Almost self-centered median and chordal graphs

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Abstract

Almost self-centered graphs were recently introduced as the graphs with exactly two non-central vertices. In this paper we characterize almost self-centered graphs among median graphs and among chordal graphs. In the first case P_4 and the graphs obtained from hypercubes by attaching to them a single leaf are the only such graphs. Among chordal graph the variety of almost self-centered graph is much richer, despite the fact that their diameter is at most 3. We also discuss almost self-centered graphs among partial cubes and among k-chordal graphs, classes of graphs that generalize median and chordal graphs, respectively. Characterizations of almost self-centered graphs among these two classes seem elusive.

Key words: radius, diameter, almost self-centered graph, median graph, chordal graph

1 Introduction

Centrality notions lie in the very center of (discrete) location theory, self-centered graphs [1, 5, 14, 21] forming a prominent theoretical model, see also the survey [4]. Their importance lie in the fact that the maximum eccentricity of any vertex is as small as possible which in turn allows different efficient locations of the emergency facilities at central locations. In some situations, however, we would like to have certain resources not to lie in the center of a graph. With this motivation, almost self-centered graphs were introduced in [17] as the graphs with exactly two non-central vertices. In the seminal paper constructions that produce almost self-centered graphs are described, and embeddings of graphs into smallest almost self-centered graphs are considered. In the present paper we continue these studies by considering almost self-centered graphs among median graphs, chordal graphs, and their generalizations.

Median graphs are probably the most extensively studied class in all metric graph theory. For a survey on median graphs dealing with their characterizations, location theory, and related structures see [16], and for more on these graphs see the book [11] and recent papers [6, 12]. Here we only emphasize that despite the fact that median graphs are bipartite, they are intimately connected with triangle-free graphs [13].

Chordal graphs, defined as the graphs having no induced cycles of length greater than 3, are by far the most investigated class of graphs, see e.g. [2]. They have been studied from numerous aspects and generalized in several ways. A very natural generalization are the so-called k-chordal graphs, in which by definition the longest induced cycles are of length k. The largest common subclass of chordal and median graphs are trees, indicating the tree-like structure of both classes. On the other hand, chordal and median graphs have a common generalization through the so-called cage-amalgamation graphs [3], for which certain tree-like equalities were proven that generalize such equalities in median graphs (counting the numbers of induced hypercubes) and in chordal graphs (counting the numbers of induced cliques).

The paper is organized as follows. In the next section definitions needed and concepts considered are collected. In Section 3 self-centered and almost self-centered median graphs are characterized and related partial cubes are considered. In Section 4 we concentrate on chordal graphs and prove that the diameter of a chordal almost self-centered graph is not more than 3. We follow with a characterization of almost self-centered chordal graph and provide several infinite subclasses of them. In the final section k-chordal graphs are considered and proved that the diameter of a k-chordal almost self-centered graph with $k \geq 4$ is at most k. A characterization of k-chordal almost self-centered graphs remains an open problem.

2 Preliminaries

The *distance* considered in this paper is the usual shortest path distance d. A shortest path between vertices u and v will also be called a *u,v-geodesic*. The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v is the distance to a farthest vertex from v. A vertex v is said to be an *eccentric vertex of u* if $d(u,v) = \mathrm{ecc}(u)$. The *radius* $\mathrm{rad}(G)$ of G and the *diameter* $\mathrm{diam}(G)$ of G are the minimum and the maximum eccentricity, respectively. A vertex u with $\mathrm{ecc}(u) = \mathrm{rad}(G)$ is called a *central vertex*, and it is *diametrical* if $\mathrm{ecc}(u) = \mathrm{diam}(G)$ holds. A graph G is *self-centered graph* if all vertices are central (equivalently, all vertices are diametrical), and is *almost self-centered graph* if the center of G consists of $|V(G)| - 2$ vertices.

For a connected graph and an edge xy of G we denote

$$W_{xy} = \{w \in V(G) \mid d(x,w) < d(y,w)\}.$$

Note that if G is a bipartite graph then $V(G) = W_{ab} \cup W_{ba}$ holds for any edge ab. Next, for an edge xy of G let U_{xy} denote the set of vertices that are in W_{xy} and have a neighbor in W_{yx}. Sets in a graph that are U_{xy} for some edge xy will be called *U-sets*. Similarly we define *W-sets*.

The *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (g,h) is adjacent to the vertex (g',h') whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The Cartesian product is commutative and associative, the product of n copies of K_2 is the n-dimensional *hypercube* or *n-cube* Q_n. With Q_n^+ we denote the graph obtained from Q_n by attaching a pendant vertex to a vertex of Q_n, while Q_n^- denotes the graph obtained from Q_n by removing one of its vertices. (Q_n is vertex-transitive, hence these two graphs are well-defined.) It is straightforward to see that if G and H are self-centered graphs, then so is $G \,\Box\, H$.

A (connected) graph G is a *median graph* if for any three vertices x, y, z there exists a unique vertex that lies in $I(x,y) \cap I(x,z) \cap I(y,z)$. (Here $I(u,v)$ denotes the set of vertices on all u,v-geodesics, that is, the *interval* between u and v.) If uv is an edge of a median graph, then the set of edges between U_{uv} and U_{vu} form a matching. Two of the most important classes of median graphs are trees and hypercubes. For the next result see [11, Lemma 12.20]:

Proposition 2.1 *Let G be a median graph. Then G is a hypercube if and only if $W_{uv} = U_{uv}$ holds for any edge uv of G.*

A subgraph H of G is *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$ and a graph G is a *partial cube* if it is an isometric subgraph of some Q_n, see [11, 22]. It is well-known that median graphs are partial cubes but not the other way around.

3 Almost self-centered median graphs and partial cubes

We begin with the following strengthening of a result of Mulder from [20] asserting that the same conclusion holds provided each vertex of a median graph has a unique diametrical vertex.

Proposition 3.1 *Let G be a median graph. Then G is self-centered if and only if G is a hypercube.*

Proof. Clearly, hypercubes are self-centered. For the converse it suffices (in view of Proposition 2.1) to prove that $W_{uv} = U_{uv}$ holds for any edge uv of G.

Suppose on the contrary that $W_{uv} \setminus U_{uv} \neq \emptyset$ for some edge $uv \in E(G)$. Then, since W_{uv} is connected, there exists a vertex x in $W_{uv} - U_{uv}$ that is adjacent to a vertex y in U_{uv}. Let y' be the neighbor of y in W_{vu}. Then, since $W_{uv} = W_{yy'}$ and $U_{uv} = U_{yy'}$, we may without loss of generality assume that $u = y$, that is, we may assume that x is adjacent to u. Note that any eccentric vertex $\overline{u}$ of u lies in W_{vu}, for otherwise $\text{ecc}(v) > ecc(u)$. But then $d(x, \overline{u}) = 1 + d(u, \overline{u})$, a contradiction. Hence $W_{uv} = U_{uv}$ holds. $\square$

Theorem 3.2 *Let G be a median graph. Then G is almost self-centered if and only if G is either P_4 or Q_n^+ for some $n \geq 1$.*

Proof. It is straightforward to see that P_4 and Q_n^+, $n \geq 1$, are almost self-centered graphs.

Let now G be an arbitrary almost self-centered median graph. Since G is not self-centered, Propositions 3.1 and 2.1 imply that there exists an edge $uv \in E(G)$ and a vertex $x \in W_{uv} - U_{uv}$. Select the edge uv and the vertex x such that $xu \in E(G)$. Let $\overline{u}$ be an eccentric vertex of u.

Case 1: $\overline{u} \in W_{vu}$.

Then $d(u, \overline{u}) = d$ and $d(x, \overline{u}) = d + 1$, that is, x and $\overline{u}$ are diametrical vertices in this case. We claim that $W_{uv} - U_{uv} = \{x\}$ and suppose on the contrary that there exists $y \in W_{uv} - U_{uv}$, $y \neq x$. Suppose first that y is adjacent to $y' \in U_{uv}$. Let y'' be the neighbor of y' in U_{vu}. If an eccentric vertex $\overline{y'}$ of y' lies in W_{vu}, then $d(y, \overline{y'}) = d + 1$, a contradiction. Therefore, $\overline{y'} \in W_{uv}$. Then $d(y'', \overline{y'}) = d + 1$ hence $\overline{y'} = x$ and $y'' = \overline{u}$. Then $d(y, v) = d + 1$, another contradiction.

Hence u is the only vertex of U_{uv} that has a neighbor in $W_{uv} - U_{uv}$. It is now clear that x is the unique vertex from $W_{uv} - U_{uv}$ for otherwise we would have more than two diametrical vertices or diameter bigger than $d + 1$. So we have proved that $W_{uv} - U_{uv} = \{x\}$.

Assume $\overline{u} \in W_{vu} - U_{vu}$. We are going to show that $W_{vu} - U_{vu} = \{\overline{u}\}$. Let y be an arbitrary vertex from $W_{vu} - U_{vu}$ with a neighbor $z \in U_{vu}$. Let $\overline{z}$ be an eccentric vertex of z. Then $\overline{z} \in W_{uv}$, for otherwise, the neighbor of z in U_{uv}

would have eccentricity $d+1$. Since $d(y,\overline{z}) = d+1$, we find that $\overline{z} = x$ and $y = \overline{u}$. We conclude that $W_{vu} - U_{vu} = \{\overline{u}\}$ by the same reasons as above. Let the neighbor of $\overline{u}$ in U_{vu} be z'' and let z' be the neighbor of z'' in U_{uv}. If an eccentric vertex of z' lies in W_{uv}, then the eccentricity of $\overline{u}$ is $d+2$, which is not possible. So an eccentric vertex $\overline{z'}$ of z' has to lie in W_{vu}. If $\overline{z'} \in U_{vu}$, then $\overline{u}$ has an eccentric vertex (the neighbor of $\overline{z'}$ in U_{uv}) different from x. Therefore, $\overline{z'} = \overline{u}$. This means that $d = 2$ and consequently $G = P_4$.

Suppose $\overline{u} \in U_{vu}$. By similar arguments as before $W_{vu} = U_{vu}$. Clearly, an eccentric vertex of $y \in U_{uv}$ lies in U_{vu}, for any y. Since $G - x = \langle U_{uv} \rangle \,\square\, K_2$, we also have that an eccentric vertex of $y' \in U_{vu}$ lies in U_{uv}, for any y'. Then $G - x$ is a hypercube by Proposition 2.1 and therefore $G = Q_n^+$.

Case 2: $\overline{u} \in W_{uv}$.
In this case, $\overline{u} \notin U_{uv}$, for otherwise u would have eccentricity $d+1$. By interchanging the roles of x and v we are in Case 1. $\square$

In the rest of the section we consider (almost) self-centered partial cubes. Even cycles form an example of self-centered partial cubes, and we can expect that the list of (almost) self-centered graphs will be considerably larger than for median graphs. However, their characterization seems difficult, just as it is difficult to characterize regular (in particular cubic) partial cubes, see [9, 15, 18]. We give a construction, based on an expansion procedure, that gives rise to new self-centered partial cubes from smaller ones. Again, as for median graphs, they give rise to almost self-centered graphs by adding a pendant vertex. However this are not the only almost self-centered partial cubes as we will see at the end of the section.

Let G_1 and G_2 be isometric subgraphs of a graph G such that $G_1 \cup G_2 = G$ and $G' = G_1 \cap G_2 \neq \emptyset$. Note that there is no edge from $G_1 \backslash G'$ to $G_2 \backslash G'$. Then the *expansion of G with respect to G_1 and G_2* is the graph H defined as follows. Take disjoint copies of G_1 and G_2 and connect every vertex from G'

in G_1 with the same vertex of G' in G_2 with an edge. It is not hard to see that copies of G' in G_1 and in G_2 and new edges between those two copies form the Cartesian product $G' \Box K_2$. Chepoi [7] has shown that G is a partial cube if and only if G can be obtained from K_1 by a sequence of expansions. Similar expansion theorem was shown for median graphs earlier by Mulder [19].

We call the expansion H of G with respect to G_1 and G_2 a *diametrical expansion* whenever for any diametrical pair of vertices u and $\overline{u}$ of G either both $u, \overline{u} \in V(G')$ or $u \in V(G_1 \backslash G')$ and $\overline{u} \in V(G_2 \backslash G')$.

Theorem 3.3 *Let G be a self-centered partial cube and let H be obtained from G by a diametrical expansion. Then H is a self-centered partial cube.*

Proof. Let H be a diametrical expansion of G with respect to G_1 and G_2 where $G' = G_1 \cap G_2$. Then H is a partial cube by Chepoi's theorem.

Let $\mathrm{ecc}_G(g) = d$ for any $g \in V(G)$. Let h be an arbitrary vertex of H. Then h must be in either $G_1 \backslash G'$, $G_2 \backslash G'$ or $G' \Box K_2$. First we assume that $h \in V(G_1 \backslash G')$. Then $\overline{h}$ must be in $G_2 \backslash G'$, since H is a diametrical expansion. Also $\mathrm{ecc}_H(h) = d + 1$. Namely, to see $\mathrm{ecc}_H(h) \geq d + 1$ we can take the same path as in G between h and $\overline{h}$ that is extended by a new edge of expansion, and in addition $\mathrm{ecc}_H(h) > d+1$ would yield a contradiction with $\mathrm{ecc}_G(h) = d$. By symmetry we also have $\mathrm{ecc}_H(h) = d + 1$ for $h \in V(G_2 \backslash G')$. Next let $h \in V(G' \Box K_2)$. Then $h, \overline{h} \in V(G')$. Let $\overline{h_1} \in V(G_1)$ and $\overline{h_2} \in V(G_2)$ be two copies of $\overline{h}$ in H. If $h \in G_1$ then $d_H(h, \overline{h_2}) = d + 1$ and if $h \in G_2$ then $d_H(h, \overline{h_1}) = d + 1$. Again $\mathrm{ecc}_H(h) > d + 1$ would yield a contradiction with $\mathrm{ecc}_G(h) = d$ and we have $\mathrm{ecc}_H(h) = d + 1$. □

An example of diametrical expansion is shown in Figure 1. Since Q_3 is a self-centered partial cube, so is the expanded graph on the right hand side of the figure.

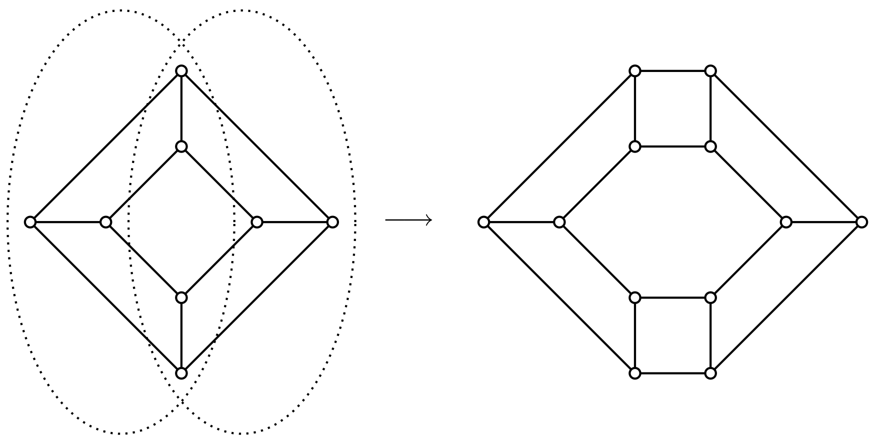

Figure 1: A diametrical expansion of Q_3

It is easy to see (by induction for instance) that if a partial cube is obtained from K_1 by a series of diametrical expansions, then every vertex has a unique diametrical vertex. We can obtain almost self-centered partial cubes from self-centered partial cube G by attaching a pendant vertex to a vertex with the unique diametrical vertex in G. However this is not the only possibility. Another family of almost self-centered partial cubes arise from Q_n^- by attaching a pendant vertex to a vertex of degree $n-1$. Note that for $n = 2$ we get the sporadic example P_4 of median graphs.

We can generalize the above idea as follows. Let G be a self-centered graph, and let G_1 and G_2 be isometric subgraphs of G such that the expansion of G with respect to G_1 and G_2 is "almost diametrical", that is, there is exactly one pair $(u, \overline{u})$ of diametrical vertices with the property $u \in G_1 \backslash G'$ and $\overline{u} \in G'$ and for all other diametrical pairs the condition of diametrical expansion holds. Such an expansion does not produce self-centered graphs, but if we attach a pendant vertex to u we get an almost self-centered graph.

4 Chordal graphs

In this section we characterize almost self-centered chordal graphs. For this purpose we first show:

Theorem 4.1 *Let G be a chordal, almost self-centered graph. Then* $\mathrm{diam}(G) \leq 3$.

Proof. Suppose on the contrary that G is a chordal almost self-centered graph with diam $= k \geq 4$. Let x and $\overline{x}$ be diametrical vertices with $\mathrm{ecc}(x) = \mathrm{ecc}(\overline{x}) = k$ and let $P : (x =)u_0u_1 \ldots u_k(= \overline{x})$ be an $x, \overline{x}$-diametrical path. Then $\mathrm{ecc}(u) = k - 1$ for all the other vertices $u \in V(G) - \{x, \overline{x}\}$. Hence there exists $\overline{u_2}$ with $d(u_2, \overline{u_2}) = k - 1$.

Since $d(u_2, \overline{u_2}) = k - 1$, we have $k - 3 \leq d(x, \overline{u_2}) \leq k - 1$. Let $Q : (x =)v_0v_1 \ldots v_q(= \overline{u_2}), k - 3 \leq q \leq k - 1$, be a shortest $x, \overline{u_2}$-path. Note that it is possible that $v_1 = u_1$, but all other vertices of P and Q are different. For if a vertex u_s, where $s \geq 2$, belongs to both P and Q, then u_2 is an inner vertex on a $x, \overline{u_2}$-geodesic of length at most $k - 1$, a contradiction with $d(u_2, \overline{u_2}) = k - 1$.

Let $R : (\overline{x} =)w_0w_1 \ldots w_r(= \overline{u_2})$, $2 \leq r \leq k - 1$, be a $\overline{u_2}, \overline{x}$-geodesic (note that the case when $\overline{x}$ and $\overline{u_2}$ are adjacent is not excluded). Let v_j and u_i be the first vertices of Q and P, respectively, that are also on R. Suppose $u_1 \neq v_1$. The x, u_i-subpath of P, the x, v_j-subpath of Q, and the u_i, v_j-subpath of R form a cycle C. Clearly, u_2 does not form a chord with any vertex of P, Q, and R since $d(u_2, \overline{u_2}) = k - 1$, with a possible exception of v_1 and w_1. The later case is possible only when $d(\overline{x}, \overline{u_2}) = k - 1$, since otherwise $d(u_2, \overline{u_2}) < k - 1$ which is not possible. But then clearly $d(x, \overline{x}) = 4$.

Let first $k > 4$. If u_2v_1 is not a chord, then u_2 has no chords on C and we are done since also u_1u_3 is not a chord. So we may assume that u_2v_1 is a chord (which also includes the case when $u_1 = v_1$). Then let C' be a cycle obtained from u_2v_1 and a longer u_2, v_1-path on C. Then u_2 has no chords on C' and u_3 and v_1 are not adjacent since $d(x, \overline{x}) = k$. Hence the same contradiction again.

Finally let $k = 4$. If u_2w_1 is not a chord we have the same contradiction as before. So let $u_2w_1 \in E(G)$ and let C' be a cycle obtained from u_2, w_1-path on C that does not contains $\overline{x}$ and u_2w_1. If $u_2v_1 \notin E(G)$ no chord on C' starts in u_2. We get a contradiction, since edge u_1w_1 would destroy $d(x, \overline{x}) = 4$. Hence $u_2v_1 \in E(G)$ and let C'' be a cycle obtained from edge u_2v_1 and u_2, v_1-path on C' that does not contains u_1. (Note that if $u_1 = v_1$ we have $C'' = C'$.) But again u_2 has no chords on C'' and v_1w_1 is again not possible since $d(x, \overline{x}) = 4$, a final contradiction. □

We introduce the class $\mathcal{C}$ of chordal graphs as follows. Let G' be a chordal graph with diameter at most 2, and let $V(G') = X + Y + Z$ (where $+$ stands for the disjoint union of sets) such that for any $v \in V(G')$, we have $d(v, X) \leq 1$ and $d(v, Y) \leq 1$, with only Z being possibly empty. (Note that it means any vertex from X must have a neighbor in Y and vice versa and a vertex in Z, if any, must have a neighbor in both X and Y.) Let G be obtained from G' by adding two new vertices x and y, and edges between x and all vertices from X, and y and all vertices from Y. Clearly, any graph G, obtained in such a way is chordal with diameter 3, and we say it belongs to the class $\mathcal{C}$. It is also clear that only x and y are diametrical vertices, and all other vertices have the same eccentricity, making G almost self-centered chordal graph.

The class $\mathcal{C}$ is relatively rich. It includes the graphs, obtained from a clique by adding two vertices with disjoint neighborhoods in the clique (say, P_4 as the smallest example. Another subclass is obtained from the join $K_n \circ \overline{K_m}$ of the complete graph K_n and totally disconnected graph $\overline{K_m}$, by adding two simplicial vertices, whose neighborhoods are disjoint subcliques of K_n.

Theorem 4.2 *Let G be a chordal graph. Then G is almost self-centered if and only if G is either $K_n - e$ or it belongs to $\mathcal{C}$.*

Proof. If G is either $K_n - e$ or in $\mathcal{C}$, it is clearly almost self-centered.

For the converse, let G be an almost self-centered chordal graph. By Theorem 4.1, G has diameter at most 3. Assume $\mathrm{diam}(G) = 2$. Since G is not a complete graph, there exist two non adjacent simplicial vertices x and y in G by Dirac's theorem [8]. Clearly $N(x) \cap N(y) \neq \emptyset$. In addition, if there is a vertex $z \notin N[x] \cap N[y]$ then $\mathrm{ecc}(z) = 2$ which is a contradiction with G being almost self-centered. By the same reasoning, we find that $N(x) = N(y)$ induces a clique. Hence G is isomorphic to $K_n - e$.

Suppose $\mathrm{diam}(G) = 3$. Then by a result of Farber and Jamison [10] there exist two simplicial vertices x and y with $d(x, y) = \mathrm{diam}(G) = 3$. Let $X = N(x)$ and $Y = N(y)$. Since $d(x, y) = 3$, $X \cap Y = \emptyset$ and let $Z = V(G) - (N[x] \cup N[y])$. Clearly, the subgraph G' induced by $V(G) - \{x, y\}$ is chordal and its diameter is at most 2. Note that for any $v \in V(G')$, $\mathrm{ecc}_G(v) = 2$. If $v \in X$ then there must be a vertex $w \in Y$ such that $vw \in E(G)$, otherwise $d(v, y) = 3$. We infer that $d(v, Y) = 1$, and similarly we find that $d(w, X) = 1$ for any $w \in Y$. If $z \in Z$, again by eccentricity 2 of vertices from G', we find that $d(z, X) = 1$ and $d(z, Y) = 1$. We derive that G belongs to the class $\mathcal{C}$. $\square$

5 k-chordal graphs

A graph G is k-chordal if every cycle C of length greater than k has a chord. The chordality of G is the smallest k such that G is k-chordal. For k-chordal graphs Theorem 4.1 naturally extends:

Theorem 5.1 *Let G be a k-chordal almost self-centered graph with $k \geq 4$. Then* $\mathrm{diam}(G) \leq k$.

Proof. Suppose on the contrary that G is a k-chordal almost self-centered graph with diam(G) $= r \geq k + 1$. Let x and $\overline{x}$ be diametrical vertices with

$\text{ecc}(x) = \text{ecc}(\overline{x}) = r$ and let $P : (x =)u_0u_1 \dots u_r(= \overline{x})$ be a $x, \overline{x}$-diametrical path. Then $\text{ecc}(u) = r - 1$ for all the other vertices $u \in V(G) - \{x, \overline{x}\}$. For $a = \lceil \frac{k-1}{2} \rceil + 1$ there exists $\overline{u_a}$ with $d(u_a, \overline{u_a}) = r - 1$.

Since $d(u_a, \overline{u_a}) = r - 1$, we have $r - a - 1 \leq d(x, \overline{u_a}) \leq r - 1$. Let $Q : (x =)v_0v_1 \dots v_q(= \overline{u_a}), r-a-1 \leq q \leq r-1$, be a shortest $x, \overline{u_a}$-path. Note that it is possible that $v_i = u_i$ for $1 \leq i < a$, but all other vertices of P and Q are different. For if a vertex u_s, $s \geq a$, belongs to both P and Q, then u_a is an inner vertex on a shortest $x, \overline{u_a}$-path of length at most $r - 1$, a contradiction with $d(u_a, \overline{u_a}) = r - 1$.

Let $R : (\overline{x} =)w_0w_1 \dots w_t(= \overline{u_a})$, $1 \leq t \leq r - 1$ be a shortest $\overline{u_a}, \overline{x}$-path. Let u_ℓ be the last vertex common to P and Q, while v_p and $u_s = w_{r-s}$ be the first vertices of Q and P, respectively, that are also on R. Suppose that there exists a chord u_av_b for some $\ell \leq b \leq p$. Then $q - b + 1 \geq d(u_a, \overline{u_a}) = r - 1$ and hence $b \leq q - r + 2 \leq r - 1 - r + 2 = 1$. Clearly $u_av_0 = u_ax \notin E(G)$ and $u_av_1 \in E(G)$ imply a contradiction with $d(x, \overline{x}) = r \geq k + 1$. Hence there is no chord u_av_b. Similarly, if there exists a chord u_aw_b for some $r - s \leq b \leq p$, we have $t - b + 1 \geq r - 1$ and again $b \leq 1$. As before edge u_aw_0 is not possible, but edge u_aw_1 can exists when $r = 5$ and $k = 4$. For $r > 5$ this is not possible since we violate $d(x, \overline{x}) = r$.

Assume first that $u_aw_1 \notin E(G)$. Fix edges u_bv_c, u_dw_e, and v_fw_g, $f \geq c$, $g \geq e$, and $b < a < d$, as follows. Let $b < a$ be the biggest number with an edge u_bv_y and among all such edges let c also be the biggest number. Note that such an edge always exists, since $u_{\ell+1}v_\ell$ is such an edge. Similarly let $d > a$ be smallest number with an edge u_dw_y and among all such edges choose e to be the biggest number. Again such an edge exists since $u_{s-1}u_s$ is of that type. Finally let $f \geq c$ be the smallest number such that edge v_fw_g exists, where $g \geq e$ is also small as possible. Since $v_{p-1}v_p$ is a candidate for this edge, there is no problem with the existence of such an edge. We construct cycle C

as follows

$$u_b v_c \xrightarrow{Q} v_f w_g \xrightarrow{R} w_e u_d \xrightarrow{P} u_b.$$

By minimality or maximality of b, c, d, e, f, g it is clear that C is chordless. We gain the contradiction by showing that C has length $> k$.

First note that since $d(x, \overline{x}) = r$, v_f is not adjacent to w_g for $g < r - f - 1$. Thus $g \geq r - f - 1$. Clearly

$$|C| = d - b + f - c + g - e + 3 \geq r + 2 + d - b - c - e.$$

We will show that $2 + d - b - c - e \geq 0$ or equivalently $d - b + 2 \geq c + e$. Let P' be a path $u_a \xrightarrow{P} u_d w_e$ and P'' a path $u_a \xrightarrow{P} u_b v_c$. Note that $P' \cap P'' = \{u_a\}$ and that the length of both is $d - b + 2$. If $c > |P''|$ we have $q = d(x, \overline{u_a}) > |P''| + q - c \geq r - 1$, a contradiction. Similarly if $e > |P'|$ we have $t = d(\overline{x}, \overline{u_a}) > |P'| + t - e \geq r - 1$, a contradiction again. Thus $c \leq |P''|$ and $e \leq |P'|$ and $d - b + 2 \geq c + e$ follows. Hence C is a chordless cycle of length $|C| \geq r + 2 + d - b - c - e \geq r > k$ which is not possible in k-chordal graphs.

Finally let $u_a w_1 \in E(G)$. Then $r = 5$, $k = 4$, and in this case $a = 3$. Since $d(u_3, \overline{u_3}) = 4$ it is easy to see that u_2v_2, u_2v_1, u_1v_1, and u_1v_0 are all possible edges of type u_bv_c. Instead of u_dw_e we take u_3w_1 and for edge v_fw_g we have the following possibilities: v_1w_3, v_2w_2, v_2w_3, v_3w_2, and v_3w_3 (whenever this vertices exists). It is easy to see that combining this edges we always get a chordless cycle of length at least 5, which is not possible in 4-chordal graphs. □

As a direct consequence of Theorem 5.1 we get:

Corollary 5.2 *If G is an almost self-centered graph of chordality k, then $diam(G) \leq k$.*

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