On the geodetic number and related metric sets in Cartesian product graphs*

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Abstract

A set S of vertices of a graph G is a geodetic set if every vertex of G lies in at least one interval between the vertices of S. The size of a minimum geodetic set in G is the geodetic number of G. Upper bounds for the geodetic number of Cartesian product graphs are proved and for several classes exact values are obtained. It is proved that many metrically defined sets in Cartesian products have product structure and that the contour set of a Cartesian product is geodetic if and only if their projections are geodetic sets in factors.

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1 Introduction

In the varied field of metric graph theory studies are often motivated by classical concepts from other fields, say by convexity and boundary sets from Euclidian spaces. Different boundary sets (contour, eccentric, peripheral) in graphs were investigated in [3, 5, 9, 10], while in [8] a certain convexity number of graphs was studied. Some other concepts have a more graph-theoretic flavour, which is certainly true for geodetic sets and the geodetic number of a graph introduced in [13], and studied further in [2, 11, 12].

Two of the mentioned concepts were combined in [3, 4] where the following question was posed: for which graphs G the contour set of G is also its geodetic set? For several classes of graphs the answer (positive or negative) is known. On the other hand, the problem is open for bipartite graphs [4] and in particular for median graphs [1].

Knowing that the shortest-path metrics in Cartesian product graphs is in straightforward correlation with the metrics in factor graphs, it is natural to consider metrically defined concepts in the Cartesian product setting, see [6, 7, 14, 16, 17]. Geodetic sets in prisms which are the simplest Cartesian products of two graphs (products of the form $G \Box K_2$) were studied already in [12] and [15]. Recently the geodetic number of the Cartesian product of arbitrary two connected graphs was investigated [16], and sharp lower and upper bounds were obtained for $G \Box H$.

In this paper we further study the presented concepts in Cartesian product graphs. In the next section we consider upper bounds for the geodetic number $g(G \Box H)$ of the Cartesian product of graphs G and H. We give a short proof of the main result from [16] and improve the bound for products whose factors G and H possess so-called linear geodetic sets of sizes $g(G)$ and $g(H)$, respectively. We also present several classes of graphs that enjoy this property. In Section 3 we obtain several exact values for $g(G \Box H)$. We prove in particular that $g(G \Box H) = \max\{g(G), g(H)\}$ holds for any trees G and H. The question of when $g(G \Box G) = g(G)$ is also partially answered. Then, in Section 4, we consider the boundary, the contour, the eccentric and the peripheral sets in arbitrary Cartesian product of graphs and show that they all have a product structure. In the concluding section we show that the contour set of a Cartesian product of graphs $G \Box H$ is a geodetic set of this product if and only this holds in both G and H.

We next introduce the key concepts of this paper.

All graphs in this paper are connected and finite. The *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the usual shortest path distance. If the graph G will be clear from the context we will abbreviate $d_G(u, v)$ to $d(u, v)$. A shortest u, v-path is called a *u, v-geodesic*. The *interval* $I_G[u, v]$ between vertices u, v of a graph G is the set of vertices of all shortest paths between u and v in G. Again, we will sometimes write $I[u, v]$ for $I_G[u, v]$ when G will be clear.

Let S be a set of vertices of a graph G. Then the *geodetic closure* $I_G[S]$ is the union of intervals between all pairs of vertices from S, that is, $I_G[S] = \cup_{u,v \in S} I_G[u, v]$. A set S of vertices of G is a *geodetic set* in G if $I_G[S] = V(G)$. The size of a minimum geodetic set in a graph G is called the *geodetic number* of G and denoted $g(G)$.

The *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ in which vertices (g, h) and (g', h') are adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The most important metric property of the Cartesian product operation is that for any graphs G and H,

$$d_{G \,\Box\, H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h')\,.$$

We also recall that the Cartesian product is associative and commutative with K_1 as its unit. For further information on this graph product see [14].

2 Upper bounds on the geodetic number

In this section we consider upper bounds on the geodetic number of Cartesian product graphs. We first give a short proof of a general upper bound proved for the first time in [16]. Then we introduce the so-called linear geodetic sets and show that the general upper bound can be improved, roughly speaking, by a factor of 2, provided that factor graphs contain linear minimum geodetic sets.

For the announced short proof we recall the following lemma that is a well-known part of the folklore. It is around at least since the book [17] has been published.

Lemma 2.1 *Let $X = G \,\Box\, H$ be the Cartesian product of (connected) graphs G and H and let (g, h) and (g', h') be vertices of X. Then $I_X[(g, h), (g', h')] = I_G[g, g'] \times I_H[h, h']$. Moreover, $I_X[(g, h), (g', h')] = I_X[(g', h), (g, h')]$.*

Using Lemma 2.1 we now give a short argument for the following theorem.

Theorem 2.2 ([16]) *Let G and H be graphs with $g(G) = p \geq g(H) = q \geq 2$. Then $g(G \,\Box\, H) \leq pq - q$.*

Proof. Set $X = G \,\Box\, H$. Let $S = \{g_1, \ldots, g_p\}$ and $T = \{h_1, \ldots, h_q\}$ be geodetic sets of G and H, respectively. Set

$$U = (S \times T) \setminus \cup_{i=1}^{q} \{(g_i, h_i)\} .$$

We claim that $I_X[U] = V(X)$. Let (g, h) be an arbitrary vertex of X. Then there exist indices i and i' such that $g \in I_G[g_i, g_{i'}]$, and there are indices j and j' such that $h \in I_H[h_j, h_{j'}]$. Since $p, q \geq 2$ we may assume that $i \neq i'$ and $j \neq j'$. Indeed, if, say $g = g_i \in S$, then select i' to be an arbitrary index from $\{1, \ldots, p\}$ different from i. Set $B = \{(g_i, h_j), (g_i, h_{j'}), (g_{i'}, h_j), (g_{i'}, h_{j'})\}$.

Suppose that one of the vertices from B is not in U. We may without loss of generality assume $(g_i, h_j) \notin U$. This means that $i = j$. Therefore $i' \neq j$ and $i \neq j'$. Then we infer that $(g, h) \in I_X[(g_i, h_{j'}), (g_{i'}, h_j)]$. Otherwise all vertices from B are in U. But then $(g, h) \in I_X[(g_i, h_j), (g_{i'}, h_{j'})]$. $\square$

Note that if G and H are graphs with $g(G) = g(H) = 2$ then Theorem 2.2 implies $g(G \,\Box\, H) = 2$.

In the proof of the upper bound of Theorem 2.2 not much of the structure of the Cartesian product is used. As it was demonstrated in [16] the bound cannot be improved in general. However, under some additional assumption(s) one can employ the product structure. We next give an example of such a result.

Let G be a graph and let $S = \{x_1, \ldots, x_k\}$ be a geodetic set of G. We say that S is a *linear geodetic set* if for any $x \in V(G)$ there exists an index i, $1 \leq i < k$, such that $x \in I[x_i, x_{i+1}]$.

Theorem 2.3 *Let G and H be graphs on at least two vertices with $g(G) = p$ and $g(H) = q$. Suppose that both G and H contain linear minimum geodetic sets. Then*

$$g(G \,\Box\, H) \leq \left\lfloor \frac{pq}{2} \right\rfloor .$$

Proof. Set $X = G \,\Box\, H$ and let $S = \{g_1, \ldots, g_p\}$ and $T = \{h_1, \ldots, h_q\}$ be linear geodetic sets of G and H, respectively. Set

$$U = (S \times T) \setminus \bigcup_{\substack{i+j \\ \text{even}}} \{(g_i, h_j)\} .$$

We claim that U is a geodetic set of X. Let (g, h) be an arbitrary vertex of X. Since S is linear, there exists an index i, $1 \leq i < p$, such that $g \in I_G[g_i, g_{i+1}]$, and because T is linear, there exists an index j, $1 \leq j < q$, such that $h \in I_H[h_j, h_{j+1}]$.

Suppose that $i + j$ is odd. Then $(i + 1) + (j + 1)$ is odd as well, hence $(g_i, h_j) \in U$ and $(g_{i+1}, h_{j+1}) \in U$. Because $(g, h) \in I_X[(g_i, h_j), (g_{i+1}, h_{j+1})]$ it follows that (g, h) lies in the geodetic closure of U. Suppose next that $i+j$ is even. Then both $i+(j+1)$ and $(i+1)+j$ are odd and thus $(g_i, h_{j+1}) \in U$ and $(g_{i+1}, h_j) \in U$. Since by Lemma 2.1, $(g, h) \in I_X[(g_i, h_{j+1}), (g_{i+1}, h_j)]$, also in this case (g, h) lies in the geodetic closure of U.

To conclude the proof observe that $|U| = \lfloor pq/2 \rfloor$. □

Many graphs admit linear minimum geodetic sets, complete graphs and graphs G with $g(G) = 2$ are obvious instances of such graphs. For another example consider complete bipartite graphs $K_{n,m}$ with $n, m \geq 4$. It is known [12] and easy to see that $g(K_{n,m}) = 4$. Moreover, selecting the first two vertices of a minimum geodetic set from one bipartition set and the last two vertices from the other yields a linear geodetic set.

An example of a graph G that does not admit a linear minimum geodetic set is shown in Fig. 1.

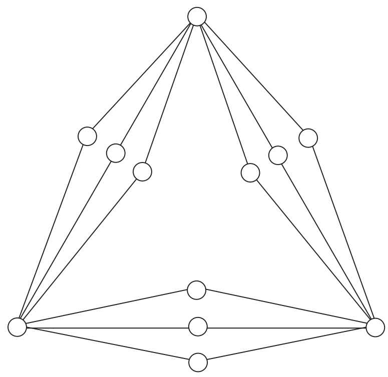

Figure 1: Graph that does not admit a linear geodetic set

Clearly, $g(G) = 3$, where the vertices of large degree form a unique minimum geodetic set, but this set is not linear. Similar examples of graphs that do not admit linear minimum geodetic sets can be obtained from the cycle C_{3k}, $k \geq 2$, to which three vertices are evenly attached.

3 Exact geodetic numbers

In this section we give several exact geodetic numbers of Cartesian product graphs. In all the cases, the value coincides with the bound from the following theorem.

Theorem 3.1 ([16]) *For any graphs G and H, $g(G \,\Box\, H) \geq \max\{g(G), g(H)\}$.*

We begin with the following application of Theorem 2.3 (and Theorem 3.1):

Corollary 3.2 *Let G be a graph on at least two vertices that admits a linear minimum geodetic set and let H be a graph with $g(H) = 2$. Then $g(G \,\Box\, H) = g(G)$.*

Proof. Use Theorem 2.3 for the upper bound and Theorem 3.1 for the lower bound. $\square$

For the next exact result we introduce the following property of geodetic sets. Let G be a graph. If S is a geodetic set of G such that

$$\forall u \in V(G) \setminus S, \forall v, w \in S : u \in I[v, w], \tag{1}$$

we say that S is a *complete geodetic set* of G. (Clearly any complete geodetic set is also a linear geodetic set.)

Proposition 3.3 *Let G and H be nontrivial graphs both having a complete minimum geodetic set. Then $g(G \,\Box\, H) = \max\{g(G), g(H)\}$.*

Proof. Set $X = G \,\Box\, H$ and let $S = \{g_1, \ldots, g_p\}$ and $T = \{h_1, \ldots, h_q\}$ be complete minimum geodetic sets of G and H, respectively. Without loss of generality we may assume that $p \geq q$, and note that $q \geq 2$. Set

$$U = \{(g_1, h_1), (g_2, h_2), \ldots, (g_q, h_q), (g_{q+1}, h_q), \ldots, (g_p, h_q)\}.$$

We claim that U is a geodetic set of X. We distinguish four cases (in which we will use Lemma 2.1 several times without referring to it).

Case 1: $g \in S$ and $h \in T$. Then there are indices i, j such that $g = g_i$ and $h = h_j$. We may assume $i \neq j$, otherwise $(g, h) \in U$. Then note that $(g, h) \in I_X[(g_j, h_j), (g_i, h_\sigma)]$, where $\sigma = i$ if $i \leq q$, and $\sigma = q$ if $i > q$.

Case 2: $g \in S$ and $h \notin T$. Then $g = g_i$ for some $i \in \{1, \ldots, p\}$. Let σ be defined as in the previous case, and let $j \leq q$ be an integer different from σ. Since $h \in I[h_j, h_\sigma]$ we derive $(g, h) \in I_X[(g_j, h_j), (g_i, h_\sigma)]$.

Case 3: $g \notin S$ and $h \in T$. Then $h = h_j$ for some $j \in \{1, \ldots, q\}$. Let $i \leq q$ be an integer different from j. Then clearly $(g, h) \in I_X[(g_i, h_i), (g_j, h_j)]$.

Case 4: $g \notin S$ and $h \notin T$. In this case we have $(g, h) \in I_X[(g_1, h_1), (g_2, h_2)]$.

Hence U is a geodetic set, and since $|U| = p = \max\{g(G), g(H)\}$ we derive that U is a minimum geodetic set and $g(X) = p$. □

Examples of graphs having a complete minimum geodetic set include complete graphs, stars, and graphs with geodetic number 2. Odd cycles are examples of graphs that admit a linear minimum geodetic set but not a complete minimum geodetic set. Note that $g(C_{2n+1}) = 3$ for any $n \geq 1$. A set of three vertices in $g(C_{2n+1})$, comprised of any vertex and its two eccentric vertices, is a minimum geodetic set which is linear, but not complete as soon as $n \geq 2$.

Yet another class of graphs that achieve the lower bound of Theorem 3.1 are trees. Note that the unique minimum geodetic set of a tree is the set of its leaves [12]. For the proof we need the following two simple properties.

Lemma 3.4 *Let T be a tree and L the set of its leaves.*

(P1) *If $x \in L$ and $u \in V(T)$ then there exists $y \in L$ (different from x) such that $u \in I_T[x, y]$.*

(P2) *If $x, y \in L$ and $u \in I_T[x, y]$ then for any $z \in L$ we have $u \in I_T[x, z]$ or $u \in I_T[y, z]$.*

Proof. (P1) is clear. For (P2) note that $T - u$ is a disconnected graph, and since $u \in I_T[x, y]$ we infer that x and y belong to distinct connected components of $T - u$. Let C be the component of $T - u$ in which z lies, and clearly not both x and y are in C. Now, if $x \notin C$ then $u \in I_T[x, z]$, otherwise $u \in I_T[y, z]$. □

Theorem 3.5 *For any trees T_1 and T_2,*

$$g(T_1 \,\Box\, T_2) = \max\{g(T_1), g(T_2)\}\,.$$

Proof. Let $p = g(T_1) \geq g(T_2)$ and denote by L_1, L_2 the sets of leaves of T_1 and T_2, respectively. Let $f : L_1 \to L_2$ be an arbitrary surjective mapping from $L_1 = \{x_1, \ldots, x_p\}$ onto the set of leaves of T_2. We claim that $S = \{(x_i, f(x_i)) \,|\, i = 1, \ldots, p\}$ is a geodetic set of $T_1 \,\Box\, T_2$.

Consider an arbitrary vertex $(g, h) \in V(T_1 \,\Box\, T_2)$. Clearly there exist $x_i, x_j \in L_1$ such that $g \in I_{T_1}[x_i, x_j]$. If $h \in I_{T_2}[f(x_i), f(x_j)]$ then by Lemma 2.1 we get $(g, h) \in I[(x_i, f(x_i)), (x_j, f(x_j))]$ as desired. Now suppose $h \notin I_{T_2}[f(x_i), f(x_j)]$. By (P1) from Lemma 3.4 there exists $y \in L_2$ such that $h \in I_{T_2}[f(x_i), y]$. Then by (P2) we infer that also $h \in I_{T_2}[f(x_j), y]$. Choose any $x \in f^{-1}(y)$. Since $g \in I_{T_1}[x_i, x_j]$ we get from (P2) that $g \in I_{T_1}[x_i, x]$

or $g \in I_{T_1}[x_j, x]$. In the first case we derive $(g, h) \in I[(x, y), (x_i, f(x_i))]$ and in the second case $(g, h) \in I[(x, y), (x_j, f(x_j))]$. Hence S is a geodetic set of $T_1 \Box T_2$ with $|S| = p = |L_1|$. $\Box$

In the rest of this section we consider the geodetic number of the square of a graph G, that is, of $G \Box G$. A natural question arises from Theorem 3.1 of when is $g(G \Box G) = g(G)$.

We say that a set of vertices S in a graph G is a *double geodetic set* if for any pair of vertices $x, y \in V(G)$ there exist vertices $a, b \in S$ such that both $x, y \in I[a, b]$. Clearly any double geodetic set is a geodetic set. Converse is of course not true in general (for instance the unique minimum geodetic set of a graph from Fig. 1 is not a double geodetic set). The obvious fact that every complete geodetic set is a double geodetic set implies that a graph can have a double geodetic set which is at the same time a minimum geodetic set (e.g. complete graphs, stars, graphs with geodetic number 2, etc.). Trees present another example of graphs that have a double minimum geodetic set.

Proposition 3.6 *If a graph G has a double minimum geodetic set then $g(G \Box G) = g(G)$.*

Proof. Let $S = \{x_1, \ldots, x_k\}$ be a double minimum geodetic set of a graph G (with $g(G) = k$). Hence, given a vertex $(g, h) \in V(G) \times V(G)$, there exist vertices $x_i, x_j \in S$ such that both $g, h \in I_G[x_i, x_j]$. By Lemma 2.1 we derive that $(g, h) \in I_{G \Box G}[(x_i, x_i), (x_j, x_j)]$. Therefore $\{(x_i, x_i) \mid i = 1, \ldots, k\}$ is a geodetic set of G and so $g(G \Box G) = k$. $\Box$

We wonder if the converse of Proposition 3.6 is true. The following observation is easy. If the set $\{(x_i, x_i) \mid i = 1, \ldots, k\}$ is a geodetic set of $G \Box G$ and $g(G) = k$ (implying $g(G \Box G) = g(G)$), then $\{x_i \mid i = 1, \ldots, k\}$ is a double geodetic set of G. However, could it happen that $G \Box G$ would not have a minimum geodetic set of such a (diagonal) structure?

4 Contour and other boundary sets

In [4] several variations of boundary/peripheral sets were considered. In this section we study the structure of these sets in Cartesian product graphs and show that they all have a product structure. This will be in particular used in the last section where the question of when the contour set of a product is its geodetic set is considered.

Let G be a graph. The *eccentricity* of a vertex $u \in V(G)$ is defined as $\mathrm{ecc}(u) = \max\{d(u,v) : v \in V(G)\}$. Given $u, v \in V(G)$, the vertex v is called an *eccentric vertex* of u if no vertex in V is further away from u than v, that is, if $d(u,v) = \mathrm{ecc}(u)$.

Lemma 4.1 *For every* $(g,h) \in V(G \,\square\, H)$, $\mathrm{ecc}_{G \square H}(g,h) = \mathrm{ecc}_G(g) + \mathrm{ecc}_H(h)$.

Proof. Let $(g,h) \in V(G \,\square\, H)$ and let (g',h') be an eccentric vertex of (g,h), that is, $d_{G \square H}((g',h'),(g,h)) = \mathrm{ecc}_{G \square H}(g,h)$. We claim that g' is an eccentric vertex of g in G. Suppose to the contrary that there exists $g'' \in V(G)$ such that $d_G(g'',g) > d_G(g',g)$. Then we have $d_{G \square H}((g'',h'),(g,h)) = d_G(g'',g) + d_H(h',h) > d_G(g',g) + d_H(h',h) = d_{G \square H}((g',h'),(g,h))$, a contradiction with the assumption. In a similar way we can prove that h' is an eccentric vertex of h in H. We derive $\mathrm{ecc}_{G \square H}(g,h) = d_{G \square H}((g',h'),(g,h)) = d_G(g',g) + d_H(h',h) = \mathrm{ecc}_G(g) + \mathrm{ecc}_H(h)$. $\square$

As usual, $N_G(v)$ stands for the set of neighbors of a vertex $v \in V(G)$. Then the *boundary* $\partial(G)$ of a graph G is the set

$$\partial(G) = \{v \in V(G) \mid \exists u \in V(G) \text{ such that } \forall w \in N_G(v) : d(u,w) \le d(u,v)\}.$$

The *contour set* $\mathrm{Ct}(G)$ of a graph G is defined by

$$\mathrm{Ct}(G) = \{v \in V(G) \mid \mathrm{ecc}(u) \le \mathrm{ecc}(v)\,, \forall u \in N_G(v)\}.$$

The *eccentricity* $\mathrm{Ecc}(G)$ of a graph G is the set

$$\mathrm{Ecc}(G) = \{v \in V(G) \mid \exists u \in V(G) \text{ such that } \mathrm{ecc}(u) = d(u,v)\}.$$

Finally, the *periphery* $\mathrm{Per}(G)$ of a graph G is the set

$$\mathrm{Per}(G) = \{v \in V(G) \mid \mathrm{ecc}(u) \le \mathrm{ecc}(v) \text{ for all } u \in V(G)\}.$$

Theorem 4.2 *For any graphs G and H:*
(i) $\partial(G \,\square\, H) = \partial(G) \times \partial(H)$,
(ii) $\mathrm{Ct}(G \,\square\, H) = \mathrm{Ct}(G) \times \mathrm{Ct}(H)$,
(iii) $\mathrm{Ecc}(G \,\square\, H) = \mathrm{Ecc}(G) \times \mathrm{Ecc}(H)$,
(iv) $\mathrm{Per}(G \,\square\, H) = \mathrm{Per}(\mathrm{G}) \times \mathrm{Per}(H)$.

Proof. (i) Suppose $(g,h) \in \partial(G \,\square\, H)$ and $g \notin \partial(G)$. Then for every $g' \in V(G)$ there exists $g'' \in N_G(g)$ such that $d_G(g',g'') > d_G(g',g)$. Now consider a vertex $(g'',h) \in N_{G \square H}(g,h)$. For arbitrary $h' \in V(H)$ we derive $d_{G \square H}((g',h'),(g'',h)) = d_G(g',g'') + d_H(h',h) > d_G(g',g) + d_H(h',h) =$

$d_{G\,\Box\,H}((g',h'),(g,h))$, a contradiction with the assumption $(g,h) \in \partial(G\,\Box\,H)$. Thus $g \in \partial(G)$. Similarly, we prove that $h \in \partial(H)$.

Now, let $g \in \partial(G)$ and $h \in \partial(H)$. Thus there exists a vertex $g' \in V(G)$ such that for every $g'' \in N_G(g)$, $d_G(g',g'') \leq d_G(g',g)$ and there is a vertex $h' \in V(H)$ such that for every $h'' \in N_H(h)$, $d_H(h',h'') \leq d_G(h',h)$. Let (g'',h'') be an arbitrary vertex from $N_{G\,\Box\,H}(g,h)$. We may without loss of generality assume that $g''g \in E(G)$ and $h'' = h$. Then $d_{G\,\Box\,H}((g',h'),(g'',h'')) = d_G(g',g'') + d_H(h',h'') \leq d_G(g',g) + d_H(h',h) = d_{G\,\Box\,H}((g',h'),(g,h))$ and $(g,h) \in \partial(G\,\Box\,H)$.

(ii) Let $(g,h) \in \mathrm{Ct}(G\,\Box\,H)$. Suppose g is not a contour vertex in G. Then there exists $g' \in N_G(g)$ such that $\mathrm{ecc}_G(g') > \mathrm{ecc}_G(g)$ which implies $\mathrm{ecc}_{G\,\Box\,H}(g',h) = \mathrm{ecc}_G(g') + \mathrm{ecc}_H(h) > \mathrm{ecc}_G(g) + \mathrm{ecc}_H(h) = \mathrm{ecc}_{G\,\Box\,H}(g,h)$ (by Lemma 4.1), a contradiction.

Conversely, let $g \in \mathrm{Ct}(G)$ and $h \in \mathrm{Ct}(H)$. Suppose that $(g,h) \notin \mathrm{Ct}(G\,\Box\,H)$. Then there is a vertex $(g',h') \in V(G\,\Box\,H)$ adjacent to (g,h) such that $\mathrm{ecc}_{G\,\Box\,H}(g',h') > \mathrm{ecc}_{G\,\Box\,H}(g,h)$. Since (g,h) and (g',h') are adjacent we can without loss of generality assume that $g = g'$ and $hh' \in E(H)$. Thus from $\mathrm{ecc}_{G\,\Box\,H}(g',h') > \mathrm{ecc}_{G\,\Box\,H}(g,h)$ by Lemma 4.1 follows that $\mathrm{ecc}_H(h') > \mathrm{ecc}_H(h)$, a contradiction with the assumption.

(iii) Suppose that for a vertex $(g,h) \in V(G\,\Box\,H)$ there exists a vertex (g',h') such that $\mathrm{ecc}_{G\,\Box\,H}(g',h') = d_{G\,\Box\,H}((g',h'),(g,h))$. Hence

$$\mathrm{ecc}_G(g') + \mathrm{ecc}_H(h') = d_G(g,g') + d_H(h,h').$$

We claim that $\mathrm{ecc}_G(g') = d_G(g,g')$. Indeed, if $\mathrm{ecc}_G(g') > d_G(g',g)$, this implies $\mathrm{ecc}_H(h') < d_G(h',h)$ which is impossible. Hence $\mathrm{ecc}_G(g') = d_G(g,g')$, that is $g \in \mathrm{Ecc}(G)$, and analogously we find that $h \in \mathrm{Ecc}(H)$.

Let $g \in \mathrm{Ecc}(G)$ and $h \in \mathrm{Ecc}(H)$. Thus there are vertices g' and h' in $V(G)$ and $V(H)$, respectively, such that $\mathrm{ecc}_G(g') = d_G(g',g)$ and $\mathrm{ecc}_H(h') = d_H(h',h)$. By Lemma 4.1 it is again easy to derive that $\mathrm{ecc}_{G\,\Box\,H}(g',h') = d_{G\,\Box\,H}((g',h'),(g,h))$.

(iv) First we prove $\mathrm{Per}(G\,\Box\,H) \subseteq \mathrm{Per}(G) \times \mathrm{Per}(H)$. Suppose $g \notin \mathrm{Per}(G)$. Then there exists $g' \in V(G)$ such that $\mathrm{ecc}_G(g') > \mathrm{ecc}_G(g)$. Let h be an arbitrary vertex in $V(H)$. Then $\mathrm{ecc}_{G\,\Box\,H}(g',h) = \mathrm{ecc}_G(g') + \mathrm{ecc}_H(h) > \mathrm{ecc}_G(g) + \mathrm{ecc}_H(h) = \mathrm{ecc}_{G\,\Box\,H}(g,h)$, thus $(g,h) \notin \mathrm{Per}(G\,\Box\,H)$. We infer $g \in \mathrm{Per}(G)$ and we prove $h \in \mathrm{Per}(H)$ analogously.

Now suppose $g \in \mathrm{Per}(G)$ and $h \in \mathrm{Per}(H)$. Then by Lemma 4.1 it easily follows that $\mathrm{ecc}_{G\,\Box\,H}(g,h) \geq \mathrm{ecc}_{G\,\Box\,H}(g',h')$ for all $(g',h') \in V(G\,\Box\,H)$, that is $(g,h) \in \mathrm{Per}(G\,\Box\,H)$. □

5 Contour sets as geodetic sets

In [3] and [4] graphs G in which $\mathrm{Ct}(G)$ is a geodetic set were studied. It is known that this holds for chordal graphs and distance-hereditary graphs, and it is an open problem whether it is true for bipartite graphs. In this short concluding section we consider the problem with respect to Cartesian product graphs.

First a lemma about geodetic sets in products that could be of independent interest. For $G \,\square\, H$ let p_G and p_H be the projection maps onto factors, that is, for $u = (g, h) \in V(G \,\square\, H)$ let $p_G(u) = g$ and $p_H(u) = h$.

Lemma 5.1 *Let X be a geodetic set of $G \,\square\, H$. Then $p_G(X)$ and $p_H(X)$ are geodetic sets of G and H, respectively.*

Proof. Let $g \in V(G)$. If $g \in p_G(X)$ there is nothing to be proved. So suppose $g \notin V(G)$. Let h be an arbitrary vertex of H. As X is a geodetic set of $G \,\square\, H$, there are vertices (g', h') and (g'', h'') in X such that $(g, h) \in I_{G \,\square\, H}[(g', h'), (g'', h'')]$. Since $g \notin p_G(X)$ it follows that $g' \neq g$ and $g'' \neq g$. Therefore also $g' \neq g''$. Let Q be a $(g', h'), (g'', h'')$-geodesic containing (g, h). Then $p_G(Q)$ is a g', g''-geodesic containing g. As $g', g'' \in p_G(X)$ the proof is complete. $\square$

The converse of Lemma 5.1 need not hold, consider $G = H = C_4$. Let g, g', g'' be different vertices in G and $\{h, h'\} \subset V(H)$, $d_H(h, h') = 2$. Then $\{g, g', g''\}$ and $\{h, h'\}$ are geodetic sets in factors while $X = \{(g, h), (g', h'), (g'', h)\}$ is not a geodetic set in $G \,\square\, H$.

On the other hand, the following equivalence can be easily verified.

Lemma 5.2 *Sets $S \subseteq V(G)$ and $T \subseteq V(H)$ are geodetic sets of G and H, respectively, if and only if $S \times T$ is a geodetic set of $G \,\square\, H$.*

Proof. Let $(g, h) \in V(G \,\square\, H)$. Since S is a geodetic set in G there exist $g', g'' \in S$ such that $g \in I_G[g', g'']$, and similarly, $h \in I_H[h', h'']$ for some $h', h'' \in T$. By Lemma 2.1, $(g, h) \in I_{G \,\square\, H}[(g', h'), (g'', h'')]$. The converse follows from Lemma 5.1. $\square$

Theorem 5.3 $\mathrm{Ct}(G \,\square\, H)$ *is a geodetic set of $G \,\square\, H$ if and only if* $\mathrm{Ct}(G)$ *and* $\mathrm{Ct}(H)$ *are geodetic sets of G and H, respectively.*

Proof. By Theorem 4.2 (ii) we have $\mathrm{Ct}(G \,\square\, H) = \mathrm{Ct}(G) \times \mathrm{Ct}(H)$. Combining this formula with Lemma 5.2 the theorem follows. $\square$

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