Graphs with Builder-Blocker visibility number two, and the game on grids and block graphs

Pakanun Dokyeesun[a,*] Sandi Klavžar [a,b,c,†]

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[a] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Abstract

In the Builder-Blocker mutual-visibility game, Builder and Blocker alternately select (as long as possible) an unmarked vertex of a given graph preserving the property that selected vertices form a mutual-visibility set. The goal of Builder is that the game lasts as long as possible, while the goal of Blocker is exactly the opposite. Assuming that both players played optimally, the number of vertices selected in the game played on a graph G in which Builder starts is denoted by $\mu_{\rm g}(G)$. It is proved that $\mu_{\rm g}(G) = 2$ if and only if every vertex in G is incident to a bridge. This corrects an error in [11, Proposition 2.6(i)], which claims that trees are the only graphs with this property. The two invariants are determined for several thin grids. It is also demonstrated that it can happen that if Builder starts the game in a block graph, then his first (optimal) move must be a non-simplicial vertex.

Keywords: mutual-visibility problem; Builder-Blocker mutual-visibility game; grid graphs; block graphs

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*papakanun@gmail.com

†Corresponding author: sandi.klavzar@fmf.uni-lj.si

1 Introduction

The graph mutual-visibility problem, which was introduced in 2022 by Di Stefano [10], has received a great deal of attention to date; see [1–9,12–17] and reference therein. The problem is as follows.

Given a set X of vertices in a graph $G = (V(G), E(G))$, two vertices u and v are *mutually-visible* with respect to X, shortly *X-visible*, if there is a shortest u,v-path P such that $V(P) \cap X \subseteq \{u, v\}$. The set X is a *mutual-visibility set* if each pair of vertices from X are X-visible. A largest mutual-visibility set of G is a *μ-set*, the cardinality of a μ-set is the *mutual-visibility number* $\mu(G)$ of G.

In [11], the mutual-visibility problem was examined from the perspective of game theory in the following way. Two players, named Builder and Blocker, play the *Builder-Blocker mutual-visibility game* on a graph G by alternately selecting a not yet played vertex on G. If Builder starts the game, we speak of the *B-game*, otherwise of the *B'-game*. At each step, the set of so far selected vertices must form a mutual-visibility set. The game ends when no more moves are possible. The goal of Builder is that the game lasts as long as possible, while the goal of Blocker is exactly the opposite. Assuming that both players played optimally, the number of vertices selected in the B-game is the *Builder-game mutual-visibility number* $\mu_{\rm g}(G)$ of G, the parallel invariant for the B'-game is the *Blocker-game mutual-visibility number* $\mu'_{\rm g}(G)$ of G.

The rest of the article is organized as follows. At the end of this section, we define some additional concepts needed. In the main result of the next section, we prove that $\mu_{\rm g}(G) = 2$ if and only if every vertex in G is incident to a bridge. In this way an error from [11, Proposition 2.6(i)] is corrected where it is claimed that trees are the only graphs G with the property $\mu_{\rm g}(G) = 2$. Then, in Section 3 we consider [11, Problem 5.3] which asks to determine $\mu_{\rm g}(P_n \,\square\, P_m)$ and $\mu'_{\rm g}(P_n \,\square\, P_m)$ for any n and m. We solve the problem for several thin grids, in particular, we determine $\mu'_{\rm g}(P_3 \,\square\, P_n)$ for all $n \geq 3$. In the final section we pay our attention to [11, Problem 5.1] which asks to determine $\mu_{\rm g}(G)$ and $\mu'_{\rm g}(G)$ for an arbitrary block graph G. The main message of our investigation is that surprisingly, it can happen that in the B-game, the first (optimal) move of a Builder must be a non-simplicial vertex. Along the way it is demonstrated that $\mu_{\rm g}(G)$ can be arbitrarily smaller than $\mu(G)$ in the class of block graphs.

We now give some more definitions and notation. For a positive integer n, we will use the notation $[n] = \{1, \ldots, n\}$. Unless stated otherwise, all the graphs considered are connected. Recall also that an edge e of a connected graph G is a bridge if $G - e$ is disconnected. A vertex u of G is *simplicial* if the set of its neighbors induces a complete subgraph. The set of all simplicial vertices of G is denoted by $S(G)$ and the cardinality of $S(G)$ by $s(G)$. The distance $d_G(u,v)$ between vertices $u,v \in V(G)$ is the standard shortest-path distance. A subgraph H of G is *isometric*, if for any $u,v \in V(H)$ it holds $d_H(u,v) = d_G(u,v)$. The *Cartesian product* of graphs G and H has vertices $V(G) \times V(H)$; two vertices (g,h) and (g',h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Cartesian products of paths are called *grids*. A graph G is a *block graph* if every maximal biconnected component (block) is a clique.

We will adopt the convention that the consecutive moves of Builder will be denoted by $b_1, b_2, \ldots$, and the consecutive moves of Blocker by $b'_1, b'_2, \ldots$. Since each of the games played on a graph G ends with a mutual-visibility set of G, we have the following observations to be used later on:

$$\mu_{\rm g}(G) \le \mu(G) \quad \text{and} \quad \mu'_{\rm g}(G) \le \mu(G)\,. \tag{1}$$

2 Graphs with small game mutual-visibility numbers revisited

A tree T is called *starlike* if T contains at most one vertex of degree at least 3. In [11, Proposition 2.6] it was asserted that if G is a connected graph of order at least two, then

$$\mu_{\rm g}(G) = 2 \iff G \text{ is a tree, and} \tag{2}$$
$$\mu'_{\rm g}(G) = 2 \iff G \text{ is a starlike tree.} \tag{3}$$

In this section, we demonstrate that (2) is not correct. More precisely, it turns out that besides trees, there are additional graphs having the Builder-game mutual-visibility number equal to 2. As a starting counterexample, consider the graphs G_n, $n \ge 3$, obtained from C_n by attaching a pendant vertex to each vertex of C_n. If Builder first selects a vertex u of the cycle of G_n, then Breaker replies by selecting the pendant vertex adjacent to u, which finishes the game. Similarly, if Builder first selects a pendant vertex, then Breaker replies by selecting its adjacent vertex. In any case, the game finishes after two moves, hence $\mu_{\rm g}(G_n) = 2$ for every $n \ge 3$.

To correct the characterization of (2), we first give the following auxiliary result.

Lemma 2.1 *If the vertices u and v lie on a common isometric cycle in a graph G, then there exists a vertex w in G such that $\{u,v,w\}$ is a mutual-visibility set in G.*

Proof. Let C be an isometric cycle of G such that $u,v \in V(C)$. Then $d_C(u,v) = d_G(u,v)$ due to the isometric nature of C. Let P be a shortest u,v-path along C

and let X be the set of inner vertices of P. Note that if u and v are adjacent, then $X = \emptyset$. Let now w be a vertex from $V(C) \setminus (X \cup \{u, v\})$ such that

$$d_{C \setminus X}(u, w) = \lfloor d_{C \setminus X}(u, v) \rfloor / 2\,,$$
$$d_{C \setminus X}(v, w) = \lceil d_{C \setminus X}(u, v) \rceil / 2\,.$$

The isometry of C implies that $d_G(u, w) = d_C(u, w) = d_{C \setminus X}(u, w)$, as well as $d_G(v, w) = d_C(v, w) = d_{C \setminus X}(v, w)$. We can conclude that $\{u, v, w\}$ is a mutual-visibility set in G. $\square$

Theorem 2.2 *Let G be a connected graph of order at least two. Then $\mu_{\rm g}(G) = 2$ if and only if every vertex in G is incident to a bridge.*

Proof. Assume that every vertex in G is incident to a bridge. Since G is a nontrivial connected graph, $\mu_{\rm g}(G) \geq 2$. Let Builder start the game at an arbitrary vertex u of G. By the assumption, u is adjacent to a vertex v such that uv is a bridge. After Blocker replies with the vertex v, the game ends. Hence, $\mu_{\rm g}(G) = 2$.

Conversely, assume that $\mu_{\rm g}(G) = 2$. Suppose, by contradiction, that G contains a vertex u which is incident to no bridge. Let Builder start the game by selecting the vertex u, and let Blocker respond by playing a vertex v. Let P be an arbitrary shortest u, v-path and let x be the neighbor of u on P. Note that it is possible that $x = v$. Since ux is not a bridge, $G - ux$ contains a u, x-path. Among all such paths, select a shortest one, say Q. Then Q together with the edge ux forms an isometric cycle C of G. Indeed, if would not be isometric, then Q would not be a shortest u, x-path in $G - ux$.

If $v \in V(C)$, then by Lemma 2.1, there is a vertex w in C such that $\{u, v, w\}$ is a mutual-visibility set. As it is Builder's turn, the vertex w can thus be selected, a contradiction with $\mu_{\rm g}(G) = 2$. Hence we may assume in the rest that $v \notin V(C)$ and distinguish two cases.

Case 1: $V(C) \cap V(P) = \{u, x\}$.
Let x' be the neighbor of x on C different from u. We claim that $X = \{u, v, x'\}$ is a mutual-visibility set. Clearly, u and v are X-visible via the path P. Second, u and x' are X-visible since they lie on the isometric cycle C. Finally, $d_G(x', v) \leq 1 + d_G(x, v) \leq d_G(u, v)$, hence there is a shortest x', v-path avoiding u. Now, Builder can play x' as his second move, a contradiction.

Case 2: $|V(C) \cap V(P)| \geq 3$.
Let now x' be the vertex from $(V(C) \cap V(P)) \setminus \{u, v\}$ closest to v, that is, the last

vertex of P which lies on C. Since $v \notin C$, we have $x' \neq v$. Let x'' be a neighbor of x' on C which is not on P. Such a neighbor exists by the way x' has been selected. Now we can argue analogously as in Case 1 that the set $\{u, v, x''\}$ is a mutual-visibility set. Hence Builder can play x'' as his second move, a final contradiction. $\square$

The class of graphs from Theorem 2.2 is quite rich. For instance, in the above initial example G_n, $n \geq 3$, we could start with an arbitrary connected graph G and respectively attach an arbitrary non-trivial tree to each of its vertices. On the other hand, as soon as at least one vertex in a graph is incident to no bridge, then the Builder-game mutual-visibility number can be arbitrarily large. For instance, consider the graph Γ_n, $n \geq 3$, obtained from K_n by attaching a pendant to all vertices in K_n except one, say u. Then all vertices of Γ_n but u are incident to a bridge. In the B-game, Builder can start the game by selecting the vertex u. In the rest of the game, for each pendant v, either v or its neighbor in K_n will be played, hence we can conclude that $\mu_{\rm g}(\Gamma_n) = n$.

3 Games on thin grids

As already mentioned, in [11, Problem 5.3] the problem was posed to determine $\mu_{\rm g}(P_n \,\square\, P_m)$ and $\mu'_{\rm g}(P_n \,\square\, P_m)$ for any n and m. In this section we consider this problem on thin grids, more precisely on grids $P_2 \,\square\, P_n$ and $P_3 \,\square\, P_n$. We first recall the following related result which is needed later.

Theorem 3.1 [10] *Let m, n be positive integers. Then*

$$\mu(P_m \,\square\, P_n) = \begin{cases} 3; & m = n = 2, \\ 4; & m = 2, n \geq 3 \\ 5; & m = n = 3, \\ 6; & m = 3, n \geq 4, \\ 2\min\{m,n\}; & m \geq 4, n \geq 4\,. \end{cases}$$

Our first result deals with grids $P_2 \,\square\, P_n$.

Theorem 3.2 *If $n \geq 2$ is a positive integer, then*

$$\mu_{\rm g}(P_2 \,\square\, P_n) = \begin{cases} 3; & n \leq 4, \\ 4; & n \geq 5, \end{cases}$$

and

$$\mu'_{\rm g}(P_2 \,\square\, P_n) = \begin{cases} 3; & n \leq 5, \\ 4; & n \geq 6. \end{cases}$$

Proof. Let $n \geq 2$ be an integer, and set $G = P_2 \,\square\, P_n$ for the rest of the proof.

We first consider the B-game. By Theorem 2.2, we have $\mu_{\rm g}(G) \geq 3$. If $n = 2$, then it is straightforward to check that $\mu_{\rm g}(G) = 3$. Assume next that $n \in \{3, 4\}$. In these two cases, by the symmetry of $P_2 \,\square\, P_3$ and of $P_2 \,\square\, P_4$, we may assume that $b_1 \in \{(1, 1), (1, 2)\}$. If $b_1 = (1, 1)$, then Blocker selects $b'_1 = (1, 2)$, and if $b_1 = (1, 2)$, then Blocker selects $b'_1 = (1, 1)$. Hence, in any case, the vertices $(1, 1)$ and $(1, 2)$ are played in the first two moves. From here, one can check that no matter what Builder selects next, no additional move will be possible. Hence, the game ends in three moves.

Assume now that $n \geq 5$. By (1) and Theorem 3.1 we get $\mu_{\rm g}(G) \leq \mu(G) = 4$. It remains to show that $\mu_{\rm g}(G) \geq 4$. Builder's strategy is to start the game by playing $b_1 = (1, \lceil \frac{n}{2} \rceil)$. If $b'_1 = (1, j), j \in [n]$ or $b'_1 = (2, j), j \in [\frac{n}{2}]$, then Builder replies by $b_2 = (2, n)$. Otherwise, Builder reacts by $b_2 = (2, 1)$. In any case, Blocker is forced to play one more vertex, which implies that $\mu_{\rm g}(G) \geq 4$.

We now consider the B'-game. By (3) we have $\mu'_{\rm g}(G) \geq 3$. If $n = 2$, then it is straightforward to see that $\mu'_{\rm g}(G) = 3$. Assume next that $n \in \{3, 4, 5\}$. To demonstrate that $\mu'_{\rm g}(G) \leq 3$, we distinguish two subcases.

(i) If $n \in \{3, 4\}$, then set $b'_1 = (1, 2)$. Assume first that $n = 3$. By symmetry, we may assume that $b_1 \in \{(1, 1), (2, 1), (2, 2)\}$. Blocker replies $b'_2 = (1, 1)$ if possible, otherwise $b'_2 = (2, 1)$. In each case, this move finishes the game. Assume second that $n = 4$. If $b_1 \in \{(1, 1), (2, 2), (2, 3)\}$, then Blocker plays $b'_2 = (2, 1)$; if $b_1 = (2, 1)$, then Blocker plays $b'_2 = (1, 1)$; and if $b'_1 \in \{(1, 3), (1, 4), (2, 4)\}$, then Blocker selects $b'_2 = (2, 3)$. In each of the cases, the game is finished after the move b'_2.

(ii) If $n = 5$, then set $b'_1 = (1, 3)$. By symmetry, we may assume that $b_1 \in \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3)\}$. Then Blocker plays $b'_2 = (2, 2)$ if it possible, otherwise $b'_2 = (1, 1)$. Thus, the game will end in three moves.

We can conclude that $\mu'_{\rm g}(G) = 3$.

Assume $n \geq 6$. By (1) and Theorem 3.1 we get $\mu'_{\rm g}(G) \leq \mu(G) = 4$. We now show that $\mu'_{\rm g}(G) \geq 4$. Assume by symmetry that $b'_1 \in \{(1, 1), (1, 2), \ldots, (1, \lceil \frac{n}{2} \rceil)\}$. Then Builder replies $b_1 = (1, n)$. If $b'_2 \in \{(2, 1), (2, 2), \ldots, (2, n-2)\}$, then Builder plays $b_2 = (2, n)$, otherwise, $b_2 = (2, 1)$. Therefore, players have to play at least four moves to end the game as claimed. $\square$

The main result of this section reads as follows.

Theorem 3.3 *If $n \geq 3$ is a positive integer, then*

$$\mu_{\rm g}(P_3 \,\Box\, P_n) = \begin{cases} 4; & n = 3, \\ 5; & 4 \leq n \leq 5, \\ 6; & n \geq 6\,. \end{cases}$$

Proof. Let $n \geq 3$ be a positive integer and set $G = P_3 \,\Box\, P_n$ for the rest of the proof. We first start with the B-game. Assume that $n = 3$. We will provide a strategy for Blocker that forces the game to finish in four moves. We assume by symmetry that $b_1 \in \{(1,1),(1,2),(2,2)\}$. If $b_1 \neq (2,2)$, then Blocker replies $b_1' = (2,2)$. Otherwise, Blocker replies $b_1' = (1,2)$. By a straightforward case analysis we can then check that the game ends within four moves, hence $\mu_{\rm g}(G) \leq 4$. Now, we show that $\mu_{\rm g}(G) \geq 4$. Builder starts the game by playing $b_1 = (1,1)$. By symmetry, assume that $b_1' \in \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$. If $b_1' \neq (2,3)$, then Builder plays $b_2 = (2,3)$. Otherwise, Builder plays $b_2 = (3,3)$. By a case analysis we then infer that in every case, at least one playable vertex remains, therefore $\mu_{\rm g}(G) \geq 4$.

Assume now that $n = 4$. We first show that $\mu_{\rm g}(G) \leq 5$. No matter which vertex was selected first by Builder, Blocker can select one of the vertices $(2,2)$ and $(2,3)$ as the move b_1'. We now claim that an arbitrary vertex-visibility set S of G which contains one of these two vertices contains at most five elements. This claim can be again checked by a case analysis, having in mind that if $|S| = 6$ would holds, then S would contain exactly two vertices from each P_4-layer. For instance, if $(2,1) \in S$ and $(2,2) \in S$, then S can contain at most one vertex from each of the other two P_4-layers. The other cases can be verified similarly. Hence, $\mu_{\rm g}(G) \leq 5$. To show that $\mu_{\rm g}(G) \geq 5$, let assume Builder plays $b_1 = (1,1)$ in his first move. After that, if $b_1' \in \{(1,2),(1,4),(1,3)\}$, then Builder responses $b_2 = (2,4)$, and in all the other cases, Builder responses $b_2 = (1,4)$. After three moves are done according to the above strategy of Builder, a straightforward case analysis verifies that players have to play at least five moves to end the game. This completes the argument for the case $n = 4$.

Assume that $n = 5$. By the symmetry of G, we may assume without loss of generality that $b_1 \in \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$. Let $i \in [2]$. If $b_1 = (i,1)$, then Blocker replies $b_1' = (i,2)$. If $b_1 = (i,2)$, then Blocker replies $b_1' = (i,1)$. Finally, if $b_1 = (i,3)$, then Blocker replies $b_1' = (3-i,3)$. After the first two moves, exactly one of the sets $\{(1,1),(1,2)\}$, $\{(2,1),(2,2)\}$, and $\{(1,3),(2,3)\}$ has been selected by the two players. Assume that the vertices $(1,1)$ and $(1,2)$ have been selected. Then we infer that the players can select at most one more vertex from the middle P_5-layer, implying that at most five vertices will be played. By similar arguments we can reach the same conclusion also in the other two cases, that is, if either $(2,1)$ and $(2,2)$, or $(1,3)$ and $(2,3)$ were selected first. Hence $\mu_{\rm g}'(G) \le 5$. To show $\mu_{\rm g}'(G) \ge 5$, Builder proceeds similarly as in the $n = 4$ case. More precisely, Builder starts the game by $b_1 = (1,1)$. After that, if $b_1' \in \{(1,2),(1,3),(1,4),(1,5)\}$, then Builder replies by $b_2 = (2,5)$, otherwise Builder plays $b_2 = (1,5)$. After three moves are played according to the described strategy of Builder, it is straightforward to check that at least two more moves will be played by the end of the game. Hence, $\mu_{\rm g}'(G) \ge 5$ and we get $\mu_{\rm g}'(G) = 5$.

Assume that $n \ge 6$. By (1) and Theorem 3.1 we have $\mu_{\rm g}(G) \le \mu(G) = 6$. It remains to show that $\mu_{\rm g}(G) \ge 6$. Builder starts the game with $b_1 = (1, \lfloor \frac{n}{2} \rfloor)$. Let $b_1' = (i,j)$. We now distinguish the following three cases.

Case 1: $i = 1$.
In this case, Builder replies by $b_2 = (2,n)$. Assume first that $j < \lfloor \frac{n}{2} \rfloor$. In this subcase, the vertices $(2, \lfloor \frac{n}{2} \rfloor - 1), \ldots, (2, n-1)$ are no longer playable after the first three moves. Then, if $b_2' \in \{(2,1), \ldots, (2, \lfloor \frac{n}{2} \rfloor - 2)\}$, Builder replies by the move $(3,n)$. Otherwise, the second move of Blocker lies in the third P_n-layer. Then, if $b_2' \in \{(3, n-1), (3,n)\}$, then Builder replies by the move $(2,1)$, otherwise Builder replies by $(3,n)$. Assume second that $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n$. After the first three moves, the vertices $(2, j-1), \ldots, (2, n-1)$ are no longer playable. Later, if $b_2' \in \{(2,1), \ldots, (2, j-2)\}$, Builder replies by the move $(3,1)$. Otherwise, the second move of Blocker lies in the third P_n-layer, and then Builder replies $b_3 = (2,1)$. Then, having in mind that $n \ge 6$, we infer that in each of the cases Blocker is forced to play one more move; that is, exactly six moves will be played.

Case 2: $i = 2$.
Assume first that $j \le \lfloor \frac{n}{2} \rfloor$. Then Builder replies $b_2 = (2,n)$. After that, if $b_2' = (1,1)$, then Builder replies $b_3 = (3,n)$, and if $b_2' \in \{(3,1), \ldots, (3, n-2)\}$, Builder replies $b_3 = (1,n)$. Otherwise, Builder replies $b_3 = (3,1)$. Assume second that $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n$. Then Builder replies with $b_2 = (2,1)$. After that, if $b_2' = (1,n)$, then Builder replies $b_3 = (3,1)$, and if $b_2' \in \{(3,3), \ldots, (3,n)\}$, Builder replies $b_3 = (1,1)$. Otherwise, Builder replies $b_3 = (3,n)$. Since $n \ge 6$, we infer that Blocker has to play one more move in every case, which means that players need to play six moves to finish the game.

Case 3: $i = 3$.
We have a strategy for Builder by using a similar argument to Case 2. More precisely, assume first that $j \le \lfloor \frac{n}{2} \rfloor$. Then Builder replies $b_2 = (3,n)$. After that, if $b_2' = (1,1)$, then Builder replies $b_3 = (2,n)$, and if $b_2' \in \{(2,1), \ldots, (2, n-2)\}$, Builder replies $b_3 = (1,n)$. Otherwise, Builder replies $b_3 = (2,1)$. Assume second that $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n$. Then Builder replies $b_2 = (3,1)$. After that, if $b_2' = (1,n)$, then Builder replies $b_3 = (2,1)$, and if $b_2' \in \{(2,3), \ldots, (2,n)\}$, Builder replies $b_3 = (1,1)$. Otherwise,

Builder replies $b_3 = (2, n)$. Hence, we need exactly six vertices to finish the game.

It completes the proof that $\mu_{\rm g}(G) = 6$. $\square$

In a similar matter, $\mu'_{\rm g}(P_3 \,\square\, P_n)$ could also be determined, but it seems that n needs to be relatively large in order that $\mu'_{\rm g}(P_3 \,\square\, P_n) = 6$ holds, so that the complete case analysis would be more tedious. It is hence not included here, we only demonstrate that $\mu'_{\rm g}(P_3 \,\square\, P_3) = 4$ and that $\mu'_{\rm g}(P_3 \,\square\, P_4) = 5$.

Assume $n = 3$. Blocker first plays $b'_1 = (2, 2)$. By symmetry, we assume that $b_1 \in \{(1, 1), (1, 2)\}$. If $b_1 = (1, 1)$, then Blocker replies $b'_2 = (1, 2)$, and if $b_1 = (1, 2)$, then Blocker replies $b'_2 = (1, 1)$. In both cases, the vertices $(1, 1)$, $(1, 2)$, and $(2, 2)$ have been played so far. After that, the only vertex which is still playable is $(3, 3)$, hence, $\mu'_{\rm g}(G) \le 4$. It remains to show that $\mu'_{\rm g}(G) \ge 4$. By the symmetry, we may assume that $b'_1 \in \{(1, 1), (2, 1), (2, 2)\}$. If $b'_1 = (2, 2)$, then Builder selects $b_1 = (1, 1)$ and if $b'_1 = (1, 1)$, then Builder selects $b_1 = (2, 2)$. In both cases we are in the same situation and then we infer that no matter which move is b'_2, Builder can play one more vertex. Finally, if $b'_1 = (2, 1)$, the Builder replies by the move $b_1 = (1, 3)$. We can again see that afterwards two moves will be played.

Assume now that $n = 4$. Blocker starts the game by playing $b'_1 = (2, 2)$. We assume by symmetry that $b_1 \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}$. After that, if $b_1 \neq (1, 2)$, then Blocker replies $b'_2 = (1, 2)$, otherwise Blocker replies $(2, 1)$. By case analysis, we can check that the game finishes within five moves. We infer that $\mu'_{\rm g}(G) \le 5$. To show $\mu'_{\rm g}(G) \ge 5$, we assume by symmetry that $b'_1 \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. If $b'_1 \in \{(1, 1), (1, 2)\}$, then Blocker plays $b'_1 = (1, 4)$. Later, when Builder plays $b'_2 = (i, j)$, then Blocker responds by playing $b_2 = (i, k)$ where $|i - k| = 2$ if it is possible. Otherwise, she plays in the other P_4-layer. It is straightforward to see that there are at least five moves to end the game. It remains to consider the cases when $b'_1 \in \{(2, 1), (2, 2)\}$. In each of these two cases, Blocker plays $b'_1 = (2, 4)$. After that Blocker follows an analogous strategy as above to ensure that at least five vertices will be played.

4 On the game in block graphs

In [11, Problem 5.1] it was asked to determine $\mu_{\rm g}(G)$ and $\mu'_{\rm g}(G)$, where G is a block graph. In this section we shed a light on this problem by showing that the problem could indeed be difficult.

Recall first that it was shown in [11] that for a tree T, $\mu_{\rm g}(T) = 2$. The reason is simple, since after Builder plays his first move, Blocker can end the game by playing its neighbor. However, determining an optimal strategy for Builder on block graphs appears to be more complicated. In general, it is not clear how Builder should start the game. The intuition indicates that it would be best for Builder to select some simplicial vertex, but the following example shows that the optimal first move for Builder need not be simplicial.

Consider graphs H_n, where $n \geq 4$ is even, constructed as follows. Start with n copies of K_3 and select a respective vertex x_i, $i \in [n]$, in each of them. Then add all possible edges between these n vertices, so that they induce a K_n. Finally, attach $\frac{n}{2} - 1$ respective copies of K_3 to all the remaining vertices. See Figure 1.

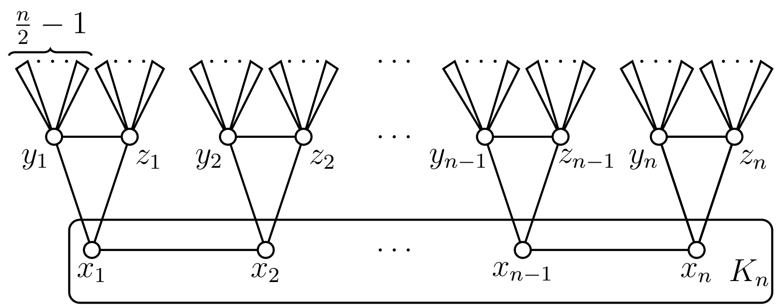

Figure 1: The graph H_n

Proposition 4.1 *If $n \geq 4$ is an even integer, then $\mu_{\rm g}(H_n) = n$.*

Proof. For each $i \in [n]$, let y_i and z_i be the other two vertices of the starting triangle of H_n containing x_i, See Figure 1 again.

We first show that $\mu_{\rm g}(H_n) \geq n$. For this sake, let's assume that Builder plays the vertex x_1 in the first move. We now distinguish several cases depending on the first move of Blocker.

If $b_1' = y_1$, then Builder plays an arbitrary simplicial vertex which is adjacent to z_1. Note that after this move, the only vertices playable are all the simplicial vertices (but the one already played) adjacent to z_1. Moreover, all these vertices will be played by the end of the game, hence the total number of vertices played is $2(n/2 - 1) + 2 = n$. By symmetry, the same conclusion holds if $b_1' = z_1$. Next, if b_1' is some simplicial vertex adjacent to y_1 or z_1, then Builder replies by z_1 or y_1, respectively, and then by the above argument, n vertices will be played.

Assume next that $b_1' = y_i$ where $i \neq 1$. Then Builder plays an arbitrary simplicial vertex which is adjacent to z_i. In the rest of the game, all simplicial vertices adjacent to z_i will be played, which means that at least n vertices are played by the end of the game. By symmetry, the same result holds when $b_1' = z_i$. If b_1' is some simplicial vertex adjacent to y_i or z_i, then Builder replies by playing z_i or y_i, respectively, and then at least n vertices will be played.

Finally, assume that $b'_1 = x_i$, $i \neq 1$. Then Builder selects an arbitrary simplicial vertex which is adjacent to y_j where x_j is unplayed. Then at least n vertices are played to finish the game. We can conclude that $\mu_{\rm g}(H_n) \geq n$.

It remains to show that $\mu_{\rm g}(H_n) \leq n$. By the symmetry we may assume that $b_1 \in \{x_1, y_1, w\}$, where w is a simplicial vertex adjacent to y_1. Consider first the case $b_1 = x_1$. In this case Blocker replies y_1. After that, the only vertices playable are z_1 and all the simplicial vertices adjacent to z_1. Note that if Builder plays z_1, then the game ends immediately, and if Builder plays an arbitrary simplicial vertex which is adjacent to z_1, then the game will end when n vertices are played. Second, if $b_1 = y_1$, then Blocker replies x_1. The above argument implies that the game finishes within n moves. The last case to consider is $b_1 = w$. Now Blocker replies by playing y_1, and the game will end in three moves. □

In [10, Theorem 4.2] it was proved that if G is a (connected) block graph, then $\mu(G) = s(G)$. Note that $s(H_n) = n(n/2 - 1) = n(n-2)/2$, hence Proposition 4.1 demonstrates that $\mu_{\rm g}(G)$ can be arbitrary smaller than $\mu(G)$ in the class of block graphs.

For an edge $uv \in V(G)$, let

$$\mathcal{V}(uv) = \{w \in V(G) :\ u, w \text{ and } v, w \text{ are } \{u,v\}\text{-visible}\}\,.$$

If G is a graph and S a subset of vertices of G, then by $\mu_{\rm g}(G \,|\, S)$ we mean the Builder-Blocker visibility game number of the graph G, where the set of vertices S has already been played, and these vertices are counted as moves in the game. Now, we pose the problem whether the formula

$$\mu_{\rm g}(G) = \max_{u \in S(G)} \min_{\substack{v \notin S(G)\\ v \sim u}} \mu_{\rm g}(G[\mathcal{V}(uv)] \,|\, \{u,v\})$$

holds true for block graphs G in which every block contains at least one simplicial vertex.

To prove this formula, it suffices to prove that in an optimal game, b_1 is some simplicial vertex of G, and that b'_1 is a non-simplicial neighbor of b_1. As we have seen in Proposition 4.1, the first assertion does not hold for arbitrary block graphs. Note that in the block graph H_n from Proposition 4.1, no vertex of a triangle block induced by $\{x_i, y_i, z_i\}$ is simplicial.

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of Interest Statement

The authors declare no conflict of interest.

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