycles in partial cubes. In the final section the main result (Theorem 4.1) is obtained in two ways and a short proof of the characterization of almost-median graphs using the almost-quadrangle property is given.

## 2 Preliminaries

A subgraph  $H$  of a graph  $G$  is *isometric* if  $d_H(u, v) = d_G(u, v)$  holds for any vertices  $u, v \in H$  and is *convex* if for any vertices  $u, v \in H$ , every  $u, v$ -shortest path in  $G$  lies in  $H$ . For  $X \subseteq V(G)$  we will also say that it is isometric/convex if the subgraph induced on  $X$  is isometric/convex. The *interval*  $I(u, v)$  between vertices  $u$  and  $v$  of a connected graph  $G$  consists of the vertices of  $G$  that lie on shortest  $u, v$ -paths.

A *partial cube* is a graph  $G$  that isometrically embeds into some  $d$ -cube  $Q_d$ . In other words,  $G$  is a partial cube if for some  $d \geq 1$  its vertices can be labeled with strings  $\{0, 1\}^d$  such the distance function of  $G$  coincides with the Hamming distance between the strings.

A graph  $G$  is a *median graph* if to every triple  $u, v, w$  of its vertices there is a

unique vertex  $x$  such that  $d(u, x) + d(x, v) = d(u, v)$ ,  $d(v, x) + d(x, w) = d(v, w)$  and  $d(u, x) + d(x, w) = d(u, w)$ . Median graphs are partial cubes.

For an edge  $uv$  of a graph  $G$  let  $W_{uv} = \{w \mid d(u, w) < d(v, w)\}$ . Let  $F_{uv}$  be the set of edges between  $W_{uv}$  and  $W_{vu}$ , and let  $U_{uv}$  be the set of those vertices of  $W_{uv}$  than have a neighbor in  $W_{vu}$ . If  $G$  is a partial cube, then the sets  $W_{uv}$  and  $W_{vu}$  partition  $V(G)$ . Moreover, the sets  $F_{uv}$ ,  $uv \in E(G)$ , partition  $E(G)$ . The sets  $F_{uv}$  are also known as  $\Theta$ -classes of  $G$  since they coincide with the partition of  $E(G)$  induced by the Djoković-Winkler relation  $\Theta$ . A partial cube  $G$  is called *almost-median* if for any  $uv \in E(G)$  the set  $U_{uv}$  is isometric and is called *semi-median* if for any  $uv \in E(G)$  the set  $U_{uv}$  is connected.

Nonempty isometric subgraphs  $G_1$  and  $G_2$  form an *isometric cover* of a graph  $G$  provided that  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2)$ . If  $G$  is connected then  $G_1 \cap G_2 \neq \emptyset$  for every isometric cover  $G_1, G_2$ . Suppose  $G_1, G_2$  is an isometric cover of  $G$ . For  $i = 1, 2$ , let  $\tilde{G}_i$  be an isomorphic copy of  $G_i$ , and for a vertex  $u \in G_1 \cap G_2$ , let  $u_i$  be the corresponding vertex in  $\tilde{G}_i$ . The *expansion* of  $G$  with respect to  $G_1, G_2$  is the graph  $\tilde{G}$  obtained from the disjoint union of  $\tilde{G}_1$  and  $\tilde{G}_2$ , where for each  $u \in G_1 \cap G_2$  the vertices  $u_1$  and  $u_2$  are joined by a new edge in  $\tilde{G}$ . We say that the expansion is *isometric* provided that  $G_1 \cap G_2$  is an isometric graph. Chepoi [5] proved that a graph is a partial cube if and only if it can be obtained from  $K_1$  by a sequence of expansions.

Let  $F$  be a  $\Theta$ -class of a partial cube  $G$ . Then the *contraction* of  $G$  with respect to  $F$  is the graph  $G'$  obtained from  $G$  by contraction every edge of  $F$ . This operation reverses the expansion, in particular  $G'$  is also a partial cube.

### 3 Convex cycles

Here is our key lemma:

**Lemma 3.1** *Let  $G$  be a partial cube and  $F$  a  $\Theta$ -class of  $G$  containing edges  $e$  and  $f$ . Then either (a) there exists an edge  $g \in F$ ,  $g \neq e, f$ , such that  $d_G(e, f) = d_G(e, g) + d_G(g, f)$ , or (b)  $e$  and  $f$  are contained in a unique shortest cycle, and this cycle is convex in  $G$ .*

**Proof.** Assume that there is no edge  $g \in F$  satisfying (a). Let  $e = e_1e_2$  and  $f = f_1f_2$ , where  $d_G(e_1, f_1) = d_G(e, f) = d_G(e_2, f_2)$ . Consider a shortest cycle  $C$  containing  $e$  and  $f$ . Then  $C$  consists of  $e$ , a shortest path  $A$  from  $e_1$  to  $f_1$ ,  $f$ , and a shortest path  $B$  from  $f_2$  to  $e_2$ . We first prove the following:

**Claim:** If  $a \in A$  and  $b \in B$  then every shortest path between  $a$  and  $b$  passes through  $e$  or through  $f$ .

Every shortest path  $T$  between  $a$  and  $b$  contains an edge  $g$  from  $F$ . Suppose that  $g \neq e, f$  and consider  $I(e_1, f_2)$ . By the way  $C$  is selected,  $a, b \in I(e_1, f_2)$ . As intervals in partial cubes are convex, it follows that  $g_1 \in I(e_1, f_2)$  as well. But then, having

in mind that  $W_{e_1e_2}$  is isometric,

$$d(e_1, f_1) + 1 = d(e_1, f_2) = d(e_1, g_1) + d(g_1, f_2) = d(e_1, g_1) + d(g_1, f_1) + 1$$

and hence  $d(e_1, f_1) = d(e_1, g_1) + d(g_1, f_1)$ . As we have assumed that there is no edge  $g$  satisfying (a), the claim is proved.

As a consequence of the claim, we now have that  $C$  is first isometric and then also convex. Indeed, suppose this is not the case and select a shortest geodesic  $R$  between vertices of  $C$  that is not contained in  $C$ . Let  $a$  and  $b$  be the endpoints of  $R$ . By the assumption, none of the edges (or intermediate vertices) of  $R$  lie on  $C$ . Hence it follows from the claim that  $a$  and  $b$  are both on  $A$  or both on  $B$ . Without loss of generality assume  $a, b \in A$ . Since  $A$  is a geodesic,  $R$  cannot be shorter than the path between  $a$  and  $b$  along  $A$ . This shows that  $C$  is isometric. To show convexity, consider the cycle  $C'$  obtained from  $C$  where the part between  $a$  and  $b$  is substituted by  $R$ . Then  $C'$  has the same property as  $C$ , in particular it is also isometric. Let  $ac$  be the first edge on  $R$ . The edge  $t$  opposite to  $ac$  on  $C'$  lies on  $B$  which is common for  $C$  and  $C'$ . In particular,  $t$  is opposite on  $C$  to an edge  $ac'$ . Since  $ac$  and  $t$  are opposite in an isometric cycle, they are in the same  $\Theta$ -class. Similarly,  $ac'$  and  $t$  are in the same  $\Theta$ -class. It follows that  $ac$  and  $ac'$  are in the same  $\Theta$ -class, a contradiction.

Note that since we have proved that  $C$  is convex,  $A$  is the only shortest path between  $e_1$  and  $f_1$  and similarly  $B$  is the only shortest path between  $e_2$  and  $f_2$ . Hence  $C$  is the unique shortest cycle containing  $e$  and  $f$ .

Note also that the claims (a) and (b) exclude each other. Indeed, if we have an intermediate edge  $g \in F$ , then there is a shortest path from  $e_1$  to  $f_1$  via  $g_1$  and similarly from  $e_2$  to  $f_2$  via  $g_2$ . This produces a shortest cycle which is not convex.  $\square$

The structure of convex cycles was in [9] encoded as follows. Let  $F$  be a  $\Theta$ -class of a partial cube  $G$ . Then the  $F$ -zone graph,  $Z_F$ , is the graph with  $V(Z_F) = F$ , vertices  $f$  and  $f'$  being adjacent in  $Z_F$  if they belong to a common convex cycle of  $G$ . We now extend the concept of the zone graph to its weighted version as follows. To every such edge  $Z_F$  we assign the weight  $\frac{k-2}{2}$ , where  $k = |C|$  and  $C$  is the convex cycle representing the edge. Note that this weight is exactly the distance in  $G$  between the two edges of  $F$  lying on  $C$ . Moreover, Lemma 3.1 immediately implies the following:

**Corollary 3.2** *The weighted distance in  $Z_F$  between  $e, f \in F$  coincides with  $d_G(e, f)$ . Furthermore, if there is an edge connecting  $e$  and  $f$ , then this is the unique shortest weighted path between  $e$  and  $f$ .*

## 4 The main result

We are now ready for the main result of this note.

**Theorem 4.1** *Let  $G$  be a partial cube. Then  $G$  is almost-median if and only if  $G$  contains no convex cycles of length more than four.*

**Proof.** It was proven in [2] that every almost-median graph has no convex cycle of length more than four. Hence we just need to show the converse. Suppose  $G$  is a partial cube without convex cycles of length more than four. According to the definition of almost-median graphs we need to verify that  $U_{ab}$  is isometric for every edge  $ab$ . Let  $F$  be the  $\Theta$ -class containing  $ab$ . Suppose that there are edges  $e, f \in F$  such that  $d_G(e_1, f_1) < d_{U_{ab}}(e_1, f_1)$ , where  $e_1$  and  $f_1$  are endpoints of  $e$  and  $f$  lying in  $U_{ab}$ . We may assume that  $e$  and  $f$  are selected so that  $d_G(e_1, f_1)$  is minimal. This condition forces that there is no intermediate edge  $g \in F$  as in case (a) of Lemma 3.1. By this lemma it follows that  $e$  and  $f$  lie in a unique shortest cycle which is convex. By our assumption, this cycle cannot be of length more than four, which means that  $e_1$  and  $f_1$  are adjacent, a contradiction.  $\square$

Another way to deduce Theorem 4.1 is to apply Corollary 3.2. Indeed, the theorem follows from the lemma since in the zone graph all the edge-weights are 1 in the absence of long convex cycles. Therefore, the weighted distance in  $Z_F$  is the same as the path distance.

We conclude this note with a short proof (based on Theorem 4.1) of the characterization of almost-median graphs due to Peterin [10]. For it we need the following definition. A graph  $G$  satisfies the *almost-quadrangle property* if for any vertices  $u, w, x, y$  such that  $d(u, x) = d(u, y) = k = d(u, w) - 1$  and  $w$  is adjacent to  $x$  and  $y$ , there exists an edge  $ab$  such that  $axwb$  is an induced  $C_4$  of  $G$  and  $d(u, a) = k - 1$ .

**Corollary 4.2** ([10, Corollary 6]) *Let  $G$  be a partial cube. Then  $G$  is almost-median if and only if  $G$  satisfies the almost-quadrangle property.*

**Proof.** Suppose  $G$  is almost-median and let  $u, w, x, y$  be vertices of  $G$  such that  $d(u, x) = d(u, y) = k = d(u, w) - 1$  and  $w$  is adjacent to  $x$  and  $y$ . Let  $P$  be a shortest  $x, u$ -path and  $Q$  a shortest  $y, u$ -path. Let  $F$  be the  $\Theta$ -class of  $G$  containing  $xw$ . Then  $Q$  contains an edge  $x'w' \in F$ , where  $d(x', x) < d(x', w)$ . Note that  $x' \in I(u, x)$ . Since  $G$  is almost-median,  $U_{xw}$  contains a geodesic  $P'$  between  $x$  and  $x'$ . Let  $a$  be the neighbor of  $x$  on  $P'$  and let  $b$  be the neighbor of  $a$  in  $U_{wx}$ . Then  $ab$  is the required edge, hence  $G$  satisfies the almost-quadrangle property.

Conversely, suppose  $G$  satisfies the almost-quadrangle property. If  $G$  contained a convex cycle of length more than four, then we could choose  $u$  and  $w$  to be opposite vertices on the cycle and then by the convexity  $a$  must be on the cycle and hence  $b$  must also be on the cycle, a contradiction. We conclude that  $G$  is almost-median by Theorem 4.1.  $\square$

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