

Square-Edge Graphs

Sandi Klavžar *

Department of Mathematics, PEF, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
email: sandi.klavzar@uni-lj.si

Riste Škrekovski †

Department of Mathematics, University of Ljubljana
Jadranska 19, 1111 Ljubljana, Slovenia
email: riste.skrekovski@fmf.uni-lj.si

Aleksander Vesel *

Department of Mathematics, PEF, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
email: vesel@uni-mb.si

Abstract

An edge of a graph is called a square-edge if it lies in exactly one 4-cycle. A graph G is a square-edge graph if it contains a sequence of square-edges whose removal produces a spanning tree of G . Cube-free median graphs can be characterized as square-edge graphs which contain no Q_3^- as a subgraph. Among square-edge graphs, the class of partial cubes coincides with the class of semi-median graphs. A recognition algorithm for square-edge graphs of complexity $O(a(G)|E(G)|)$ is also presented, where $a(G)$ is the arboricity of a graph G .

*Supported by the Ministry of Science and Technology of Slovenia under the grant J1-0498-0101.

†Supported by the Ministry of Science and Technology of Slovenia under the grant J1-0502-0101.

1 Introduction

An edge of a graph G is a *square-edge* if it lies in exactly one 4-cycle of G . Square-edges were introduced in [7] in the context of median graphs and partial cubes. From [7] we recall the following result to be used in the sequel.

Theorem 1.1 *Let e be a square-edge of a median graph G . Then $G - e$ is a median graph. Conversely, let e be a square-edge of a graph G and let $G - e$ be a median graph. If e does not lie in a subgraph of G isomorphic to Q_3^- then G is a median graph.*

Here we continue the investigation of the concept of square-edges and proceed as follows. In the rest of this section we recall the necessary concepts and notations. In Section 2 we introduce square-edge graphs which, for instance, include cube-free median graphs. We prove that square-edge graphs are bipartite and contain $m - n + 1$ 4-cycles, where n is the number of vertices of a given graph and m the number of its edges. In Section 3 we show that cube-free median graphs can be characterized as square-edge graphs that contain no Q_3^- as a subgraph. We also give a short argument for a result from [6] and prove that among square-edge graphs, partial cubes and semi-median graphs coincide. In Section 4 we present a recognition algorithm for square-edge graphs of complexity $O(a(G)m)$, where $a(G)$ is the arboricity of G .

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or, if $a = b$ and $xy \in E(H)$. n -cube Q_n is the Cartesian product of n copies of the complete graph on two vertices K_2 . Q_3 is also shortly called the *cube*, and a graph is *cube-free* if it contains no Q_3 as a subgraph. Q_3^- denotes the graph obtained from the 3-cube by removing one of its vertices.

The *distance* $d_G(u, v)$ between vertices u and v of a graph G will be the usual shortest path distance. A subgraph H of a graph G is an *isometric* subgraph, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$ and H is *convex* if for all $u, v \in V(H)$ all the shortest u, v -paths lie in H . *Partial cubes* are isometric subgraphs of n -cubes. A *median graph* is a connected graph such that, for every triple of its vertices, there is a unique vertex lying on a geodesic (i.e. shortest path) between each pair of the triple.

For an edge ab of a graph G let

$$\begin{aligned}
W_{ab} &= \{w \mid w \in V(G), d_G(w, a) < d_G(w, b)\}, \\
U_{ab} &= \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\}, \\
F_{ab} &= \{uv \mid u \in U_{ab}, v \in U_{ba}\}.
\end{aligned}$$

A partial cube G is a *semi-median* graph if the sets U_{ab} induce connected subgraphs. Median graphs form a proper subclass of semi-median graphs, and, by the definition, semi-median graphs form a proper subclass of partial cubes [5].

Let G' be a graph and let G'_1 and G'_2 be subgraphs of G' such that $V(G'_1) \cap V(G'_2) \neq \emptyset$ and $V(G'_1) \cup V(G'_2) = V(G')$. Assume in addition that G'_1 and G'_2 are isometric subgraphs of G' and that there is no edge between a vertex of $G'_1 \setminus G'_2$ and a vertex of $G'_2 \setminus G'_1$. An *expansion* of a graph G' (with respect to G'_1 and G'_2) is a graph G , obtained from G' in the following way.

- (i) Replace each vertex $v \in V(G'_1) \cap V(G'_2)$ by adjacent vertices v_1 and v_2 .
- (ii) Join v_1 and v_2 to all neighbors of v in $V(G'_1) \setminus V(G'_2)$ and $V(G'_2) \setminus V(G'_1)$, respectively.
- (iii) If $v, u \in V(G'_1) \cap V(G'_2)$ are adjacent in G' , then join v_1 to u_1 and v_2 to u_2 .

Let $G'_0 = G'_1 \cap G'_2$. Then the expansion is *connected (convex)* if G'_0 is connected (convex). In 1978 Mulder [8, 9] proved his convex expansion theorem: A graph is a median graph if and only if it can be obtained from the one vertex graph by a sequence of convex expansions. Later Chepoi [2] proved the analogous result for partial cubes. They can be obtained by a sequence of expansions from the one vertex graph. Finally, semi-median graphs can be obtained by a sequence of connected expansions from the one vertex graph [5].

The Djoković-Winkler's relation Θ introduced implicitly in [4] and explicitly in [10] is defined on the edge-set of a graph in the following way. Two edges $e = xy$ and $f = uv$ of a graph G are in relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Relation Θ need not be transitive, in fact Winkler [10] proved, that a bipartite graph is a partial cube if and only if Θ is transitive. Hence is partial cubes and in particular in semi-median graphs and median graphs, Θ is an equivalence relation.

For $X \subseteq V(G)$ we write $\langle X \rangle$ for the subgraph of G induced by X .

2 Square-edge graphs

Let G be a graph and suppose that there exists a sequence of connected graphs $G = G_j, G_{j-1}, \dots, G_0 = T$, and a sequence of edges e_j, e_{j-1}, \dots, e_1 , where

- (i) $G_i = G_{i+1} - e_{i+1}$, for $i = j-1, j-2, \dots, 0$,
- (ii) e_i is a square-edge of G_i , for $i = j, j-1, \dots, 1$, and
- (iii) T is a spanning tree of G .

Then we say that G is a *square-edge graph* and the sequence of edges e_j, e_{j-1}, \dots, e_1 is a *square-edge sequence*. Example of a square-edge graph is given in Fig. 1.

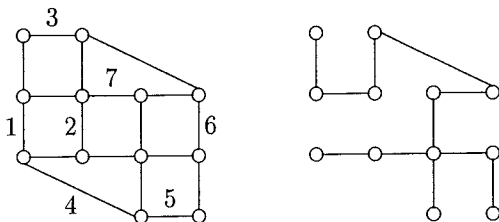


Figure 1: Square-edge graph with its square-edge sequence

Proposition 2.1 *Square-edge graphs are bipartite.*

Proof. Let G be a square-edge graph with a corresponding square-edge sequence e_j, e_{j-1}, \dots, e_1 . Let $G = G_j, G_i = G_{i+1} - e_{i+1}$ for $j-1 \geq i \geq 0$ and set $G_0 = T$. Clearly, T is bipartite. We need to show that G_{i+1} is bipartite provided that G_i is such. Let $e_{i+1} = uv$. As G_i is bipartite, $d_{G_i}(u, v) = 3$. This means that u and v belong to different bipartition sets of G_i and so G_{i+1} is bipartite as well. \square

Note that square-edge graphs contain no $K_{2,3}$ as a subgraph. Indeed, every edge of $K_{2,3}$ is contained in two 4-cycles, thus no edge of $K_{2,3}$ will be removed in an eliminating process. Analogously we see that a square-edge graph contains no Q_3 as a subgraph.

Trees and Cartesian products of two paths are simple examples of square-edge graphs. In order to find more interesting examples we first state the following lemma that follows by a simple induction from the already

mentioned fact that semi-median graphs can be obtained by a sequence of connected expansions from the one vertex graph.

Lemma 2.2 *Let G be a $(K_2 \square C_{2t})$ -free semi-median graph, $t \geq 2$. Then G can be obtained from the one vertex graph by a connected expansion procedure, in which every expansion step G'_0 is a tree.*

We continue with another result from our lemma department.

Lemma 2.3 *Let G be a graph and let F be its Θ -equivalence class with at least two edges. Then F contains at least two square-edges in each of the following two cases:*

- (i) G is a $(K_2 \square C_{2t})$ -free semi-median graph, $t \geq 2$;
- (ii) G is a cube-free median graph.

Proof. (i) Let F be a Θ -equivalence class of G with at least two edges. Let $ab \in F$. It follows from Lemma 2.2 that $\langle U_{ab} \rangle$ is a tree, hence it contains at least two vertices of degree 1, say x and y . If xx' and yy' are the edges from F then by the expansion theorem we conclude that xx' and yy' are square-edges.

(ii) If G is a median graph, then G is also a semi-median graph. Moreover, if G is cube-free then it can be obtained from the one vertex graph by a (convex) expansion procedure, in which every expansion step is done with respect to a convex cover with a convex tree as intersection, cf. [6]. Thus G is a $(K_2 \square C_{2t})$ -free semi-median graph, which means that we are in the first case. \square

Corollary 2.4 *Cube-free median graphs are square-edge graphs. In particular, the Cartesian product $T_1 \square T_2$ of trees T_1 and T_2 is a square-edge graph.*

Proof. Let G be a cube-free median graph. Then by Lemma 2.3 (ii) it has a square-edge e and by Theorem 1.1 the graph $G - e$ is a cube-free median graph. Induction completes the proof that cube-free median graphs are square-edge graphs.

Cartesian products of median graphs are median graphs. As trees are median graphs and $T_1 \square T_2$ is clearly cube-free, $T_1 \square T_2$ is a cube-free median graph. \square

Another interesting property of square-edge graphs is the following.

Proposition 2.5 *Let G be a square-edge graph with n vertices and m edges. Then the number of 4-cycles in G is equal to $m - n + 1$.*

Proof. Let e_j, e_{j-1}, \dots, e_1 be a square-edge sequences of G and let T be the appropriate spanning tree. It is not hard to see that deletion of a square-edge destroys exactly one square and creates no new squares. Since $j = m - n + 1$ and T has no 4-cycles, the proof easily follows. \square

To show that the converse of Proposition 2.5 does not hold in general, consider the graph G which we obtain from $C_6 \square C_6$ by adding a path of length two between diagonal vertices of one of its squares. Clearly, G has 37 vertices, 74 edges and $36 + 2 = 38$ squares and so $m - n + 1 = 38$. (We note that one can find easier examples of this type, but the one presented here is also tiled, cf. the next section.)

We conclude this section with the following result, which is in particular useful from the algorithmic point of view.

Theorem 2.6 *Let G be a square-edge graph and let e^* be a square-edge of G . Then $G^* = G - e^*$ is a square-edge graph.*

Proof. Suppose that the theorem is false and let G be a counterexample with $|E(G)|$ as small as possible. Let $S = e_j, e_{j-1}, \dots, e_1$ be an arbitrary square-edge sequence of G and let $G_j = G$ and $G_i = G_{i+1} - e_{i+1}$ for $0 \leq i \leq j - 1$.

Suppose first that $e^* = e_k$ for some $1 \leq k \leq j$. Clearly, as G^* does not have a square-edge sequence, $k < j$. We claim that $S^* = e_j, e_{j-1}, \dots, e_{k+1}, e_{k-1}, \dots, e_1$ is a square-edge sequence of G^* . Since $e^* = e_k$, it is enough to see that e_j, \dots, e_{k+1} form a part of a square-edge sequence in G^* . Assume thus that e_t cannot be removed from $G_t - e^*$ for some t with $j > t \geq k + 1$. Thus, e_t must lie in a square containing e_k . However, since S is a square-edge sequence of G , this implies that $e_k = e^*$ lies on two different 4-cycles of G , which is not possible. It follows that S^* is a square-edge sequence of G^* , a contradiction.

We may thus assume that e^* does not belong to any square-edge sequence of G . Let a, b and c be the edges which, together with e^* , form the unique square containing e^* . Clearly, at least one of the edges a, b and c belongs to S . We may assume that $a = e_k$ is the first among these three edges that appears in S . If $k = j$, then it is easy to see that e^*, e_{j-1}, \dots, e_1 is a square-edge sequence of G containing e^* , which is not possible. Therefore $k < j$. Note that e^* is a square-edge in G_k . Now, by the minimality assumption, we may assume that $G_k - e^*$ is a square-edge graph and that

there is a square-edge sequence of G_k of the form $e^*, e'_{k-1}, \dots, e'_1$. But then $e_t, \dots, e_{k+1}, e^*, e'_{k-1}, \dots, e'_1$ is a square-edge sequence of G containing e^* . This final contradiction completes the proof. \square

3 Square-edge graphs and partial cubes

In this section we present some results which demonstrate that square-edge graphs are not only interesting as such, but can also be used in studying (subclasses of) partial cubes. We begin with the following characterization of cube-free median graphs, which follows from Corollary 2.4. (See [1] for two additional characterizations of this class of graphs.)

Proposition 3.1 *A graph G is a cube-free median graph if and only if G is a square-edge graph and contains no Q_3^- as a subgraph.*

Proof. Let G be a cube-free median graph. Then G is also Q_3^- -free, for otherwise a Q_3^- would give rise to a 3-cube. G is a square-edge graph by Corollary 2.4.

Conversely, if G is Q_3^- -free, then it is clearly cube-free. Moreover, by Theorem 1.1 G is also a median graph and so the induction on $|E(G)|$ completes the proof. \square

Let G be a cube-free median graph with n vertices, m edges and k equivalence classes of the relation Θ . Then, by Corollary 2.4, G has a square-edge sequence. Note that whenever we remove a square-edge, the number of Θ -classes increases by one. We end up with a spanning tree, thus finally we have $n - 1$ Θ -classes. Since we have removed $m - n + 1$ edges, we conclude that $k + (m - n + 1) = n - 1$, i.e. $2n - m - k = 2$. Thus, in a cube-free median graph we have

$$2n - m - k = 2,$$

a result established in [6].

We can not apply the above argument to semi-median graphs, since semi-median graphs are not hereditary for removing square-edges. Consider, for instance, the semi-median graph G from Fig. 2. Any edge e of its outer 6-cycle is a square-edge, but $G - e$ is not a semi-median graph. (In fact, $G - e$ is not even a partial cube.)

For the sake of simplicity we will next consider a cycle also as the set of its edges. An *even* graph is a graph whose every vertex has even

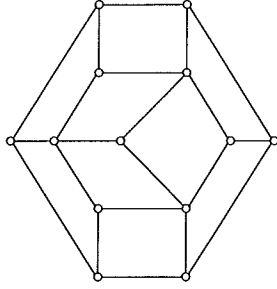


Figure 2: A semi-median graph

degree. Let H be a even subgraph of a graph G . Then a set of 4-cycles $\mathcal{F} = \{C_1, C_2, \dots, C_p\}$ in G is a *tiling* of H if

$$H = C_1 \oplus C_2 \oplus \dots \oplus C_p .$$

Call a graph G *tilted* if every cycle of G has a tiling. (Or in other words, G has a cycle basis comprised of 4-cycles.)

Proposition 3.2 *Square-edge graphs are tiled.*

Proof. Let G be a square-edge graph with a corresponding square-edge sequence e_j, e_{j-1}, \dots, e_1 . Let $G = G_j$, $G_i = G_{i+1} - e_{i+1}$ for $j - 1 \geq i \geq 0$ and set $G_0 = T$. Denote by C_i the unique 4-cycle in G_i , which contains e_i . Note that e_i do not belongs to none of cycles C_{i-1}, \dots, C_1 . This implies that cycles C_j, \dots, C_1 are independent. Since $j = m - n + 1$ it follows that these cycles comprise cycle basis. \square

Using Proposition 3.2 we can prove the following result which connects partial cubes with semi-median graphs.

Theorem 3.3 *Let G be a square-edge graph. Then G is a partial cube if and only if G is a semi-median graph.*

Proof. Since semi-median graphs are partial cubes, we only need to show that a square-edge graph which is a partial cube is also a semi-median graph. By Proposition 3.2 it suffices to show that a tiled partial cube is a semi-median graph.

Thus let G be a tiled partial cube and assume that G is not a semi-median graph. Then there exists an edge $ab \in E(G)$ such that $\langle U_{ab} \rangle$ is not

connected. Let $\langle U_{ab}^1 \rangle$ and $\langle U_{ab}^2 \rangle$ be two different connected components of $\langle U_{ab} \rangle$, and let $\langle U_{ba}^1 \rangle$ and $\langle U_{ba}^2 \rangle$ be the appropriate connected components of $\langle U_{ba} \rangle$. Let u_1v_1 and u_2v_2 be edges from F_{ab} with $u_i \in \langle U_{ab}^i \rangle$ and $v_i \in \langle U_{ba}^i \rangle$ for $i = 1, 2$. We may assume that u_1 and u_2 are as close as possible under the above assumptions. Then there is an induced (in fact even isometric) cycle $C = u_1Pu_2v_2Qv_1u_1$ of G such that the path P lie in $W_{ab} \setminus U_{ab}$.

Observe first that the length of C is at least 6. Since G is tiled, there exists a set of cycles $\mathcal{C} = \{C_1, \dots, C_p\}$ ($p \geq 2$) of G such that $C = C_1 \oplus \dots \oplus C_p$. Note that, if the 4-cycle C_i has an edge in F_{ab} , then it has exactly two edges in F_{ab} . Since u_1v_1 and u_2v_2 are the only two edges of C which are in F_{ab} there exists a subset $\mathcal{C}_r = \{C_{i_1}, \dots, C_{i_r}\}$ of \mathcal{C} , where every C_{i_j} has two edges in F_{ab} and $F_{ab} \cap E(C_{i_1} \oplus \dots \oplus C_{i_r}) = \{u_1v_1, u_2v_2\}$. But then we observe that in $C_{i_1} \oplus \dots \oplus C_{i_r}$ there is a u_1, u_2 -path whose every inner vertex is in $\langle U_{ab} \rangle$. This is a contradiction, for we assumed that the vertices u_1, u_2 are in different connected components of $\langle U_{ab} \rangle$. \square

4 Recognizing square-edge graphs

In this section we present an algorithm of complexity $O(a(G)m)$ which recognizes square-edge graphs, and find a square-edge sequence, if one exists. The algorithm depends on the work of Chiba and Nishizeki [3].

The first part of our algorithm is basically analogous to Algorithm C4 from [3]. It finds all the quadrangles of a given graph and prepares data structures for the second part. For each vertex v of a graph it finds all the quadrangles containing v . Let w be a vertex with $d_G(v, w) = 2$. Then, as square-edge graphs contain no $K_{2,3}$ as a subgraph, there are at most two potential neighbors of both w and v . They are denoted by f_w and s_w in the algorithm. If on the vertices v, w, f_w, s_w we find an induced triangle, the graph is rejected, since a square-edge graph is bipartite. Every quadrangle, found by the algorithm, is stored in the set Q_c which contains its four edges. The indices of the quadrangles containing the edge e are stored in L_e .

In the second part the algorithm constructs a square-edge sequence. The array I represents the list of indices of Q . The indices are sorted such that the quadrangles with a square-edge are at the beginning of I . Applying Theorem 2.6 the algorithm at each step tries to find a quadrangle containing a square-edge. If no such edge is found, the graph is rejected; otherwise the array S representing the square-edge sequence is augmented and the sets L_e are updated for every edge e of the quadrangle involved. If the updated set L_e contains a square-edge, index of its quadrangle is set at the

beginning of the list I . The procedure is repeated until $I = \emptyset$. If we end up with a tree, the graph is accepted.

Procedure SEQUENCE(G);
 { Let $G = (V, E)$ be a connected graph. }
begin

1. Finding quadrangles and computing L -sets

$c := 0$; { Quadrangles counter }
 Sort the vertices in V in a way that $d(v_1) \geq d(v_2) \geq \dots \geq d(v_n)$;
for each vertex $v \in V$ **do begin** $f_v := 0$; $s_v := 0$; **end**;
for $i := 1$ to n **do begin**
 for each vertex u adjacent to v_i **do**
 for each vertex $w \neq v_i$ adjacent to u **do begin**
 if $v_i w \in E$ **then** REJECT;
 if $f_w = 0$ **then** $f_w := u$
 else if $s_w = 0$ **then** $s_w := u$ **else** REJECT;
 end;
 for each vertex w with $s_w \neq 0$ **do begin**
 if $f_w s_w \in E$ **then** REJECT;
 $c := c + 1$; $Q_c = \{ v_i f_w, v_i s_w, w f_w, w s_w \}$;
 $L_{v_i, f_w} := L_{v_i, f_w} \cup \{c\}$; $L_{v_i, s_w} := L_{v_i, s_w} \cup \{c\}$;
 $L_{w, f_w} := L_{w, f_w} \cup \{c\}$; $L_{w, s_w} := L_{w, s_w} \cup \{c\}$;
 end;
 for each vertex w with $f_w \neq 0$ **do begin** $f_w := 0$; $s_w := 0$; **end**;
 Delete the vertex v_i from G and let G be the new graph;
end;

2. Searching for a square-edge sequence

Sort the indices of Q such that if exists $e \in Q_i$ with $|L_e| = 1$, then put i in the beginning of list I ;
 $k := 1$;
while there is an index in I **do begin**
 Let i be the first index of I ;
 Let $e_j, j \in \{1, 2, 3, 4\}$ be the edges of Q_i such that $|L_{e_1}| = 1$;
 if $|L_{e_1}| \neq 1$ **then** REJECT;
 $S_k := e_1$; $k := k + 1$;
 for $j := 2$ to 4 **do** $L_{e_j} := L_{e_j} \setminus \{i\}$;
 for $j := 2$ to 4 **do**
 if $|L_{e_j}| = 1$ **then** put $\ell \in L_{e_j}$ at the beginning of the list I ;
 Delete i from I ;
end;
if $G \setminus S$ is a tree **then** ACCEPT **else** REJECT;
end.

Recall that the *arboricity* $a(G)$ of a graph G is the minimum number of edge-disjoint spanning forests into which G can be decomposed.

Proposition 4.1 *Let G be a graph with m edges. Then Algorithm SEQUENCE decides in $O(a(G)m)$ time whether G is a square-edge graph and finds a square-edge sequence if one exists.*

Proof. The correctness of the algorithm follows from the above discussion.

Since the first part of the algorithm is similar to Algorithm C4, it can be shown along the same lines as in [3] that this part of the above algorithm obtains all the quadrangles of G in $O(a(G)m)$ time.

The second part of the algorithm can be divided into three steps. In the first step the indices of the quadrangles are ordered such that the quadrangles with a square-edge are at the beginning of the list. The number of quadrangles is bounded with $O(a(G)m)$, thus this ordering can be done within the same time.

Concerning the while loop, note that the body of it can be executed in a constant time. Therefore, the total running time is again bounded with the number of quadrangles.

In the last step we need to check if a given graph is a tree. This can clearly be done in $O(n)$ time. We conclude that the total running time of the algorithm is bounded by $O(a(G)m)$. \square

Corollary 4.2 *Let G be a cube-free median graph on n vertices. Then Algorithm SEQUENCE obtains a square-edge sequence of G in $O(n)$ time.*

Proof. We have seen that the first part of SEQUENCE runs in the same time as Algorithm C4. It was shown in [3] that the complexity of C4 is at most:

$$O(m) + O(n) + O\left(\sum_{uv \in E(G)} \min\{d(u), d(v)\}\right).$$

Let S be a square-edge sequence of G and let T be a spanning tree obtained after deleting the edges of S . Then the summation from the above can be written as

$$O\left(\sum_{uv \in S} \min\{d(u), d(v)\}\right) + \sum_{uv \in T} \min\{d(u), d(v)\}.$$

In Section 2 we have shown that $m = 2n - 2 - k$ holds in a cube-free median graph. Hence $m = O(n)$. Therefore the complete running time is bounded as

$$O(m) + O(n) + O\left(\sum_{i=1}^n d(v_i) + \sum_{i=1}^n d(v_i)\right) \leq O(n).$$

Since the running time of the second part of the SEQUENCE is bounded by the number of quadrangles (that cannot exceed the number of edges) the assertion follows. \square

References

- [1] Bandelt, H. J., Dählmann, A., Schütte, H.: Absolute retracts of bipartite graphs. *Discrete Appl. Math.* **16** (1987) 191–215.
- [2] Chepoi, V.: d -Convexity and isometric subgraphs of Hamming graphs. *Cybernetics* **1** (1988) 6–9.
- [3] Chiba, N., Nishizeki T.: Arboricity and subgraph listing algorithms. *SIAM J. Comput.* **14** (1985) 210–223.
- [4] Djoković, D.: Distance preserving subgraphs of hypercubes. *J. Combin. Theory Ser. B* **14** (1973) 263–267.
- [5] Imrich, W., Klavžar, S.: A convexity lemma and expansion procedures for bipartite graphs. *European J. Combin.* **19** (1998) 677–685.
- [6] Klavžar, S., Mulder, H. M., Škrekovski, R.: An Euler-type formula for median graphs. *Discrete Math.* **187** (1998) 255–258.
- [7] Klavžar, S., Škrekovski, R.: On median graphs and median grid graphs. *Discrete Math.* **219** (2000) 287–293.
- [8] Mulder, H. M.: The structure of median graphs. *Discrete Math.* **24** (1978) 197–204.
- [9] Mulder, H. M.: *The Interval Function of a Graph*. Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.
- [10] Winkler, P.: Isometric embeddings in products of complete graphs. *Discrete Appl. Math.* **7** (1984) 221–225.