Characterizing posets for which their natural transit functions coincide*

Boštjan Brešar
Faculty of Electrical Engineering and Computer Science
University of Maribor, Slovenia
bostjan.bresar@uni-mb.si

Manoj Changat
Department of Futures Studies, University of Kerala
Trivandrum-695 034, India
mchangat@gmail.com

Sandi Klavžar
Department of Mathematics and Computer Science
FNM, University of Maribor, Slovenia
sandi.klavzar@uni-mb.si

Joseph Mathews
Department of Mathematics, St.Berchmans College
Changanassery-686 101, Kerala, India
jose_chingam@yahoo.co.in

Antony Mathews
Department of Futures Studies, University of Kerala
Trivandrum-695 034, India
sonykandans@yahoo.co.in

Prasanth G. Narasimha-Shenoi
Department of Mathematics, Government College, Chittur
Palakkad-678 104, India
gnprasanth@gmail.com

*Work supported by the Ministry of Science of Slovenia and by the Ministry of Science and Technology of India under the bilateral India-Slovenia grants BI-IN/06-07-002 and DST/INT/SLOV-P-03/05, respectively.

Abstract

The standard poset transit function of a poset P is a function T_P that assigns to a pair of comparable elements the interval between them, while $T_P(x,y) = \{x,y\}$ for a pair x,y of incomparable elements. Posets in which the standard poset transit function coincides with the shortest-path transit function of its cover-incomparability graph are characterized in three ways, in particular with forbidden subposets.

Keywords: transit function, ranked poset, underlying graph, geodesic interval, induced-path interval.

1 Introduction

The notion of convexity has been extended from Euclidean spaces to several different mathematical structures. In the last several decades an abstract theory of convex structures was developed. It is based on just two (or three in the infinite case) natural conditions, imposed to a family of subsets of a given set. The theory is comprehensively surveyed in the van de Vel's book [13], where the so-called interval convexity turns out as one of its universal concepts.

As shown by Calder in [3], pertinent properties of a mapping $I : X \times X \rightarrow 2^X$ that yields a convexity on a set X are the *symmetry law* ($I(x,y) = I(y,x)$ for all $x,y \in X$) and the *extensive law* ($x,y \in I(x,y)$). These two laws constitute all axioms for the formal *interval convexity* with I as the *interval operator* [13]. By adding also the idempotent law ($I(x,x) = \{x\}$ for all $x \in X$), one gets the concept of a transit function as defined by Mulder [10]. His purpose was to generalize geodesic (and some other) intervals in graphs and other discrete structures, including posets. Furthermore, for a set X on which a transit function T is defined, the *underlying graph* G_T was introduced as the graph with X as its vertex set, where distinct u and v in X are adjacent whenever $|T(u,v)| = 2$.

In a poset $P = (X, \leq)$ by the notion of interval $[x,y]$ one usually assumes that x and y are two comparable elements, and $[x,y] = \{z : x \leq z \leq y\}$. This function can be extended to all pairs of elements in a poset, by setting $T_P(x,y) = \{x,y\}$ if x and y are incomparable, and $T_P(x,y) = [x,y]$ otherwise [7, 10, 13]. This yields an interval convexity in the general sense as described above. Clearly T_P is also a transit function on P and is called the *standard poset transit function*.

The underlying graph G_{T_P} of the standard poset transit function T_P was studied in [1] under the name *cover-incomparability graph* of a poset P.

(For other connections between posets and graphs we refer to [12].) In G_{T_P} the most common transit function is given by the geodesic intervals (formed by vertices on shortest paths between pairs in $V(G_{T_P})$), which is denoted I_{G_P}. The natural question appears, in which posets both transit functions (interval convexities) T_P and I_{G_P} coincide. In this note we characterize such posets in two ways, once by listing forbidden (induced) subposets, and in the other case as certain maximal ranked posets. Interestingly, in such posets also the induced-path transit function J_{G_P} in the underlying graph coincides with the standard poset transit function of P, which is, in fact, characteristic for these posets.

In the next section we fix terminology and state some simple observations. In Section 3 we prove the main theorem. In the concluding we show how can the theorem be extended from finite to chain-finite, countable posets.

2 Preliminaries

A *transit function* on a non empty set V is a function $T : V \times V \to 2^V$ satisfying the following axioms:

(t1) $u \in T(u, v)$ for any u and $v \in V$.
(t2) $T(u, v) = T(v, u)$ for all u and $v \in V$.
(t3) $T(u, u) = \{u\}$ for all $u \in V$.

Prominent examples of transit functions on a connected graph $G = (V, E)$ are the geodesic transit function (studied extensively in [9]):

$$I_G(u, v) = \{w \in V \mid w \text{ lies on a shortest } u, v\text{−path in } G\},$$

the induced-path transit function [8, 5]:

$$J_G(u, v) = \{w \in V \mid w \text{ lies on an induced } u, v\text{−path in } G\},$$

and the all-paths transit function A, which is the coarsest path transit function on G [4, 6]:

$$A_G(u, v) = \{w \in V \mid w \text{ lies on some } u, v\text{−path in } G\}.$$

Recall [2, 11] that a *ranked poset* is a poset $P = (V, \leq)$ that is equipped with a rank function $\rho : V \to \mathbb{Z}$ satisfying:

- ρ is constant on all minimal elements of P (usually with value -1 or 0), and

- ρ preserves covering relations: if b covers a then $\rho(b) = \rho(a) + 1$.

A ranked poset P is said to be *complete* if for every i, every element of rank i covers all the elements of rank $i - 1$.

Let $P = (V, \leq)$ be a poset. Then the poset $Q = (V', \leq')$ is an *induced subposet* of P if $V' \subseteq V$ and if for any elements u and v of V', $u \leq' v$ if and only if $u \leq v$.

Recall that the *height* of a poset P is the maximum size of a chain in P, and a poset is called *bipartite* if its height is at most 2 [2].

Let $P = (V, \leq)$ be a poset, then the graph $G_P = (V, E)$ is called the *cover-incomparability graph* of P if $ab \in E$ whenever a covers b, or b covers a, or a and b are incomparable. In other words, the edge set of G_P is the union of the edge set of the cover graph of P and the incomparability graph of P. Note that the cover-incomparability graph G_P of a poset P is just the underlying graph of the standard poset transit function. Hence G_{T_P} would be an appropriate notation for the cover-incompatibility graph (as stated in the introduction) which we will shorten to G_P in the sequel. We denote by I_{G_P} the geodesic transit function in the cover-incompatibility graph G_P, and J_{G_P} the induced-path transit function in G_P.

It can be easily verified that for any poset P, the graph G_P is connected [1]. Let u, v be a pair of incomparable elements of a poset P. If P has more than two elements, then it is clear that there exists another element z in P which either covers or is incomparable with one of u or v or both u and v. In all cases $A_{G_P}(u, v)$ (where A_{G_P} stands for the all-paths transit function in G_P) contains z, but $T_P(u, v)$ consists only of u and v alone. Therefore, we have the following observation:

Remark 2.1. *Let $P = (V, \leq)$ be a poset on at least three elements. Then $T_P = A_{G_P}$ if and only if P is a chain.*

3 The theorem

In this section we characterize the posets $P = (V, \leq)$ in which $T_P = I_{G_P}$. First, if P is bipartite (that is, the size of a longest chain is at most 2), then G_P is a complete graph which readily implies:

Remark 3.1. *Let $P = (V, \leq)$ be a bipartite poset. Then $T_P = I_{G_P} = J_{G_P}$.*

In the following theorem we consider the non trivial case, when the height of a poset is greater than 2.

Theorem 3.2. *Let $P = (V, \leq)$ be a poset of height at least 3. Then the following statements are equivalent:*

(i) $T_P = I_{G_P}$*;*
(ii) $T_P = J_{G_P}$*;*
(iii) P is P_1-, P_2-, and P_3-free, see Fig. 1;
(iv) P is a complete ranked poset.

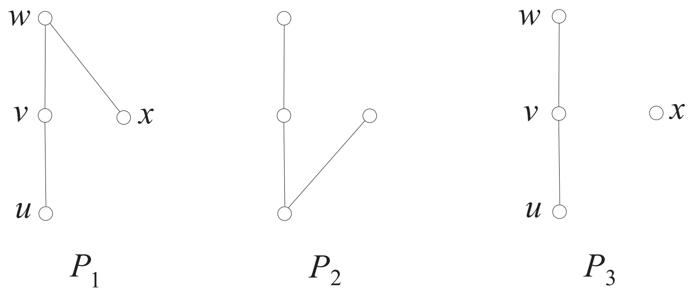

Figure 1: Forbidden subposets

In the rest of this section we prove Theorem 3.2 and begin with the following implication.

(i) $\Rightarrow$ *(iii)*. Let u, v, w, x be the elements of P that induce a subposet isomorphic to P_1, see Fig. 1. Let $x = x_0, x_1, \ldots, x_k = w$ be a x, w-chain in P. Note that $k \geq 1$, and that if $k = 1$ then $w = x_1$. Note that none of the elements x_i, $i < k$, is below v because otherwise x would be below v.

Let i, $1 \leq i \leq k$, be the smallest index such that $v \leq x_i$. Such an index exists since $v \leq w = x_k$. Then v and x_{i-1} are incomparable. Let $v = v_0, v_1, \ldots, v_j = x_i$ be a v, x_i-chain in P. Note that $j \geq 1$.

Suppose $j \geq 2$. Then consider $v_{j-2}, v_{j-1}, v_j, x_{i-1}$. Then v_{j-2} is incomparable with x_{i-1}. Indeed, if $x_{i-1} \leq v_{j-2}$ would hold, then x_i is not covering x_{i-1}. And if $v_{j-2} \leq x_{i-1}$ we would have $v \leq x_{i-1}$, which is not possible by the selection of i. Analogous argument implies that v_{j-1} is incomparable with x_{i-1}. Now observe that $d_{G_P}(v_{j-2}, x_i) = 2$ hence $x_{i-1} \in I_{G_P}(v_{j-2}, x_i)$. On the other hand, $x_{i-1} \notin T_P(v_{j-2}, x_i)$, a contradiction.

Suppose $j = 1$, that is, x_i covers v. If $i \geq 2$ then $v \in I_{G_P}(x_{i-2}, x_i)$ but $v \notin T_P(v_{i-2}, x_i)$. Let $i = 1$. Then x_1 covers x. Let y be a vertex on a u, v-chain that is covered by v. Then $y \leq x$ would imply $u \leq x$. Moreover, $x \leq y$ would mean that x is not covered by x_1. It follows that y and x are incomparable and hence $x \in I_{G_P}(y, x_i)$ but $x \notin T_P(y, x_i)$.

Thus we have proved that if P has an induced subposet P_1 then $T_P \neq I_{G_P}$. Analogous arguments also yields that a subposet P_2 implies that $T_P \neq I_{G_P}$.

Assume finally that u, v, w, x are the elements of P that induce a subposet isomorphic to P_3, see Fig. 1. Let $u = u_0, u_1, \ldots, u_k = w$ be a u, w-chain that contains v. Then $k \geq 2$ and moreover, none of the elements u_i is comparable to x. Then $x \in I_{G_P}(u, u_2)$ but $x \notin T_P(u, u_2)$.

(ii) $\Rightarrow$ *(iii)*. This implications follows by the same arguments as the implication *(i)* $\Rightarrow$ *(iii)* because a path of length 2 is a shortest path if and only if it is induced.

(iii) $\Rightarrow$ *(iv)*. Assume that P has no induced subposet isomorphic to P_1, P_2, or P_3. Let $C = u_0 \leq u_1 \leq \ldots \leq u_n$ be a longest chain in $(P, \leq)$. Then $n \geq 2$.

Define S_0 as the set consisting of the minimal elements of P. Further define S_i, $1 \leq i \leq n$, as the subset of V containing u_i and the elements of P which are incomparable with u_i and that cover u_{i-1}.

Claim 1. $u_0 \in S_0$.
This is clear because otherwise C would not be a longest chain.

Claim 2. Elements of S_i, $0 \leq i \leq n$, are pairwise incomparable.
This is clear for $i = 0$. Consider different elements v_i and w_i of S_i, $i \geq 1$. By the definition of S_i, both v_i and w_i cover u_{i-1}. But then v_i and w_i are incomparable.

Claim 3. For $i \geq 1$ any element of S_i covers every element of S_{i-1}.
Assume first $i \geq 2$ and consider elements $v_i \in S_i$ and $w_{i-1} \in S_{i-1}$. Since $v_i \in S_i$, it covers u_{i-1}. By Claim 2, u_{i-1} and w_{i-1} are incomparable and hence w_{i-1} cannot be below u_{i-2}. It follows that $v_i, u_{i-1}, u_{i-2}, w_{i-1}$ form either an induced P_2 or an induced P_3. Hence v_i and w_{i-1} are comparable. Using Claim 2 again, w_{i-1} cannot be above v_i hence there must be a chain $w_{i-1} \leq x \leq \cdots \leq v_i$.

Suppose $x \neq v_i$. Then x and u_{i-1} are incomparable. Indeed, $u_{i-1} \leq x$ would yield $u_{i-1} \leq x \leq v_i$, a contradiction with the fact that v_i covers u_{i-1}. On the other hand, $x \leq u_{i-1}$ would give us that $w_{i-1} \leq x \leq u_{i-1}$ contradicting Claim 2. So x and u_{i-1} are incomparable, but then v_i, x, w_{i-1}, u_{i-1} induce P_1. We conclude that $x = v_i$ and hence v_i covers w_{i-1}. Note that the same argument also applies if $v_i = u_i$.

It remains to prove Claim 3 for $i = 1$. Let $v_1 \in S_1$ and $w_0 \in S_0$ be arbitrary elements. Then v_1 covers u_0. By the above paragraph, u_2 covers v_1. If w_0 and v_1 are incomparable, then u_0, v_1, u_2, w_0 induce P_3 or P_1. Since

w_0 is a minimal element, we can thus only have $w_0 \leq v_1$. Let x be the first element on a w_0, v_1-chain. Then we can argue as above that $x \in S_1$, which is only possible if $x = v_1$.

Claim 4. $\bigcup_{i=0}^{n} S_i = V$.
Let v be an arbitrary element of P. Consider a maximal chain $v_0 \leq v_1 \leq \ldots \leq v_m$ in P that contains v. Clearly, $m \leq n$. Hence $v = v_i$ for some $0 \leq i \leq m$. We claim that $v \in S_i$ and prove it by induction. The element v_0 is minimal, hence $v_0 \in S_0$. Assume now that $v_{i-1} \in S_{i-1}$. Hence by Claim 3, u_i covers v_{i-1}. If u_{i-1} and v_i are incomparable then consider u_i, u_{i-1}, u_{i-2}, v_i (or, if $i = 1$, consider u_{i-1}, u_i, u_{i+1}, v_i) to obtain P_2 or P_3 (or, P_1 or P_3). So u_{i-1} and v_i are comparable and it is only possible that $u_{i-1} \leq v_i$. Consider a chain $u_{i-1} \leq x \leq v_i$. Then x and v_{i-1} are incomparable. But then u_{i-1}, x, v_i, v_{i-1} induce P_1. Hence $x = v_i = v$. Therefore v covers u_{i-1} and since in addition v_i is incomparable to u_i we conclude that $v = v_i \in S_i$.

Claim 5. For $0 \leq j \leq i - 2 \leq n - 2$, no element of S_i covers an element of S_j.
Let $v_i \in S_i$, $w_j \in S_j$, $j \leq i - 2$. Suppose v_i covers w_j. Let $v_j \leq v_{j+1} \leq \ldots \leq v_i$, where $v_k \in S_k$, be a v_j, v_i-chain. Then w_j is incomparable with $v_j, \ldots, v_{i-1}$. Hence $v_{i-2}, v_{i-1}, v_i, w_j$ induce a P_1, a contradiction that proves Claim 5.

For $v \in V$, define $\rho : V \to \mathbb{Z}$ by $\rho(v) = i$ where $v \in S_i$. Then ρ is a rank function on P. Moreover, Claim 3 implies that this ranked poset is complete.

To complete the proof note that implications $(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (ii)$ are straightforward.

4 Extension to infinite posets

Let $P = (V, \leq)$ be an arbitrary poset. Then P is called *chain-finite* if every chain between any comparable elements u and v is finite. We claim that Theorem 3.2 extends to countable, chain-finite posets with the following modifications.

By a *complete ranked poset* we now mean a poset whose elements can be partitioned into sets S_i, $i \in \mathbb{Z}$, such that every element of S_i covers every element of S_{i-1} for all i and that $S_k \neq \emptyset$ as soon as there are indices i and j such that $i < k < j$ and $S_i \neq \emptyset$, $S_j \neq \emptyset$.

Let $P = (V, \leq)$ be a countable, chain-finite poset. Then note that G_P is a connected (infinite) graph and hence I_{G_P} and J_{G_P} are well-defined.

With these additions Theorem 3.2 holds for such posets. The proof for $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$, $(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (ii)$ goes along the same lines as in the finite case. To prove $(iii) \Rightarrow (iv)$ we can define the sets S_i in a similar way. Notably, let C be an arbitrary maximal chain of P and let its elements be $\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$ (or $u_0, u_1, u_2, \ldots$ if P has minimal elements, and $\ldots, u_{-2}, u_{-1}, u_0$ if P has maximal elements). Then define S_i, $i \in \mathbb{Z}$, as the subset of V containing u_i and the elements of P which are incomparable with u_i and cover u_{i-1}.

Now in the proof of $(iii) \Rightarrow (iv)$ most of the claims go through regardless of finiteness of P. More precisely, Claim 1 is the same, or even not needed if P has no minimal and no maximal elements (if P has only maximal elements then an analogous trivial claim has to be used). Claims 2 and 5 have the same proof as above, while Claim 3 is even easier in the case, when P has no minimal elements, since the last paragraph of its proof is not needed. Hence, we only need to prove Claim 4 which is the only non trivial part of the extension to infinite posets.

Let x be an arbitrary element of a countable, chain-finite poset P. We need to show that there exists an i such that $x \in S_i$ (from this one finds that $\bigcup_i S_i = V$). First step is to show that x is incomparable with exactly one element from $\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$.

Suppose that x is incomparable with u_i and u_j, and let $i < j$. Then u_i, u_j, u_{j+1} and x induce a subposet isomorphic to P_2 or P_3, a contradiction. Next, suppose x is comparable with all elements of the maximal chain C. We claim that then the set $S = \{u_i \in C : \ u_i \leq x\}$ is nonempty and bounded from above. Suppose this is not the case. Then assuming that S is empty (resp. S not bounded from above) we quickly derive that for all $u_i \in C$ we have $x \leq u_i$ (resp. $u_i \leq x$), which is a contradiction with C being a maximal chain. Thus, let u_j be the largest element in C such that $u_j \leq x$, and note that u_j cannot be a maximal element in P because C is a maximal chain. Thus $x \leq u_{j+1}$ which again contradicts the fact that C is maximal since x can be inserted between u_j and u_{j+1}, extending the chain C. Hence x cannot be comparable to all elements of C either.

Thus x is incomparable with exactly one element of C, say u_i. If u_i is a minimal element then $i = 0$ and $x \in S_0$. Hence, let u_i not be minimal, and we need to show that x covers u_{i-1} (this will imply that $x \in S_i$).

It is clear that $u_{i-1} \leq x \leq u_{i+1}$. Let v be the first element on a chain from u_{i-1} to x, and assume that $v \neq x$. Suppose that v is incomparable with u_i. Then, since x is also incomparable with u_i we infer that u_{i-1}, u_i, v and x form an induced subposet, isomorphic to P_2 or P_3. Since this is forbidden, the only possibility is that v and u_i are comparable. If $u_i \leq v$, then by transitivity $u_i \leq x$, but u_i and v are incomparable. Also $v \leq u_i$ is not possible, since then one could extend the chain C between u_{i-1} and u_i with v and the chain between v and u_i. Since C is a maximal chain this is a contradiction, hence $v = x$. Thus x covers u_{i-1} and so $x \in S_i$. Theorem 3.2 indeed extends to countable chain-finite posets.

References

[1] B. Brešar, M. Changat, S. Klavžar, M. Kovše, J. Mathew, A. Mathews, Cover-incomparability graphs of posets, manuscript.

[2] G. Brightwell, D.B. West, Partially ordered sets, Chapter 11 in: Handbook of Discrete and Combinatorial Mathematics (K.H. Rosen, ed.) CRC Press, Boca Raton, 2000, 717–752.

[3] J. R. Calder, Some elementary properties of interval convexities, J. London Math. Soc. 3 (1971) 422–428.

[4] M. Changat, S. Klavžar, H. M. Mulder, The all path transit function of a graph, Czech. Math. J. 51 (2001) 439–448.

[5] M. Changat, J. Mathew, Induced path transit function, monotone and Peano axioms, Discrete Math. 286 (2004) 185–194.

[6] M. Changat, H. M. Mulder, G. Sierksma, Convexities related to path properties on graphs, Discrete Math. 290 (2005) 117–131.

[7] A. Mathews, J. Mathews, Transit functions on posets and lattices. In: *Convexity in Discrete Structures* (M. Changat, S. Klavžar, H. M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 105–116.

[8] M. A. Morgana, H. M. Mulder, The induced path convexity, betweenness and svelte graphs, Discrete Math. 254 (2002) 349–370.

[9] H. M. Mulder, *The Interval Function of a Graph*, Mathematisch Centrum, Amsterdam, 1980.

[10] H. M. Mulder, Transit functions on graphs (and posets). In: *Convexity in Discrete Structures* (M. Changat, S. Klavžar, H. M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 117–130.

[11] D. Stanton, D. White, *Constructive Combinatorics*, Springer, New York, 1986.

[12] W. Trotter, Graphs and partially ordered sets: recent results and new directions, Cong. Numer. 116 (1996) 253–278.

[13] M. van de Vel, *Theory of Convex Structures*, North-Holland, Amsterdam, 1993.