Fibonacci (p, r)-cubes as Cartesian products

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Abstract

The Fibonacci (p, r)-cube $\Gamma_n^{(p,r)}$ is the subgraph of Q_n induced on binary words of length n in which there are at most r consecutive ones and there are at least p zeros between two substrings of ones. These cubes simultaneously generalize several interconnection networks, notably hypercubes, Fibonacci cubes, and postal networks. In this note it is proved that $\Gamma_n^{(p,r)}$ is a non-trivial Cartesian product if and only if $p = 1$ and $r = n \geq 2$, or $p = r = 2$ and $n \geq 2$, or $n = p = 3$ and $r = 2$. This rounds a result from [Ou, Zhang, Yao, Discrete Math. 311 (2011) 1681–1692] asserting that $\Gamma_n^{(2,2)}$ are non-trivial Cartesian products.

Key words: Hypercube; Fibonacci (p, r)-cube; Cartesian product

AMS Subj. Class: 05C12, 05C76

1 Introduction

The Fibonacci (p, r)-cubes $\Gamma_n^{(p,r)}$ were introduced in [2] and unify several models of interconnection networks including hypercubes, Fibonacci cubes [8, 11], generalized Fibonacci cubes in the sense of [12, 17], and postal networks [16]. Due to the general nature of Fibonacci (p, r)-cubes, not many universal results are known about them, but recently two such results appeared. In [14] it was determined when a Fibonacci (p, r)-cube is a Z-transformation graph of a planar graph, while in [13] Ou and Zhang characterized median graphs among the Fibonacci (p, r)-cubes. (The Z transformation graph is a graph whose construction is based on perfect matchings of a given graph and play an important role in mathematical chemistry.) Along the way of characterizing median graphs among Fibonacci (p,r)-cubes it was proved in [13] that as soon as $r \le p$, the corresponding Fibonacci (p,r)-cube isometrically embeds into a corresponding hypercube.

We were primarily motivated with a result from [14] which asserts that the Fibonacci $(2,2)$-cubes can be factored with respect to the Cartesian product. In this note we give a complete characterization of the Fibonacci (p,r)-cubes that are Cartesian products:

Theorem 1 *Let $1 \le p, r \le n$. Then $\Gamma_n^{(p,r)}$ is a non-trivial Cartesian product graph if and only if $p = 1$ and $r = n \ge 2$, or $p = r = 2$ and $n \ge 2$, or $n = p = 3$ and $r = 2$. In these cases,*

- $\Gamma_n^{(1,n)} \cong Q_n$ $(n \ge 2)$,
- $\Gamma_n^{(2,2)} \cong \Gamma_{\lceil \frac{n}{2} \rceil}^{(1,1)} \,\Box\, \Gamma_{\lfloor \frac{n}{2} \rfloor}^{(1,1)}$ $(n \ge 2)$, *and*
- $\Gamma_3^{(3,2)} \cong P_3 \,\Box\, K_2$.

We proceed as follows. In the rest of this section key definitions are given. In the subsequent section we prove Theorem 1, while the last section contains a couple of suggestions for future research.

The *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ in which vertices (g,h) and (g',h') are adjacent whenever either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. This graph operation is commutative and associative. The Cartesian product of n copies of K_2 is known as the *n-dimensional hypercube* Q_n, called *n-cube* for short. Equivalently, the n-cube Q_n is the graph whose vertices are all the binary words of length n, two vertices being adjacent if they differ in exactly one coordinate. A graph G is called *prime with respect to the Cartesian product* if it has no representation as the Cartesian product of at least two non-trivial graphs. Since the only product of graphs considered here is the Cartesian product, we will simply say *prime graph* instead of prime graph with respect to the Cartesian product. For more information on the Cartesian product of graphs see [5, 10].

Let p and r be positive integers and $p, r \le n$. A *Fibonacci (p,r)-string of length n* is a binary word of length n in which there are at most r consecutive ones, and at least p zeros between two substrings of ones (of course each composed of at most r ones). The *Fibonacci (p,r)-cube* $\Gamma_n^{(p,r)}$ is the subgraph of Q_n induced on the

Fibonacci (p,r)-strings of length n. See Fig. 1 for $\Gamma_5^{(2,3)}$ and $\Gamma_6^{(4,2)}$. Note that $\Gamma_6^{(4,2)}$ is isomorphic to the bipartite wheel BW_6. In fact, it is not difficult to observe that for any $n \geq 3$ we have $\Gamma_n^{(n-2,2)} \cong BW_n$. In particular, $\Gamma_4^{(2,2)} \cong BW_4 \cong P_3 \,\square\, P_3$.

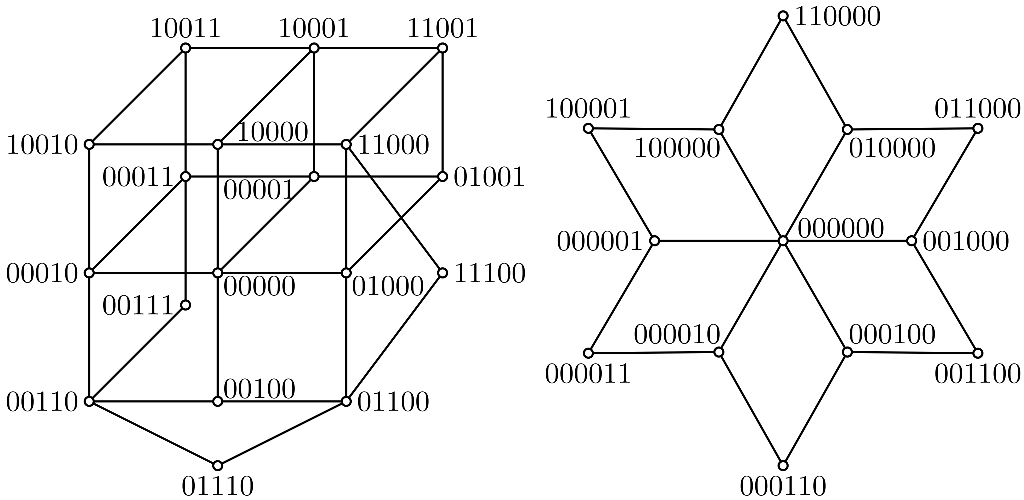

Figure 1: Fibonacci (p,r)-cubes $\Gamma_5^{(2,3)}$ and $\Gamma_6^{(4,2)}$

The *Hamming distance* $H(u,v)$ between binary words u and v (of equal length) is the number of coordinates in which they differ. It is well-known that $d_{Q_n}(u,v) = H(u,v)$ holds for any $u,v \in V(Q_n)$.

If u and v are binary words, then uv denotes its concatenation. With u^n we mean the concatenation of n copies of u. In particular, 1^n is the binary word of length n, all of its bits equal to 1, and u^0 is the empty word λ.

Finally, the set $\{1,\ldots,n\}$ will be denoted with $[n]$, the disjoint union of sets with $\uplus$, and the subgraph of G induced on $X \subseteq V(G)$ with $G[X]$.

2 Proof of Theorem 1

To begin the proof we first recall that for a graph G, the *Djoković-Winkler's relation* Θ_G [1, 15] is defined on $E(G)$ as follows: if $e = xy \in E(G)$ and $f = uv \in E(G)$, then $e\Theta f$ if $d(x,u) + d(y,v) \neq d(x,v) + d(y,u)$. Another relevant relation defined on the edge set of a graph G is τ_G, where edges uv and uw are in relation τ_G if u is the unique common neighbor of v and w. We will often omit the subscript G in Θ_G and τ_G, if the underlying graph will be clear from the context. For a relation R, let

R^* be the transitive closure of R. Let $G = G_1 \,\square\, \cdots \,\square\, G_k$ be a connected Cartesian product. For an edge uv of G let $c(uv)$ be the coordinate in which u and v differ. Edges e and f of G are, by definition, in *product relation* if $c(e) = c(f)$. With these definitions in hand we can state the following theorem due to Feder [3]. (See also [5, Theorem 23.2] for two different proofs of the result.)

Theorem 2 *If G is a connected graph, then $(\Theta_G \cup \tau_G)^*$ is a product relation.*

For our purposes the most important conclusion of this fundamental result is that a graph is prime if and only if the relation $(\Theta \cup \tau)^*$ has a single equivalence class.

Let $E_i^{(p,r)}$ be the set of edges of the Fibonacci (p,r)-cube $\Gamma_n^{(p,r)}$ whose endpoints differ in coordinate i, and set

$$
\begin{aligned}
V_i^0 &= \{u :\ u \text{ is an endvertex of an edge from } E_i^{(p,r)} \text{ with } u_i = 0\}\,,\\
V_i^1 &= \{u :\ u \text{ is an endvertex of an edge from } E_i^{(p,r)} \text{ with } u_i = 1\}\,.
\end{aligned}
$$

Ou, Zhang, and Yao proved:

Lemma 3 [14, Lemmas 7 and 8] *For any $1 \le i \le n$,*

$$\Gamma_n^{(p,r)}[V_i^0 \uplus V_i^1] \cong \Gamma_n^{(p,r)}[V_i^0] \,\square\, K_2\,.$$

Moreover, $\Gamma_n^{(p,r)}[V_i^0]$ and $\Gamma_n^{(p,r)}[V_i^1]$ are connected subgraphs of $\Gamma_n^{(p,r)}$.

Corollary 4 *If $e, f \in E_i^{(p,r)}$, then $e \Theta f$.*

Proof. Follows from Lemma 3 and [5, Lemma 13.5(i)]. The latter lemma asserts that if e and f are edges of a Cartesian product G such that (i) endpoints of e and f differ in the same coordinate i and (ii) e and f project onto the same edge of the ith factor of G, then e and f are in relation Θ. $\square$

If $n = 1$, then the only graph to be considered is $\Gamma_1^{(1,1)}$. It is isomorphic to K_2 and hence prime. When $n = 2$, we have $\Gamma_2^{(1,1)} \cong \Gamma_2^{(2,1)} \cong P_3$, and $\Gamma_2^{(1,2)} \cong \Gamma_2^{(2,2)} \cong C_4 \cong Q_2$. Hence we can assume in the rest that $n \ge 3$.

According to Theorem 2, to prove that $\Gamma_n^{(p,r)}$ is prime it suffices to show that the relation $(\Theta_{\Gamma_n^{(p,r)}} \cup \tau_{\Gamma_n^{(p,r)}})^*$ consists of a single equivalence class. By Corollary 4 all the edges of $E_i^{(p,r)}$ $(1 \le i \le n)$ are in the same Θ^*-class. We now define a binary relation

$\sim$ on the set $[n]$ by saying that $i \sim j$ if there exist edges $e \in E_i^{(p,r)}$ and $f \in E_j^{(p,r)}$ such that $e\,\tau f$. Then it follows that G is a prime graph as soon as $\sim^* = [n] \times [n]$.

Case 1: $p = 1$.
If $r = n$ then $\Gamma_n^{(1,n)} \cong Q_n$. Hence suppose that $r < n$. Let $i \in [n-1]$ and consider the following words:

$$\begin{aligned} u &= 0^k 1^j 011^{r-j-1} 0^{n-r-1-k} \\ v &= 0^k 1^j 001^{r-j-1} 0^{n-r-1-k} \\ w &= 0^k 1^j 101^{r-j-1} 0^{n-r-1-k}, \end{aligned}$$

where $j = \min\{r-1, i-1\}$ and $k = i - 1 - j$. Here we make the subword 1^j as long as possible under the constraint that the constructed words lie in $\Gamma_n^{(1,r)}$. Note that in this way the words u, v, w indeed belong to $V(\Gamma_n^{(1,r)})$. The only possible common neighbor of u and w that differs from v is the vertex $0^k 1^j 111^{r-j-1} 0^{n-r-1-k} = 0^k 1^{r+1} 0^{n-r-1-k}$, which does not not lie in $\Gamma_n^{(1,r)}$. Therefore, u is in relation τ with w and hence $i \sim (i+1)$ holds for any $i \in [n-1]$. It follows that $\sim^* = [n] \times [n]$ and therefore $\Gamma_n^{(1,r)}$ is prime.

Case 2: $p \geq 2$ and $r \geq 3$.
Consider the following words:

$$\begin{aligned} u &= 1110^{n-3}, \\ v &= 0110^{n-3}, \\ w &= 0010^{n-3}. \end{aligned}$$

Since $n \geq r \geq 3$, the vertices u, v, and w are of length n. Note also that $u, v, w \in V(\Gamma_n^{(p,r)})$. The only possible common neighbor of u and w that differs from v is the vertex 1010^{n-3}. Since this vertex of Q_n does not lie in $\Gamma_n^{(p,r)}$, it follows that u is in relation τ with w. Therefore, $1 \sim 2$. Let u', v', w' be the words obtained from u, v, w by attaching 0^s, $1 \leq s \leq n-3$, in the front, and removing the same word on the right. More precisely, $u' = 0^s 1110^{n-3-s}$, $v' = 0^s 0110^{n-3-s}$, and $w' = 0^s 0010^{n-3-s}$. Then by an analogous argument as above we get $(s+1) \sim (s+2)$. Hence $1 \sim 2, \ldots, (n-2) \sim (n-1)$. Finally, to infer that $(n-1) \sim n$, consider the words $u'' = 0^{n-3}111$, $v'' = 0^{n-3}110$, and $w'' = 0^{n-3}100$. We conclude that for any $p \geq 2$ and $r \geq 3$, the cube $\Gamma_n^{(p,r)}$ is prime.

Case 3: $r = 2$ and $p \geq 3$.
Considering the words 1000^{n-3}, 0000^{n-3}, and 0010^{n-3}, we get $1 \sim 3$. Suppose first

that $n \geq 4$. Attaching a fixed number of zeros in front of these three words and removing the same number of zeros at their ends, we also get $2 \sim 4$, ..., $(n-2) \sim n$. To see that also $1 \sim 4$, consider the words 00010^{n-4}, 00000^{n-4}, and 10000^{n-4}. Suppose $n = 3$. Then the only graph to be considered is $\Gamma_3^{(3,2)}$. This graph is obtained from Q_3 by removing vertices 111 and 101, and is isomorphic to $P_3 \,\square\, K_2$.

Case 4: $r = 1$.
From the words 100^{n-2}, 000^{n-2}, and 010^{n-2} we find out that $1 \sim 2$. Attaching zeros in front of these words and removing the same number of zeros at their ends, we also get $2 \sim 3, \ldots, (n-1) \sim n$. As before we can conclude that $\sim^* = [n] \times [n]$.

Case 5: $r = 2$ and $p = 2$.
That $\Gamma_n^{(2,2)}$ is isomorphic to $\Gamma^{(1,1)}_{\lceil \frac{n}{2} \rceil} \,\square\, \Gamma^{(1,1)}_{\lfloor \frac{n}{2} \rfloor}$ was proved in [14]. To make the proof of Theorem 1 self-contained, we give an argument shorter than the original one.

Let $n \geq 2$ and let $X = \Gamma_{\lceil \frac{n}{2} \rceil}$ and $Y = \Gamma_{\lfloor \frac{n}{2} \rfloor}$. Then the vertices of X and Y are Fibonacci strings of lengths $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$, respectively. Assign to any pair $(x, y) = (x_1 \ldots x_{\lceil \frac{n}{2} \rceil}, y_1 \ldots y_{\lfloor \frac{n}{2} \rfloor})$ of such strings the interlacing string $x * y = x_1 y_1 x_2 y_2 \ldots$, where this new string ends with either $x_{\lceil \frac{n}{2} \rceil}$ or $y_{\lfloor \frac{n}{2} \rfloor}$ depending on the parity of n. It is straightforward to verify that $x * y \in V(\Gamma_n^{(2,2)})$. Moreover, any vertex of $\Gamma_n^{(2,2)}$ is obtained in this way. We claim that $\Gamma_n^{(2,2)} \cong X \,\square\, Y$. By the above we can bijectively associate the strings $x * y$ with the vertices (x, y) of $X \,\square\, Y$. Since the distance in $\Gamma_n^{(2,2)}$ is equal to the the Hamming distance, see [13, Lemma 2], (x, y) is adjacent to (x', y') in $\Gamma_n^{(2,2)}$ if and only if $H(x * y, x' * y') = 1$. This holds if and only if either $x = x'$ and $H(y, y') = 1$, or $H(x, x') = 1$ and $y = y'$ which in turn holds if and only if either $x = x'$ and $yy' \in E(Y)$, or $xx' \in E(X)$ and $y = y'$. It follows that $\Gamma_n^{(2,2)} \cong X \,\square\, Y$.

3 Concluding remarks

In this note we have characterized Fibonacci (p, r)-cubes that admit representations as non-trivial Cartesian products. As it turned out, not many of the cubes are such. (This is in accordance with Graham's result from [4] that almost all graphs have a single Θ^*-class, which by Theorem 2 in turn implies that almost all graphs are prime.) Hence it would be interesting to investigate how close to the Cartesian product are the Fibonacci (p, r)-cubes in the sense of the theory of approximate Cartesian graph products [6, 7]. A look to Fig. 1 supports this idea.

Another simultaneous generalization of hypercubes and Fibonacci cubes was recently proposed in [9] under the name of *generalized Fibonacci cubes*. If f is a given binary word, then the generalized Fibonacci cube $Q_n(f)$ is defined as the subgraph of Q_n induced on the words that do not contain f as a subword. For instance, $\Gamma_n \cong Q_n(11)$. In view of the present note we also pose the question which of generalized Fibonacci cubes are Cartesian product of graphs.

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References

[1] D. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.

[2] K. Egiazarian, J. Astola, On generalized Fibonacci cubes and unitary transforms, Appl. Algebra Engrg. Comm. Comput. 8 (1997) 371–377.

[3] T. Feder, Product graph representations, J. Graph Theory 16 (1992) 467–488.

[4] R.L. Graham, Isometric embeddings of graphs, in: Selected Topics in Graph Theory, 3, 133–150, Academic Press, San Diego, CA, 1988.

[5] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, Second Edition, CRC Press, Boca Raton, FL, 2011.

[6] M. Hellmuth, W. Imrich, W. Klöckl, P.F. Stadler, Approximate graph products, European J. Combin. 30 (2009) 1119–1133.

[7] M. Hellmuth, W. Imrich, T. Kupka, Partial star products: A local covering approach for the recognition of approximate Cartesian product graphs, Math. Comput. Sci. 7 (2013) 255–273.

[8] W.-J. Hsu, Fibonacci cubes—a new interconnection topology, IEEE Trans. Parallel Distrib. Syst. 4 (1993) 3–12.

[9] A. Ilić, S. Klavžar, Y. Rho, Generalized Fibonacci cubes, Discrete Math. 312 (2012) 2–11.

[10] W. Imrich, S. Klavžar, D.F. Rall, Topics in Graph Theory: Graphs and Their Cartesian Product, A K Peters, Wellesley, MA, 2008.

[11] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (2013) 505–522.

[12] J. Liu, W.-J. Hsu, M.J. Chung, Generalized Fibonacci cubes are mostly Hamiltonian, J. Graph Theory 18 (1994) 817–829.

[13] L. Ou, H. Zhang, Fibonacci (p, r)-cubes which are median graphs, Discrete Appl. Math. 161 (2013) 441–444.

[14] L. Ou, H. Zhang, H. Yao, Determining which Fibonacci (p, r)-cubes can be Z-transformation graphs, Discrete Math. 311 (2011) 1681–1692.

[15] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.

[16] J. Wu, The postal network: a recursive network for parameterized communication model, J. Supercomput. 19 (2001) 143–161.

[17] N. Zagaglia Salvi, On the existence of cycles of every even length on generalized Fibonacci cubes, Matematiche (Catania) 51 (1996) 241–251.