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On median nature and enumerative properties of Fibonacci-like cubes☆

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Abstract

Fibonacci cubes, extended Fibonacci cubes, and Lucas cubes are induced subgraphs of hypercubes defined in terms of Fibonacci strings. It is shown that all these graphs are median. Several enumeration results on the number of their edges and squares are obtained. Some identities involving Fibonacci and Lucas numbers are also presented.

Keywords: (Extended) Fibonacci cube; Lucas cube; Median graph; Cartesian product; Fibonacci number; Lucas number

1. Introduction

Several classes of graphs based on Fibonacci strings were introduced in the last 10 years as models for interconnection networks. Fibonacci cubes were defined in [5,6], followed with extended Fibonacci cubes [19] and Lucas cubes [15]. Different structural properties of the Fibonacci cubes were studied in [3,9,10,16,17], while for additional information on extended Fibonacci cubes and Lucas cubes see [18] and [3], respectively.

The vertex set of the *n-cube* Q_n consists of all binary strings of length n, two vertices being adjacent if the corresponding strings differ in precisely one place. Note that $Q_1 = K_2$ and $Q_2 = C_4$. We also set $Q_0 = K_1$. The Fibonacci numbers F_n are defined with $F_1 = F_2 = 1$,

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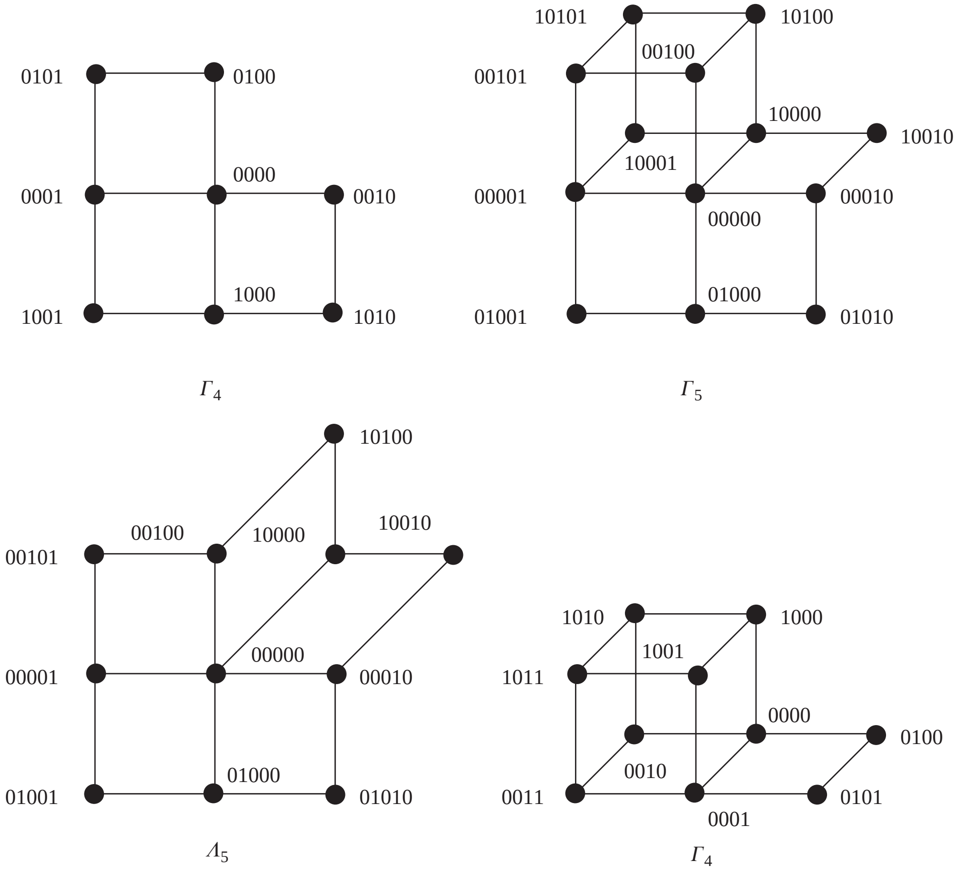

Fig. 1. Γ_4, Γ_5, Λ_5, and Γ_4^1.

$F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$, and the Lucas numbers with $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$.

A *Fibonacci string* of length n is a binary string $b_1 b_2 \ldots b_n$ with $b_i b_{i+1} = 0$ for $1 \leqslant i < n$, that is, a binary string without two consecutive ones. The *Fibonacci cube* Γ_n ($n \geqslant 1$) is the subgraph of Q_n induced by the Fibonacci strings of length n. For convenience we also set $\Gamma_0 = K_1$. Call a Fibonacci string $b_1 b_2 \ldots b_n$ a *Lucas string* if $b_1 b_n = 0$. Then the *Lucas cube* Λ_n ($n \geqslant 2$) is the subgraph of Q_n induced by the Lucas strings of length n. For $n \geqslant i \geqslant 0$, the ith *extended Fibonacci cube of order* n, Γ_n^i, is defined as follows. Let B_i be the set of all binary strings of length i. Then the vertex set V_n^i of Γ_n^i is defined recursively by $V_{n+2}^i = 0V_{n+1}^i \cup 10V_n^i$, with initial conditions $V_i^i = B_i$, $V_{i+1}^i = B_{i+1}$. Note that then $\Gamma_i^i = Q_i$ and $\Gamma_{i+1}^i = Q_{i+1}$. Observe also that $\Gamma_n^0 = \Gamma_n$.

The Fibonacci cubes Γ_4 and Γ_5, the Lucas cube Λ_5, and the extended Fibonacci cube Γ_4^1, are, together with the corresponding vertex labels, shown in Fig. 1.

For a graph G let, as usual, $V(G)$ be its vertex set and $E(G)$ its edge set. In addition, let $S(G)$ be the set of all induced squares (that is, 4-cycles) of G.

Let G be a graph. Then a *median* of vertices u, v, w is a vertex that simultaneously lies on a shortest u, v-path, a shortest u, w-path, and a shortest v, w-path. A connected graph is called a *median graph* if every triple of its vertices has a unique median. Note that n-cubes are median graphs. For more information on these graphs we refer to the survey [8], books [7,12], and classical references [1,11].

In the next section we show that all the Fibonacci-like cubes introduced above are median graphs. In particular, we construct an explicit retraction from Γ_n onto Λ_n and mention an expansion property of the Fibonacci and Lucas cubes. In the last section we use the expansion property to obtain several enumerative results concerning the number of edges and squares in these cubes, which in turn enable us to obtain some identities involving Fibonacci and Lucas numbers.

2. Fibonacci-like cubes as median graphs

Median graphs form an important and well-studied class of graphs. In this section we show that all the introduced Fibonacci-like cubes are median graphs which could be useful for studies of the related interconnection networks.

It is well known that a connected graph G is a median graph if and only if G is an induced subgraph of an n-cube such that with any three vertices of G their median in the n-cube is also a vertex of G. The result is due to Mulder [11], see also [2,14] for alternative proofs. We say a subgraph H of a graph G is *median closed* if, with any triple of vertices of H, their median is also in H.

Theorem 1. *Γ_n^i is a median graph for any $n \geqslant i \geqslant 0$, and Λ_n is a median graph for any $n \geqslant 2$.*

Proof. Clearly, any Γ_n^i, $n \geqslant i \geqslant 0$, and any Λ_n, $n \geqslant 2$, is an induced subgraph of Q_n. We first show that $\Gamma_n^0 = \Gamma_n$ is a median closed subgraph of Q_n. So let u, v, w be arbitrary vertices of Γ_n embedded into Q_n with coordinates $u_1u_2 \ldots u_n$, $v_1v_2 \ldots v_n$, and $w_1w_2 \ldots w_n$, respectively. It is well known (cf. the proof of [7, Proposition 1.29]) that the median of the triple in Q_n is obtained by the majority rule: the ith coordinate of the median is equal to the element that appears at least twice among the u_i, v_i, and w_i. Suppose that for some i, the majorities of u_i, v_i, w_i and $u_{i+1}, v_{i+1}, w_{i+1}$ are both equal to 1. But then in at least one of the vertices u, v, w we find two consecutive 1's, say $u_i = u_{i+1} = 1$, which is not possible. It follows that the median of u, v, w does not contain two consecutive 1's and so it is a vertex of Γ_n. In other words, Γ_n is a median closed subgraph of Q_n and hence a median graph by the above theorem of Mulder.

To see that Λ_n is a median graph for any $n \geqslant 2$, we use analogous argument. First, the majority of u, v, w does not contain two consecutive 1's. Moreover, the majority of u_1, v_1, w_1 and the majority of u_n, v_n, w_n cannot both be 1, for otherwise for at least one of the vertices u, v, w, the first and the last coordinates would be 1. So the median of u, v, w is a vertex of Λ_n and we are done.

In [18, Corollary 2.2] it is proved that $\Gamma_n^i = \Gamma_{n-i}^0 \square Q_i = \Gamma_{n-i} \square Q_i$. Since the Cartesian product operation preserves median graphs, and we have just shown that Γ_{n-i} is median, Γ_n^i is median as well. $\square$

One of the cornerstones in the theory of median graphs is the following theorem of Bandelt [1]: A connected graph G is a median graph if and only if G is a retract of some n-cube. Recall that a subgraph R of a graph G is a *retract* of G if there is an edge-preserving map $r : V(G) \to V(R)$ with $r(x) = x$, for all $x \in V(R)$, cf. [4]. The map r is called a *retraction*. The next result shows that there is a natural retraction from Γ_n onto Λ_n. Hence, knowing that Γ_n is a median graph, and since a composition of retractions is a retraction, this result gives an alternative argument that Λ_n is a median graph.

Proposition 2. *For any $n \geqslant 2$, Λ_n is a retract of Γ_n.*

Proof. Let the mapping $f : V(\Gamma_n) \to V(\Lambda_n)$ be defined with

$$f(u) = \begin{cases} 0u_2u_3 \ldots u_{n-1}0; & u_1 = u_n = 1, \\ u; & \text{otherwise.} \end{cases}$$

Clearly, f maps Γ_n onto Λ_n and fixes Λ_n. Thus we only need to show that f is edge-preserving. Suppose first that $uv \in E(\Gamma_n)$ where $u, v \notin \Lambda_n$. Then $u = 1u_2 \ldots u_{n-1}1$ and $v = 1v_2 \ldots v_{n-1}1$, where for exactly one index i, $2 \leqslant i \leqslant n-1$, $u_i \neq v_i$. But then $f(u) = 0u_2 \ldots u_{n-1}0$ is adjacent to $f(v) = 0v_2 \ldots v_{n-1}0$. The second case to consider is when $uv \in E(\Gamma_n)$ where $u \notin \Lambda_n$ and $v \in \Lambda_n$ (or vice versa). Then $u = 1u_2 \ldots u_{n-1}1$ and, since u is adjacent to v, we may without loss of generality assume that $v = 1v_2 \ldots v_{n-1}0$. Hence $u_i = v_i$ for any i, $2 \leqslant i \leqslant n-1$. But then $f(u) = 0u_2 \ldots u_{n-1}0$ is adjacent to $f(v) = 1u_2 \ldots u_{n-1}0$. $\square$

Another classical result on median graphs is due to Mulder. To state it some preparation is needed. A subgraph H of a graph G is called *convex* if for any vertices u, v of H, any shortest u, v-path of G lies completely in H. Let G' be a connected graph with convex subgraphs G'_1, G'_2 such that $G' = G'_1 \cup G'_2$, $G'_1 \cap G'_2$ is a nonempty convex subgraph of G', and no vertex of $G'_1 \backslash G'_2$ is adjacent to a vertex of $G'_2 \backslash G'_1$. Then the *convex expansion* of G' with respect to G'_1, G'_2 is the graph G constructed as follows. For $i = 1, 2$ let G_i be an isomorphic copy of G'_i, and for any vertex u' in $G'_1 \cap G'_2$, let u_i be the corresponding vertex in G_i, $i = 1, 2$. Then G is obtained from the disjoint union $G_1 \cup G_2$, where for each u' in $G'_1 \cap G'_2$ the vertices $u_1$and u_2 are joined by an edge. Roughly speaking, the convex expansion of G' is obtained by selecting two convex subgraphs that cover G' and have nonempty intersection, and expanding the intersection by blowing each vertex to an edge. Then Mulder [11,12] proved that a graph G is a median graph if and only if G can be obtained from K_1 by a sequence of convex expansions. In fact, a strengthening of this result is true. Calling a convex expansion *peripheral* if $G'_1 \subseteq G'_2$ or $G'_2 \subseteq G'_1$, Mulder [13] proved that G is a median graph if and only if G can be obtained from K_1 by a sequence of convex *peripheral* expansions.

It is easy to see that the Fibonacci cube Γ_n can be obtained by a convex peripheral expansion from Γ_{n-1}, where Γ_{n-2} is the peripheral subgraph. This in particular implies that

$$|E(\Gamma_n)| = |E(\Gamma_{n-1})| + |E(\Gamma_{n-2})| + |V(\Gamma_{n-2})| \tag{1}$$

and also

$$|S(\Gamma_n)| = |S(\Gamma_{n-1})| + |S(\Gamma_{n-2})| + |E(\Gamma_{n-2})|. \tag{2}$$

Similarly, the Lucas cube Λ_n is obtained by a convex peripheral expansion from Γ_{n-1}, where Γ_{n-3} is peripheral. Therefore, we have

$$|E(\Lambda_n)| = |E(\Gamma_{n-1})| + |E(\Gamma_{n-3})| + |V(\Gamma_{n-3})| \tag{3}$$

and

$$|S(\Lambda_n)| = |S(\Gamma_{n-1})| + |S(\Gamma_{n-3})| + |E(\Gamma_{n-3})|. \tag{4}$$

These recurrences will be applied in the next section.

3. Enumerative properties

In this section we count the number of edges and squares for each of the Fibonacci cubes, Lucas cubes, and extended Fibonacci cubes. Some relations involving Fibonacci and Lucas numbers are obtained along the way.

3.1. Fibonacci cubes

The Fibonacci cube Γ_n contains F_{n+2} vertices, cf. [6]. We next count the number of its edges in the following way.

Proposition 3. *For any $n \geqslant 1$,*

$$|E(\Gamma_n)| = F_{n+1} + \sum_{i=1}^{n-2} F_i F_{n+1-i}.$$

Proof. For $n = 1$ and $n = 2$ the above sum vanishes, thus we get $|E(\Gamma_1)| = 1 = F_2$ and $|E(\Gamma_2)| = 2 = F_3$. Hence the equality holds for $n = 1, 2$. Let $n \geqslant 3$ and assume that it holds for all indices smaller than n. Then, using (1), the induction assumption, and keeping in mind that $|V(\Gamma_{n-2})| = F_n$, we compute as follows:

$$\begin{aligned} |E(\Gamma_n)| &= \left(F_n + \sum_{i=1}^{n-3} F_i F_{n-i}\right) + \left(F_{n-1} + \sum_{i=1}^{n-4} F_i F_{n-1-i}\right) + F_n \\ &= F_{n+1} + \sum_{i=1}^{n-4} F_i (F_{n-i} + F_{n-1-i}) + F_{n-3}F_3 + F_n \\ &= F_{n+1} + \sum_{i=1}^{n-4} F_i F_{n+1-i} + F_{n-3}F_3 + F_n. \end{aligned}$$

Since $F_{n-3}F_3 + F_n = 2F_{n-3} + F_{n-1} + F_{n-2} = 3F_{n-3} + 2F_{n-2} = F_{n-3}F_4 + F_{n-2}F_3$, we conclude that $|E(\Gamma_n)| = F_{n+1} + \sum_{i=1}^{n-2} F_i F_{n+1-i}$. $\square$

In [15] it is proved that for $n \geqslant 1$,

$$|E(\Gamma_n)| = \frac{nF_{n+1} + 2(n+1)F_n}{5}. \tag{5}$$

(The result is there stated for $n \geqslant 2$, but also holds for $n = 1$.) Combining this formula with Proposition 3 we get the next result.

Corollary 4. *For any* $n \geqslant 1$, $n \neq 5$,

$$F_{n+1} = \frac{5\sum_{i=1}^{n-2} F_i F_{n+1-i} - 2(n+1)F_n}{n-5}.$$

Note that Corollary 4 indeed holds also for $n = 1, 2$ since in these two cases the above sum vanishes and we get $F_2 = 1 = -4/(1-5)$ and $F_3 = 2 = -6/(2-5)$.

We continue by counting the number of squares in the Fibonacci cubes.

Proposition 5. *For any* $n \geqslant 1$,

$$|S(\Gamma_n)| = -\frac{3n}{25}F_{n+1} + \left(\frac{n^2}{10} + \frac{3n}{50} - \frac{1}{25}\right)F_n.$$

Proof. The right-hand side of the equality is 0 for $n = 1$ and $n = 2$, so the assertion holds for Γ_1 and Γ_2. Let $n \geqslant 3$ and assume that it holds for all indices smaller than n. By induction, using (2) and (5), we thus have

$$\begin{aligned}
|S(\Gamma_n)| = & -\frac{3(n-1)}{25}F_n + \left(\frac{(n-1)^2}{10} + \frac{3(n-1)}{50} - \frac{1}{25}\right)F_{n-1} \\
& - \frac{3(n-2)}{25}F_{n-1} + \left(\frac{(n-2)^2}{10} + \frac{3(n-2)}{50} - \frac{1}{25}\right)F_{n-2} \\
& + \frac{(n-2)F_{n-1} + 2(n-1)F_{n-2}}{5} \\
= & -\frac{3(n-1)}{25}F_n + \frac{5n^2 - 3n - 8}{50}F_{n-1} + \frac{5n^2 + 3n - 8}{50}F_{n-2} \\
= & -\frac{3n}{25}F_{n-1} + \frac{5n^2 - 3n - 2}{50}F_n \\
= & -\frac{3n}{25}F_{n+1} + \frac{5n^2 + 3n - 2}{50}F_n. \qquad \square
\end{aligned}$$

The number of squares of Γ_n can also be expressed as follows.

Proposition 6. *For any* $n \geqslant 3$,

$$|S(\Gamma_n)| = \sum_{i=1}^{n-2} F_i |E(\Gamma_{n-1-i})|.$$

Proof. We again proceed by induction and first observe that for $n = 3$ and $n = 4$ the sum returns 1 and 3, respectively. Let b_n and c_n denote the number of edges and squares of Γ_n, respectively. Then for $n \geqslant 5$ we use (2) and proceed as follows:

$$\begin{aligned}
c_n &= \sum_{i=1}^{n-3} F_i b_{n-2-i} + \sum_{i=1}^{n-4} F_i b_{n-3-i} + b_{n-2} \\
&= F_1 b_{n-3} + \sum_{i=2}^{n-3} (F_i + F_{i-1}) b_{n-2-i} + F_1 b_{n-2} \\
&= F_1 b_{n-2} + F_2\, b_{n-3} + \sum_{i=3}^{n-2} F_i b_{n-1-i}. \qquad \square
\end{aligned}$$

Note that by combining Propositions 3 and 6, the number of squares of Γ_n can be expressed using Fibonacci numbers only.

3.2. Lucas cubes

For the Lucas cubes we have $|V(\Lambda_n)| = L_n$, see [15]. The following expression for the number of its edges was obtained using (3). However, we will follow an alternative argument.

Proposition 7. *For any $n \geqslant 2$,*

$$|E(\Lambda_n)| = \sum_{i=1}^{n-1} F_i L_{n-1-i}.$$

Proof. In [15, Proposition 4(ii)] it is shown that $|E(\Lambda_n)| = nF_{n-1}$, $n \geqslant 2$, hence we are going to show that $\sum_{i=1}^{n-1} F_i L_{n-1-i} = nF_{n-1}$. Clearly, the equality holds for $n = 2, 3$ and for the induction step we compute as follows:

$$\begin{aligned}
\sum_{i=1}^{n-1} F_i L_{n-1-i} &= L_{n-2} + L_{n-3} + \sum_{i=3}^{n-1} (F_{i-1} + F_{i-2}) L_{n-1-i} \\
&= L_{n-2} + \sum_{i=1}^{n-2} F_i L_{n-2-i} + \sum_{i=1}^{n-3} F_i L_{n-3-i} \\
&= L_{n-2} + (n-1)F_{n-2} + (n-2)F_{n-3}.
\end{aligned}$$

Since $L_{n-2} = F_{n-2} + 2F_{n-3}$, the result follows. $\square$

Combining Proposition 7 with [15, Proposition 4(ii)] we immediately get the following consequence. Since the identity is quite nice, we suspect that it could be previously known.

Corollary 8. *For any* $n \geqslant 2$,

$$F_n = \frac{1}{n-1} \sum_{i=1}^{n-1} F_i L_{n-i}.$$

For the number of squares of Λ_n we have:

Proposition 9. *For any* $n \geqslant 5$,

$$|S(\Lambda_n)| = \sum_{i=0}^{n-4} L_i |E(\Gamma_{n-3-i})|.$$

Proof. Combining Eq. (4) with Proposition 6, setting b_n for the number of edges of Γ_n, and c'_n for the number of squares of Λ_n, we have

$$\begin{aligned} c'_n &= \sum_{i=1}^{n-3} F_i b_{n-2-i} + \sum_{i=1}^{n-5} F_i b_{n-4-i} + b_{n-3} \\ &= F_1 b_{n-3} + F_2 b_{n-4} + \sum_{i=1}^{n-5} (F_i + F_{i+2}) b_{n-4-i} + b_{n-3} \\ &= L_0 b_{n-3} + L_1 b_{n-4} + \sum_{i=1}^{n-5} (F_i + F_{i+2}) b_{n-4-i}. \end{aligned}$$

Since $F_i + F_{i+2} = L_{i+1}$, the result follows. □

3.3. Extended Fibonacci cubes

As we already mentioned, $\Gamma_n^i = \Gamma_{n-i} \Box Q_i$ holds for $n \geqslant i \geqslant 0$. Therefore, counting vertices, edges, and squares of extended Fibonacci cubes reduces to the corresponding problems in Fibonacci cubes. More precisely, using only basic properties of the Cartesian product, cf. [7], the following equalities are straightforward:

$$\begin{aligned} |V(\Gamma_n^i)| &= |V(\Gamma_{n-i})| \cdot |V(Q_i)| = F_{n+2-i}\, 2^i; \\ |E(\Gamma_n^i)| &= |V(\Gamma_{n-i})| \cdot |E(Q_i)| + |E(\Gamma_{n-i})| \cdot |V(Q_i)| \\ &= F_{n+2-i}\, i 2^{i-1} + |E(\Gamma_{n-i})|\, 2^i; \\ |S(\Gamma_n^i)| &= |V(\Gamma_{n-i})| \cdot |S(Q_i)| + |E(\Gamma_{n-i})| \cdot |E(Q_i)| + |S(\Gamma_{n-i})| \cdot |V(Q_i)| \\ &= F_{n+2-i}\, i(i-1) 2^{i-3} + |E(\Gamma_{n-i})| i 2^{i-1} + |S(\Gamma_{n-i})| 2^i. \end{aligned}$$

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