Nordhaus-Gaddum and other bounds for the chromatic edge-stability number

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Abstract

Let G be a graph. The chromatic edge-stability number $\mathrm{es}_\chi(G)$ of a graph G is the minimum number of edges of G whose removal results in a graph H with $\chi(H) = \chi(G) - 1$. A Nordhaus-Gaddum type inequality for the chromatic edge-stability number is proved. Sharp upper bounds on es_χ are given for general graphs in terms of size and of maximum degree, respectively. All bounds are demonstrated to be sharp. Graphs with $\mathrm{es}_\chi = 1$ are considered and in particular characterized among k-regular graphs for $k \leq 5$. Several open problems are also stated.

Keywords: chromatic number; chromatic edge-stability; Nordhaus-Gaddum type inequality; bipartite density

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1 Introduction

The *chromatic edge-stability number* $\mathrm{es}_\chi(G)$ of a graph G is the minimum number of edges of G whose removal results in a graph with the chromatic number smaller than that of G. Since this concept is not defined for edge-less graphs, we will assume throughout that every graph considered has at least one edge. The chromatic edge-stability number was introduced by Staton [12], where upper bounds on es_χ in terms of the size of a given graph were proved for regular graphs.

Theorem 1.1 [12, Corollary 1] *Let G be a k-regular graph with $k \geq 3$. If G contains no K_{k+1}, then G has a spanning $k-1$ colorable subgraph with at least the fraction $\frac{k^2-2}{k^2}$ of the edges of G.*

The bounds are best possible for 4-regular and 5-regular graphs as can be demonstrated by several copies of Albertson-Bollobás-Tucker graphs [1]. A concept closely related to the chromatic edge-stability number is the *bipartite edge frustration* of a graph, which is defined as the smallest number of edges that have to be deleted from a graph to obtain a bipartite spanning subgraph, see [5, 6, 14, 15]. Equivalently, one can search for the maximum number of edges lying in a bipartite subgraph of a graph, see [16].

Although the chromatic edge-stability number es_χ appears like a fundamental graph coloring concept (more basic than the bipartite edge frustration), it has, rather surprisingly, received not much attention with the exception of the recent paper [9]. In this paper we fill this gap and prove several results on es_χ. In the next section we first prove an upper bound on es_χ in terms of the order and the chromatic number. Then we use this result to derive a Nordhaus-Gaddum type inequality for the chromatic edge-stability number. Then, in Section 3, we find an upper bound on es_χ for subcubic graphs in terms of their size. We also derive a general upper bound in terms of maximum degree, chromatic number and the cardinality of a smallest possible color class. In Section 4 we consider graphs G with $\mathrm{es}_\chi(G) = 1$, give several conditions on G that ensure $\mathrm{es}_\chi(G) = 1$, and characterize k-regular graphs G, $k \leq 5$, with $\mathrm{es}_\chi(G) = 1$. In the final section we rediscuss the results obtained in this paper and state several related problems.

All graphs considered are finite, simple and undirected. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices and the number of edges of G are its *order* $n(G)$ and *size* $m(G)$, respectively. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G admits a proper coloring of its vertices using k colors. A $\chi(G)$-coloring, or simply χ-coloring of G is a proper coloring of its vertices using $\chi(G)$ colors. In a χ-coloring of G, each set of vertices having the same color is called a *color class*. If G is a graph, then let $c^\star(G)$ denotes the cardinality of a smallest color class among all χ-colorings of G.

In a graph G, the degree of a vertex v is denoted by $\deg(v)$, and the maximum degree of G is denoted by $\Delta(G)$. A graph G is called k-regular if all its vertices have degree k. The complete graph of order r and the complete multi-partite graph having parts of sizes $n_1, \ldots, n_k$ are denoted by K_r and $K_{n_1,\ldots,n_k}$, respectively. For an edge e of a graph G, the graph obtained by deleting e is denoted by $G \backslash e$. An *independent set* of a graph G is a set of vertices of G such that no two vertices are adjacent. The *complement* of G is denoted by $\overline{G}$. Finally, for every positive integer n, we will use the notation $[n] = \{1, \ldots, n\}$ and we use the notation $|C_i|$ for the number of vertices in the color class C_i.

2 Nordhaus-Gaddum type inequality

In this section we prove a Nordhaus-Gaddum type inequality for the chromatic edge-stability number. For this sake we first derive an upper bound in terms of the order and the chromatic number, and for the equality case of this result determine the chromatic edge-stability number of complete multi-partite graphs. (We note in passing that Nordhaus-Gaddum type inequalities are known for many graph invariants, [2, 4, 8, 10] is just a sample of recent related results.)

Theorem 2.1 *Let G be a graph and set $n = n(G)$ and $r = \chi(G)$. Then*

$$\mathrm{es}_\chi(G) \le \begin{cases} \left\lfloor \frac{n}{r} \right\rfloor \left\lfloor \frac{n}{r} + 1 \right\rfloor; & n \equiv r-1 \pmod{r}, \\ \\ \left\lfloor \frac{n}{r} \right\rfloor^2; & \textit{otherwise}. \end{cases}$$

Moreover, the bounds are sharp.

Proof. Consider an r-coloring of G. Let e_{ij} be the number of edges between the ith and jth color classes. Obviously, $\mathrm{es}_\chi(G) \le e_{ij}$ for all distinct $i, j \in [r]$. If $r = 2$, then it is clear that the inequality holds. So, assume $r > 2$.

Suppose first that $n \equiv r - 1 \pmod{r}$. Then it is clear that $n = r\left\lfloor \frac{n}{r} \right\rfloor + r - 1$. Assume that for each pair of color classes C_1 and C_2 we have $|C_1| + |C_2| > \left\lfloor \frac{n}{r} \right\rfloor + \left\lfloor \frac{n}{r} + 1 \right\rfloor$, so at least one of C_1 or C_2 has cardinality greater than $\left\lfloor \frac{n}{r} + 1 \right\rfloor$. This means there exists at least $r-1$ color classes $C_1, C_2, \ldots, C_{r-1}$ with cardinality greater than $\left\lfloor \frac{n}{r} + 1 \right\rfloor$ and we have $|C_i| + |C_r| > 2\left\lfloor \frac{n}{r} \right\rfloor + 1$ for $i \in [r-1]$. Hence, $|C_1| + |C_2| + \cdots + |C_{r-1}| + (r-1)|C_r| > 2(r-1)\left\lfloor \frac{n}{r} \right\rfloor + (r-1)$ and so $|C_r| \ge \left\lfloor \frac{n}{r} \right\rfloor + 1$.

By assumption we have $|C_1| \ge \left\lfloor \frac{n}{r} \right\rfloor + 2$, …, $|C_{r-1}| \ge \left\lfloor \frac{n}{r} \right\rfloor + 2$. Hence we have $n = |C_1| + \cdots + |C_r| \ge r\left\lfloor \frac{n}{r} \right\rfloor + 2r - 1$, a contradiction. So, there exists at least a pair of color class C_1 and C_2 in which $|C_1| + |C_2| \leq \lfloor \frac{n}{r} \rfloor + \lfloor \frac{n}{r} + 1 \rfloor$ and it is clear that the maximum possible number of edges between these two color classes is at most $\lfloor \frac{n}{r} \rfloor \times \lfloor \frac{n}{r} + 1 \rfloor$ and hence $e_{ij} \leq \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor$ for some $i, j \in [r]$.

Assume now that $n \not\equiv r-1 \pmod r$. It is clear that $n \leq r \lfloor \frac{n}{r} \rfloor + r - 2$. Suppose for each pair of color classes C_1 and C_2 we have $|C_1| + |C_2| > 2 \lfloor \frac{n}{r} \rfloor$, so at least one of C_1 or C_2 has cardinality greater than $\lfloor \frac{n}{r} \rfloor$. This means there exists at least $r-1$ color classes $C_1, C_2, \ldots, C_{r-1}$ with cardinality greater than $\lfloor \frac{n}{r} \rfloor$ and we have $|C_i| + |C_r| > 2 \lfloor \frac{n}{r} \rfloor$ for $i \in [r-1]$. Hence, $|C_1| + |C_2| + \cdots + |C_{r-1}| + (r-1)|C_r| > 2(r-1) \lfloor \frac{n}{r} \rfloor$ and so $|C_r| \geq \lfloor \frac{n}{r} \rfloor$.

By assumption we have $|C_1| \geq \lfloor \frac{n}{r} \rfloor + 1, \ldots, |C_{r-1}| \geq \lfloor \frac{n}{r} \rfloor + 1$. Hence $n = |C_1| + \cdots + |C_r| \geq r \lfloor \frac{n}{r} \rfloor + r - 1$, a contradiction. So, there exists at least a pair of color class C_1 and C_2 in which $|C_1| + |C_2| \leq 2 \lfloor \frac{n}{r} \rfloor$ and it is clear that the maximum possible number of edges between these two color classes is less than or equal to $\lfloor \frac{n}{r} \rfloor^2$ and hence $e_{ij} \leq \lfloor \frac{n}{r} \rfloor^2$ for some $i, j \in [r]$, which completes the proof.

Examples for the equality are two complete graphs K_n glued at a vertex for the first inequality, and K_n, $n \geq 2$ for the second inequality. $\square$

For more equality cases we next give the following result obtained earlier in [9, Theorem 3.6]. Here we give an alternative, short proof of it.

Lemma 2.2 *If $k \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_k$, then* $\mathrm{es}_\chi(K_{n_1,n_2,\ldots,n_k}) = n_1 n_2$.

Proof. Set $G = K_{n_1,n_2,\ldots,n_k}$. Since $\chi(G) = k$ and since removing the edges between the parts with n_1 and n_2 vertices decreases χ, we have $\mathrm{es}_\chi(G) \leq n_1 n_2$.

If $e \in E(G)$, then since $n_1 \leq n_2 \leq \cdots \leq n_k$, the graph $G \backslash e$ has at most $n_3 \cdots n_k$ fewer subgraphs isomorphic to K_k than G. Let $F \subseteq E(G)$ with $|F| = n_1 n_2 - 1$. Then the graph $G \setminus F$ has at most $(n_1 n_2 - 1)(n_3 \cdots n_k)$ fewer subgraphs K_k than G. Since G has $n_1 \cdots n_k$ subgraphs K_k, we thus infer that $G \setminus F$ has at least one subgraph K_k and consequently $\chi(G \backslash F) = k$. Now, we conclude that $\mathrm{es}_\chi(G) = n_1 n_2$. $\square$

Let now $n \equiv r-1 \pmod r$ and let $n = n_1 + n_2 + \cdots + n_r$, where $n_1 = \lfloor \frac{n}{r} \rfloor$ and $n_2 = \cdots = n_r = \lfloor \frac{n}{r} + 1 \rfloor$. Then Lemma 2.2 asserts that $\mathrm{es}_\chi(K_{n_1,n_2,\ldots,n_r}) = \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor$. Similarly, if $n \not\equiv r-1 \pmod r$, then let $n = n_1 + n_2 + \cdots + n_r$, where $n_1 = n_2 = \lfloor \frac{n}{r} \rfloor$, and $n_3, \ldots, n_r$ are appropriate. Applying Lemma 2.2 again we get $\mathrm{es}_\chi(K_{n_1,n_2,\ldots,n_r}) = (\lfloor \frac{n}{r} \rfloor)^2$.

We are now ready for the main result of this section, the announced Nordhaus-Gaddum type inequality.

Theorem 2.3 *If G is a graph of order $n(G) \geq 3$, then*

$$\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\bar{G}) \leq \begin{cases} \frac{n(G)^2}{4} + 2; & n(G) \ even\,, \\ \\ \frac{n(G)^2}{4} + \frac{3}{4}; & otherwise\,. \end{cases}$$

Moreover, the equality holds if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Proof. Set $n = n(G)$ and let $G' = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. In the rest of the proof we will use the fact that G' is bipartite and consequently $\mathrm{es}_\chi(G') = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

Assume first that none of the graphs G and $\overline{G}$ is bipartite, that is, $\chi(G) \geq 3$ and $\chi(\overline{G}) \geq 3$. Then Theorem 2.1 implies that

$$es_\chi(G) + es_\chi(\overline{G}) \leq 2 \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor 1 + \frac{n}{3} \right\rfloor \approx \frac{2n^2 + 6n}{9} < \frac{n^2}{4} \approx \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \mathrm{es}_\chi(G')\,.$$

Hence the maximum value of $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G})$ can only be achieved when at least one of G and $\overline{G}$ is bipartite. Assume without loss of generality that G is bipartite and let $V(G) = X \cup Y$ be its bipartition, where $|X| = n_1$ and $|Y| = n_2$. Suppose further that G is not complete bipartite and set $G' = K_{n_1,n_2}$. Since the complement of G' is a disjoint union of two complete graphs, we have $\mathrm{es}_\chi(G') + \mathrm{es}_\chi(\overline{G'}) = n_1 n_2 + 2$ if $n_1 = n_2 = \frac{n}{2}$, and $\mathrm{es}_\chi(G') + \mathrm{es}_\chi(\overline{G'}) = n_1 n_2 + 1$, otherwise. On the other hand, if $n_1 = n_2 = \frac{n}{2}$, then

$$\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) \leq |E(G)| + (|E(G')| - |E(G)|) + 2\,,$$

while otherwise

$$\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) \leq |E(G)| + (|E(G')| - |E(G)|) + 1\,.$$

So, for the maximum value of $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G})$ to happen, G must be the complete bipartite graph K_{n_1,n_2}, where $n_1 + n_2 = n$. Since $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = n_1 n_2 + k$, where $k \in [2]$, the maximum value is obtained when $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, which is equal to $\frac{n^2}{4} + 2$ if n is even and $\frac{n^2}{4} + \frac{3}{4}$ otherwise. Moreover, G is the only graph for which the upper bound is attained. $\square$

3 Upper bounds involving maximum degrees

We have already mentioned that Staton [12] derived some sharp upper bounds on es_χ in terms of the size of a given graph for 4-regular and 5-regular graphs. In this section we first derive such a bound for subcubic graphs. Then we prove an upper bound on $\mathrm{es}_\chi(G)$ for an arbitrary graph in terms of $\Delta(G)$, $\chi(G)$, and the cardinality of a smallest possible color class in an optimal coloring. Using the latter invariant, a corresponding lower bound on es_χ is also proved.

In order to prove our next result, we will use a known result for which in turn we need the following concept. The *bipartite density* $b(G)$ of a graph G is defined to be the ratio of the maximum size of a bipartite subgraph of G to the size of G, that is

$$b(G) = \max\left\{\frac{m(H)}{m(G)} :\ H \text{ is a bipartite subgraph of } G\right\}.$$

Theorem 3.1 [3] *If G is a connected 3-chromatic graph, then $b(G) \geq \frac{3}{4} - \frac{1}{4m(G)}$.*

Using this result we can give a short proof of the following:

Theorem 3.2 *If G is a connected 3-chromatic graph, then $\mathrm{es}_\chi(G) \leq \frac{m(G)+1}{4}$. Moreover, the bound is sharp.*

Proof. By Theorem 3.1 we have $b(G) \geq \frac{3}{4} - \frac{1}{4m(G)}$. Let $m'(G)$ be the maximum number of edges of a bipartite subgraph of G. Since $m'(G) = m(G) - \mathrm{es}_\chi(G)$ and $b(G) = \frac{m'}{m}$, the inequality follows.

Let G_k, $k \geq 1$, be the graph of order $3k$ with $V(G_k) = \{x_1, y_1, z_1, \ldots, x_k, y_k, z_k\}$, where the vertices $x_1, y_1, \ldots, x_k, y_k$ (in that order) induce a path of length $2k-1$, and z_i is adjacent to x_i and y_i for $i \in [k]$. So $G_1 = K_3$, while G_5 is drawn in Fig. 1.

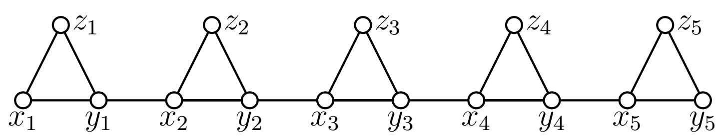

Figure 1: Graph G_5

Clearly, $m(G_k) = 4k - 1$. Moreover, as at least one edge of each of the triangles of G_k must be removed to decrease the chromatic number, $\mathrm{es}_\chi(G_k) = k$. Hence $\mathrm{es}_\chi(G_k) = k = \frac{m(G_k)+1}{4}$ which proves that the bound is sharp. $\square$

Using the concept $c^\star(G)$ we have another upper bound for $\mathrm{es}_\chi(G)$.

Theorem 3.3 *If G is a graph, then $\mathrm{es}_\chi(G) \leq \left\lfloor \frac{\Delta(G)}{\chi(G)-1} \right\rfloor c^\star(G)$. Moreover, the bound is sharp.*

Proof. For the sharpness consider the complete multipartite graph $G = K_{n,\ldots,n}$ with k partite sets. Since $\chi(G) = k$, $\Delta(G) = n(k-1)$, $c^\star(G) = n$, and $\mathrm{es}_\chi(G) = n^2$, the graph G attains the equality in the theorem.

Set $\Delta = \Delta(G)$ and $r = \chi(G)$. Let S be a color class with $|S| = c^\star(G)$ in a proper r-coloring of G. Then each vertex in S has at least one neighbor in every other color class. Indeed, otherwise we can recolor some vertices in S to obtain a χ-coloring of G with a color class whose size is smaller than $c^\star(G)$, a contradiction. By the pigeonhole principle, each vertex in S is adjacent to at least one color class with at most $\left\lfloor \frac{\Delta}{r-1} \right\rfloor$ edges. By removing these edges we can recolor each vertex of S to get a $(\chi(G)-1)$-coloring. Therefore, by deletion of at most $c^\star(G) \left\lfloor \frac{\Delta}{r-1} \right\rfloor$ edges we can reduce the chromatic number. □

We conclude the section with a lower bound on the chromatic edge-stability number which involves $c^\star$.

Theorem 3.4 *If a graph G contains a set of edges F such that $\chi(G \setminus F) < \chi(G)$, where $|F| = \mathrm{es}_\chi(G)$, and the set of endpoints of F can be partitioned into two independent sets, then $\mathrm{es}_\chi(G) \geq c^\star(G)$. Moreover, the bound is sharp.*

Proof. By the assumption, the set of endpoints of F can be partitioned into independent sets S_1 and S_2. Assume that $|S_1| < c^\star(G)$ and let $G' = G \setminus F$. Color G' with $\chi(G) - 1$ colors. Now, apply the same coloring to G, and recolor the vertices of S_1 with a new color. This yields a new χ-coloring of G. But in this coloring the set S_1 forms a color class with $|S_1| < c^\star(G)$, a contradiction. Therefore, $|S_1| \geq c^\star$. The conclusion now follows because $\mathrm{es}_\chi(G) \geq |S_1|$.

For the sharpness, let $H(n,k)$ be the graph constructed from a path P_n on n vertices and n pairwise disjoint complete graphs K_k as follows. First, make a bijection between the vertices of P_n and the complete graphs. Second, for each complete graph select a vertex and identify it with the corresponding vertex of P_n. Since $\mathrm{es}_\chi(H(n,k)) = c^\star(H(n,k)) = n$, the bound of the theorem is sharp. □

4 Graphs with $\mathrm{es}_\chi = 1$

In this section we consider graphs G for which $\mathrm{es}_\chi(G) = 1$ holds. When we say a graph G has a *singleton* color class this means there exists a χ-coloring of G in which one of the colors is used exactly once.

We begin with an infinite family of such graphs that is of independent interest. Recall that the *Mycielskian* $M(G)$ of a graph G is the graph with the vertex set

$V(G) \cup V' \cup \{w\}$, where $V' = \{x' : x \in V(G)\}$, and the edge set $E(G) \cup \{xy' : xy \in E(G)\} \cup \{wx' : x' \in V'\}$. We refer to [7, 13] for recent investigations of the Mycielskian. The most classical property of it is that $\chi(M(G)) = \chi(G) + 1$ always holds [11]. From our point of view we have:

Lemma 4.1 *If G is a graph with a singleton color class, then* $\mathrm{es}_\chi(M(G)) = 1$.

Proof. Let the vertices of $M(G)$ be as above and let $u \in V(G)$ be a vertex of G that forms a singleton color class with respect to a χ-coloring c. We may assume without loss of generality that $c(u) = 1$. Let $e = wu'$ and consider the coloring c' of $M(G) \setminus e$ defined as follows. Let $c'(w) = 1$, and for every $x \in V(G)$ let $c'(x) = c'(x') = c(x)$. Then it is straightforward to verify that c' is a proper coloring of $M(G) \setminus e$ which implies (since $\chi(M(G)) = \chi(G)+1)$) that $\chi(M(G) \setminus e) = \chi(G)$. Hence the assertion follows. $\square$

We next collect some sufficient conditions for this property.

Lemma 4.2 *Let G be a graph. Then* $\mathrm{es}_\chi(G) = 1$ *if at least one of the following conditions is fulfilled.*

(i) $\Delta(G) \le 2\chi(G) - 3$, *and G has a singleton color class,*

(ii) $n(G) \le 2\chi(G) - 2$,

(iii) G contains two vertices of degree $n(G) - 1$.

Proof. (i) Let $\chi(G) = k$ and let $\{v\}$ be a singleton color class in a χ-coloring of G. Since $\deg(v) \le 2\chi(G) - 3$ and v is adjacent to $\chi(G) - 1$ color classes, by the pigeonhole principle there exists at least one color class which is adjacent to v with exactly one edge e. By deletion of e, one can change the color of v to obtain a $(\chi(G) - 1)$-coloring for $G \backslash e$, as desired.

(ii) Since $n(G) \le 2\chi(G) - 2$, the pigeonhole principle gives a singleton color class. Moreover, $\Delta(G) \le n(G) - 1 \le (2\chi(G) - 2) - 1 = 2\chi(G) - 3$, hence (ii) follows from the already proved (i).

(iii) Let u and v be vertices of G with $\deg(u) = \deg(v) = n(G) - 1$. Since u and v are both adjacent to all the other vertices (in particular to each other), each of u and v forms a singleton color class in any χ-coloring of G. Hence $\chi(G \setminus uv) = \chi(G) - 1$. $\square$

In the main result of this section we characterize k-regular graphs G, where $k \le 5$, with the property $\mathrm{es}_\chi(G) = 1$.

Lemma 4.3 *If G is a graph with* $\mathrm{es}_\chi(G) = 1$*, then* $c^\star(G) = 1$.

Proof. Suppose G contains an edge e such that $\chi(G \setminus e) < \chi(G)$. Then coloring $G \setminus e$ with $\chi(G) - 1$ colors, and recoloring one of the endpoints of e with a new color, a singleton color class appears. □

Theorem 4.4 *Let G be a connected, k-regular graph, $k \leq 5$. Then* $\mathrm{es}_\chi(G) = 1$ *if and only if G is K_2, G is an odd cycle, or $\chi(G) > 3$ and $c^\star(G) = 1$.*

Proof. If $\chi(G) = 2$, then, as already observed, $\mathrm{es}_\chi(G) = m(G)$. Hence the only connected bipartite graph with $\mathrm{es}_\chi = 1$ is K_2.

Suppose next that $\chi(G) = 3$. If $k = 2$, then G is an odd cycle for which $\mathrm{es}_\chi = 1$. We now claim that if $\chi(G) = 3$ and $k \geq 3$, then $\mathrm{es}_\chi(G) > 1$. Suppose on the contrary that G is a k-regular graph, $k \geq 3$, with $\chi(G) = 3$ and $\mathrm{es}_\chi(G) = 1$. Let $e = uv$ be an edge such that $G' = G \backslash e$ is bipartite. Let (S_1, S_2) be the bipartition of G', let $|S_1| = s_1$ and $|S_2| = s_2$. Clearly, either $u, v \in S_1$ or $u, v \in S_2$. Assume without loss of generality that the first option holds. Since in G' every vertex of S_2 is of degree k, we have $m(G') = ks_2$. On the other hand, all vertices of S_1 but u and v are of degree k (the latter two being of degree $k - 1$), hence $m(G') = 2(k-1) + (s_1 - 2)k$. Therefore,

$$ks_2 = k(s_1 - 2) + 2(k - 1)\,.$$

But this implies that $2(k-1)$ is divisible by k, a contradiction.

We are thus left with the case when G is k-regular, $k \leq 5$, and $\chi(G) \geq 4$. We claim that $\mathrm{es}_\chi(G) = 1$ if and only if $c^\star(G) = 1$.

Suppose first that G contains an edge e such that $\chi(G \setminus e) < \chi(G)$. Then the first part is clear by Lemma 4.3. Conversely, suppose that $c^\star(G) = 1$. By Brooks theorem, $\chi(G) \in \{4, 5, 6\}$. In any case $\Delta(G) \leq 2\chi(G) - 3$, hence $\mathrm{es}_\chi(G) = 1$ holds by lemma 4.2(i). □

5 Some open problems

In this paper we have obtained several results on an old fundamental coloring concept, that has been only sporadically investigated so far. We hope that the results presented demonstrate that the concept deserves further consideration. In this respect, there are many problems that are open. Let us list a few of them.

In Theorems 2.1 and 3.3 we have proved two upper bounds on es_χ and demonstrated their sharpness. Now, we pose the following:

Problem 5.1 Characterize graphs that attain the upper bound in Theorem 2.1 and those that attain the upper bound in Theorem 3.3.

In Theorem 2.3 we have characterized graphs that attain the Nordhaus-Gaddum bound from the theorem. On the other hand, since clearly $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\bar{G}) \geq 2$, we pose the following problem:

Problem 5.2 Characterize graphs G with $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\bar{G}) = 2$.

Consider the odd cycles C_{2k+1}, $k \geq 2$. Clearly $\mathrm{es}_\chi(C_{2k+1}) = 1$. On the other hand, observe that $\chi(\bar{C}_{2k+1}) = k+1$ since the independence number of $\bar{C}_{2k+1}$ is 2. If e is an edge of $\bar{C}_{2k+1}$ between two vertices that are at distance 2 in C_{2k+1}, then $\chi(\bar{C}_{2k+1} \setminus e) = k$, so that $\mathrm{es}_\chi(\bar{C}_{2k+1}) = 1$. Hence the class of graphs from Problem 5.2 contains odd cycles of length at least 5. Additional classes of graphs that attain the equality in Problem 5.2 include $K_n \setminus E(K_{n'})$ ($n' \leq n-2$), $K_n \setminus E(W_n)$ (m even), $K_n \setminus E(C_m)$ (m odd, $m \leq n$), and split graphs.

In view of the results from Section 4 we pose:

Problem 5.3 Characterize graphs G with $\mathrm{es}_\chi(G) = 1$. In particular, for the regular case extend the classification of Theorem 4.4 to $k > 5$.

Let $\rho(G)$ denote the minimum number of edges between two color classes among all χ-colorings of a graph G. Then clearly, $\mathrm{es}_\chi(G) \leq \rho(G)$. In many cases the equality holds, in particular recall from Lemma 2.2 that $\mathrm{es}_\chi(K_{n_1,n_2,\ldots,n_k}) = \rho(K_{n_1,n_2,\ldots,n_k})$.

To see that the equality need not hold in general, consider the following example. Let $G_{2k+1,r}$, $k \geq 1$, $r \geq 2$, be the graph obtained from C_{2k+1} and $r(2k+1)$ additional vertices, where for each $i \in [2k+1]$, the vertices from the i-th group of r vertices are adjacent to the endpoints of the i-th edge of C_{2k+1}. The construction is illustrated in Fig. 2 with the graph $G_{7,3}$.

Since $\chi(G_{2k+1,r}) = 3$ and $G_{2k+1,r}$ contains $2k+1$ edge-disjoint triangles, we have $\mathrm{es}_\chi(G_{2k+1,r}) = 2k+1$. On the other hand, $\rho(G_{2k+1,r}) = 2(k+r) - 1$.

In frame of the above discussion we propose the following problem:

Problem 5.4 Characterize graphs G for which $\mathrm{es}_\chi(G) = \rho(G)$. Moreover, is it true that $\mathrm{es}_\chi(G) = \rho(G)$ holds for almost every graph G?

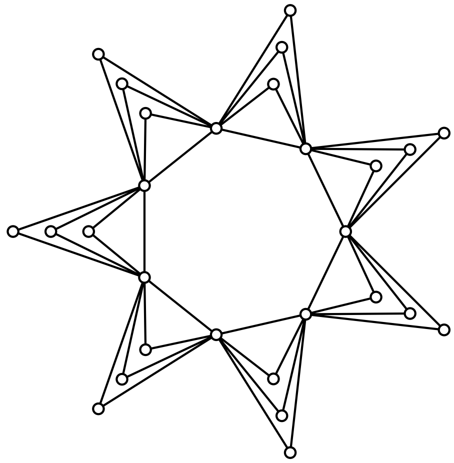

Figure 2: Graph $G_{7,3}$

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