Cancellation properties of products of graphs*

Wilfried Imrich[a] Sandi Klavžar[b,c] Douglas F. Rall[d]

[a]Montanuniversität Leoben, A-8700 Leoben, Austria

[b]Department of Mathematics and Computer Science
University of Maribor, Koroška 160, 2000 Maribor, Slovenia

[c]Institute of Mathematics, Physics and Mechanics
Jadranska 19, 1000 Ljubljana

[d]Department of Mathematics, Furman University
Greenville, SC, USA.

Abstract

This note extends results of Fernández, Leighton, and López-Presa on the uniqueness of r^{th} roots for disconnected graphs with respect to the Cartesian product to other products and shows that their methods also imply new cancelation laws.

Key words: Graph products; Cancellation property; Uniqueness of roots

AMS subject classification (2000): 05C60, 05C75

1 Introduction

In a recent paper [2] Fernández, Leighton, and López-Presa showed among others that the isomorphism of the Cartesian powers G^r and H^r implies the isomorphism of G and H. As the authors note, in the case of connected graphs their result follows immediately from the unique prime factor decomposition theorem of Sabidussi [7] and Vizing [9].

*This work was supported in part by the Ministry of Science of Slovenia under the grants J1-6150.

Unique prime factorization also implies the cancellation property for connected graphs. In this note we show the validity of the cancellation property for Cartesian products without the connectedness assumption and that the isomorphism of G^r and H^r implies that of G and H also for certain classes of infinite graphs.

We also note that the same method of proof implies the cancellation property for the strong product and, in the class of nonbipartite graphs, the direct one. Moreover the implication $G^r \cong H^r \implies G \cong H$, where powers are taken with respect to the strong or the direct product, also holds.

Interestingly, the latter results have already been shown by Lovász [5], except for the cancellation property

$$A \times C \cong B \times C \implies A \cong B.$$

Lovász proves it in the case when there are homomorphisms from A and B to C, which need not be satisfied in the case we treat.

Finally we wish to note that the implication

$$G^r \cong H^r \implies G \cong H$$

and the cancellation property hold for the lexicographic product too [3].

The note ends with a few remarks about infinite graphs.

2 Results

The main idea of the proof in [2] is based on the fact that finite graphs form a commutative semiring with unit K_1 with respect to Cartesian multiplication and disjoint union, see [4, page 30]. Every connected graph is uniquely representable as a product of prime graphs. This allows an embedding of all (connected or disconnected) finite graphs into a polynomial ring $\mathcal{R}$ with integer coefficients that is compatible with Cartesian multiplication and disjoint union. The indeterminants are just the graphs that are indecomposable with respect to the Cartesian product. Every finite graph G is thus uniquely representable in $\mathcal{R}$ as a polynomial $P(G)$ with positive coefficients.

Theorem 2.1 *Let A, B, C be finite graphs such that $A \,\square\, C \cong B \,\square\, C$. Then $A \cong B$.*

Proof. It is easy to see that $P(A \,\square\, C) = P(A) \cdot P(C)$. Clearly,

$$P(A \,\square\, C) = P(B \,\square\, C)$$

whence

$$P(A) \cdot P(C) = P(B) \cdot P(C)\,,$$

and thus $P(A) = P(B)$ by the cancellation property in $\mathcal{R}$. Therefore $A \cong B$. $\square$

The same approach can also be used for direct and strong products of graphs. Recall that both the direct product $G \times H$ and the strong product $G \boxtimes H$ of graphs G and H have the same vertex sets as the Cartesian product. Two vertices of the direct product are joined by an edge whenever their projections form edges in both factor graphs. The edge set of the strong product is the union of the edge sets of the Cartesian and the direct product.

Just as the Cartesian product, the strong product also has the unique prime factor decomposition property for connected graphs [1], cf. also [4]. Again K_1 is a unit and we have distributivity with respect to the disjoint union. Thus, as in the case of the Cartesian product, we obtain a commutative semiring that can be embedded into $\mathcal{R}$, which allows the derivation of the same results. We have thus found an easy proof of the following proposition:

Proposition 2.2 *Let G^r and H^r be powers of G and H with respect to the strong product. If $G^r \cong H^r$, then $G \cong H$. Furthermore, if A, B, C are graphs such that $A \boxtimes C \cong B \boxtimes C$, then $A \cong B$.*

Let us consider the direct product now. It is convenient to consider this product in the class of simple graphs with loops, which we will denote by Γ_0. The following theorem of Lovász [5] holds:

Theorem 2.3 *Let $G, H \in \Gamma_0$. If $G^r \cong H^r$, where powers are taken with respect to the direct product, then $G \cong H$. Furthermore, if $A, B, C \in \Gamma_0$ and if there are homomorphisms from A and B to C, then $A \times C \cong B \times C$, implies $A \cong B$.*

Let us see how this compares with the results we can achieve by our method. In order to obtain a commutative semiring of graphs with unit that can be embedded into $\mathcal{R}$ we have to allow loops and restrict attention to nonbipartite graphs. The reason is, that K_1 is not a unit for the direct product, but the one vertex graph K_1^* with a loop is. Furthermore, for the direct product the unique prime factorization theorem does not hold in general, but it does hold for connected nonbipartite graphs in Γ_0. Thus, the class of nonbipartite graphs in Γ_0 is a commutative semiring with respect to the direct product and the disjoint union, and K_1^* as a unit. Every connected graph in this class has a unique prime factorization with respect to the direct product [1, 6], and this allows an embedding into $\mathcal{R}$ as before. We thus obtain the following results:

Proposition 2.4 *Let the nonbipartite graphs G^r and H^r be powers of G and H with respect to the direct product. If $G^r \cong H^r$, then $G \cong H$.*

Theorem 2.5 *If A, B, C are nonbipartite graphs such that $A \times C \cong B \times C$, then $A \cong B$.*

The proposition is a special case of the result of Lovász, but Theorem 2.5 is somewhat stronger than the second part of Theorem 2.3.

We show now that Theorem 2.3 implies Proposition 2.2. The reason is that the strong product $G \boxtimes H$ of two graphs can be obtained by addition of loops to every vertex of G and H, multiplication of the resulting graphs with respect to the direct product, and subsequent deletion of the loops. The strong product can thus be considered as special case of the direct one. Moreover, since every graph with a loop is nonbipartite, we do not need such a restriction for the strong product. This already takes care of the first part of Proposition 2.2. The second part follows since there are always homomorphisms from A and B into C if C has at least one loop. (Simply map A and B into such a vertex and its loop).

For the sake of completeness we wish to remark that Proposition 2.2 also holds for the lexicographic product, see [3] or [4]. The lexicographic product is the only noncommutative standard product of graphs. For its definition and further properties we also refer to [4].

3 Concluding remarks

The Cartesian product of infinitely many nontrivial graphs is disconnected, every one of its connected components is called a weak Cartesian product. It is well known, see e.g. [4] that every connected (finite or infinite) graph has a unique prime factorization with respect to the weak Cartesian product. This implies, that the r^{th} root of a connected graph with respect to that product is unique if it exists. One can presumably use this to show that the equality of G^r and H^r, where powers are taken with respect to the Cartesian product, implies the equality of G and H also in the case of infinite graphs if G and H have only finitely many components.

Clearly, the cancellation law does not hold for infinite graphs, not even for connected ones. For more information on infinite graphs we refer to [8].

References

[1] W. Dörfler and W. Imrich, Über das starke Produkt von endlichen Graphen (German), Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II 178 (1970) 247–262.

[2] A. Fernández, T. Leighton, and J. L. López-Presa, Containment propeties of product and power graphs, Discrete Appl. Math., in press: doi:10.1016/j.dam.2006.06.001.

[3] W. Imrich, Assoziative Produkte von Graphen (German), Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II 180 (1972) 203–239.

[4] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, 2000.

[5] L. Lovász, On the cancellation law among finite relational structures, Period. Math. Hungar. 1 (1971) 145–156.

[6] R. McKenzie, Cardinal multiplication of structures with a reflexive relation, Fund. Math. 70 (1971) 59–101.

[7] G. Sabidussi, Graph multiplication, Math. Z. 72 (1960) 446–457.

[8] V. Trnková and V. Koubek, Isomorphisms of products of infinite graphs, Comment. Math. Univ. Carolin. 19 (1978) 639–652.

[9] V.G. Vizing, The Cartesian product of graphs, Comp. El. Syst. 2 (1966) 352–365.