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An almost complete description of perfect codes in direct products of cycles ☆

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Abstract

Let $G = \bigtimes_{i=1}^{n} C_{\ell_i}$ be a direct product of cycles. It is proved that for any $r \geqslant 1$, and any $n \geqslant 2$, each connected component of G contains an r-perfect code provided that each ℓ_i is a multiple of $r^n + (r+1)^n$. On the other hand, if a code of G contains a given vertex and its canonical local vertices, then any ℓ_i is a multiple of $r^n + (r+1)^n$. It is also proved that an r-perfect code ($r \geqslant 2$) of G is uniquely determined by n vertices, and it is conjectured that for $r \geqslant 2$ no other codes in G exist other than the constructed ones.

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1. Introduction

The study of codes in graphs presents a wide generalization of the problem of the existence of (classical) error-correcting codes. In general, for a given graph G we search for a subset X of its

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vertices such that the r-balls centered on vertices from X form a partition of the vertex set of G. Hamming codes and Lee codes correspond to codes in the Cartesian product of complete graphs and cycles, respectively.

The study of codes in graphs was initiated by Biggs [1] who rightly noticed that the class of all graphs is too general a setting, and hence restricted himself to distance-transitive graphs. Kratochvíl continued the study of (perfect) codes in graphs, see [12–14] and references therein. For instance, in [12] he proved the remarkable result that there are no nontrivial 1-perfect codes over complete bipartite graphs with at least three vertices. (Here "over" means with respect to the Cartesian product powers of such graphs.) Perfect codes in graphs arising from interconnection networks were studied in [15]. For recent results about codes in specific classes of graphs see [2, 11], and for the complexity point of view we refer to [3,13].

While the Cartesian product of graphs covers some classical error-correcting codes, the direct product of graphs is another interesting graph product with respect to codes in graphs. This graph product is one of the four standard graph products [6] and is *the* product in the categorical sense, for instance in studies of graph mappings [4]. For practical purposes it is important that nonbipartite, connected graphs have unique prime factor decomposition with respect to the direct product and that this decomposition can be found in polynomial time [5].

Specific direct products have been used in several applications. For example, the diagonal mesh studied by Tang and Padubirdi in [18] is a multiprocessor network representable as the direct product of two odd cycles, while the underlying graph of a fault-tolerant computational array from [16] is a connected component of the direct product of two paths of equal length. In this setting a natural problem to consider is an efficient resource placement and thus, in turn, perfect codes.

Jha [9] built r-perfect codes in the direct product of two cycles $C_{\ell_1} \times C_{\ell_2}$, where both ℓ_1 and ℓ_2 are multiples of $r^2 + (r+1)^2$. In another paper [8] he followed with a construction of r-perfect codes in the direct product of three cycles $C_{\ell_1} \times C_{\ell_2} \times C_{\ell_3}$, where each ℓ_i is a multiple of $r^3 + (r+1)^3$. Thus a natural question appears: what about products of cycles with more than three factors?

In this paper—in Section 3—we extend Jha's results to any number of cycles by proving that for any $r \geqslant 1$, and any $n \geqslant 2$, each connected component of $\bigtimes_{i=1}^{n} C_{\ell_i}$ contains an r-perfect code provided that each ℓ_i is a multiple of $r^n + (r+1)^n$. In the last section we address the question whether such codes can also be constructed for cycles of other lengths. We prove that if an r-perfect code P of a connected component of $\bigtimes_{i=1}^{n} C_{\ell_i}$, where $r \geqslant 2$, $n \geqslant 2$, and $\ell_i \geqslant 2r+2$, contains $\mathbf{0}$ and the so-called canonical local vertices for $\mathbf{0}$, then every ℓ_i is a multiple of $r^n + (r+1)^n$. Moreover, the same conclusion holds if a perfect code is isomorphic to P. This enables us to conclude that for the direct product of four cycles no other codes exist. (For two and three factors this is proved in [7].) Based on these results we conclude the paper with a conjecture that no other codes exist.

2. Preliminaries

In this section we introduce the terminology and notation needed in this paper.

For a graph G the *distance* $d_G(u,v)$, or briefly $d(u,v)$, between vertices u and v, is defined as the number of edges on a shortest u,v-path. A set $C \subseteq V(G)$ is an *r-code* in G if $d(u,v) \geqslant 2r+1$ for any two distinct vertices $u, v \in C$. The code C is called an *r-perfect code* if for any $u \in V(G)$ there is exactly one $v \in C$ such that $d(u,v) \leqslant r$. Note that C is an 1-perfect code if and only if the closed neighborhoods of its elements form a partition of $V(G)$.

Let G be a graph, u a vertex of G, and $r \geqslant 0$. The *r-ball* $B(u,r)$ with center u and diameter r is defined as $B(u,r) = \{x \mid d_G(u,x) \leqslant r\}$. In this terminology $C \subset V(G)$ is an r-perfect code if and only if the r-balls $B(u,r)$, where $u \in C$, form a partition of $V(G)$.

The *direct product* $G \times H$ of graphs G and H is the graph defined on the Cartesian product of the vertex sets of the factors. Two vertices (u_1,u_2) and (v_1,v_2) are adjacent whenever $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$. The direct product of graphs is commutative and associative in a natural way. Hence, for graphs $G_1,\dots,G_n$ we may write

$$G = G_1 \times \cdots \times G_n = \bigtimes_{i=1}^{n} G_i$$

without parentheses, and the vertices of G can be represented as vectors $\mathbf{v} = (v_1,\dots,v_n)$, where $v_i \in V(G_i)$, $1 \leqslant i \leqslant n$. For $G = \bigtimes_{i=1}^{n} G_i$ we will use d_i to denote the distance function in the ith factor graph G_i, that is, $d_i = d_{G_i}$.

The direct product of two nontrivial graphs is connected if and only if both factors are connected and at least one of them is not bipartite [19], cf. also [6]. If both factors are connected and bipartite, then their direct product consists of two connected components. Hence the direct product $G = \bigtimes_{i=1}^{n} C_{\ell_i}$ is connected if and only if at most one of the ℓ_i's is even. Otherwise, G consists of 2^{k-1} isomorphic connected components, where k is the number of ℓ_i's that are even, cf. [10]. Since a direct product of graphs is vertex transitive if and only if every factor is vertex transitive, cf. [6], all the direct products considered in this paper are vertex transitive. Hence we will often implicitly, without loss of generality, assume that $\mathbf{v} = \mathbf{0}$ is a fixed arbitrary vertex of the product considered.

For the cycle C_k ($k \geqslant 3$) we will always assume $V(C_k) = \{0,1,\dots,k-1\}$. Whenever applicable, the computations will be done modulo k, that is, modulo the length of the appropriate cycle.

We conclude the preliminaries with three observations concerning the distance function in the direct product of cycles. The first lemma follows immediately from the definition of the product.

Lemma 2.1. *Let $G = \bigtimes_{i=1}^{n} C_{\ell_i}$. Then vertices* $\mathbf{a}$ *and* $\mathbf{b}$ *of G are adjacent if and only if for every $i = 1,\dots,n$ either $a_i = b_i + 1$ or $a_i = b_i - 1$.*

If two vertices of the direct product of cycles are close, their distance has the following simple interpretation.

Lemma 2.2. *Let $r \geqslant 1$, let $\ell_i \geqslant 2r$ $(1 \leqslant i \leqslant n)$, and let $G = \bigtimes_{i=1}^{n} C_{\ell_i}$. If $d_G(\mathbf{a},\mathbf{b}) \leqslant r$ then either $d_i(a_i,b_i)$ is odd for all i, or $d_i(a_i,b_i)$ is even for all i. Moreover,*

$$d(\mathbf{a},\mathbf{b}) = \max_{1 \leqslant i \leqslant n} \bigl\{d_i(a_i,b_i)\bigr\}.$$

Proof. Suppose that $d_j(a_j,b_j)$ is odd and $d_k(a_k,b_k)$ is even for some indices $j \neq k$. Since $d(\mathbf{a},\mathbf{b}) \leqslant r$ and the length of all cycles is $\geqslant 2r+1$, for a neighbor $\mathbf{x} = (x_1,\dots,x_n)$ of $\mathbf{a}$, either $d(\mathbf{x},\mathbf{b}) \geqslant d(\mathbf{a},\mathbf{b})$, or $d_j(x_j,b_j)$ is even and $d_k(x_k,b_k)$ is odd. Thus for all vertices $\mathbf{x}$ on an $\mathbf{a},\mathbf{b}$-shortest path the parity of $d_j(x_j,b_j)$ and $d_k(x_k,b_k)$ do not match. However, this is not possible as for the neighbor $\mathbf{y}$ of $\mathbf{b}$ of an $\mathbf{a},\mathbf{b}$-shortest path we have $d_i(y_i,b_i) = 1$ for all i by Lemma 2.1. The last statement is now straightforward. $\square$

Regardless of the lengths of the cycles C_{ℓ_i} and the distances $d_i(a_i, b_i)$ we can state the following lemma.

Lemma 2.3. *Let $G = \times_{i=1}^{n} C_{\ell_i}$. Let $\mathbf{a}, \mathbf{b} \in G$, where $|a_i - b_i| = h_i$, $1 \leqslant i \leqslant n$. If either h_i is odd for all i, or h_i is even for all i, then*

$$d(\mathbf{a}, \mathbf{b}) \leqslant \max_{1 \leqslant i \leqslant n} \{h_i\}.$$

3. Constructing codes in products of several cycles

In this section we prove that (each connected component of) the direct product of n cycles contains an r-perfect code, $r \geqslant 1$, provided that the length of each cycle is a multiple of $r^n + (r+1)^n$. For a given $r \geqslant 1$ we define $s = 2r + 1$ and use this notation throughout the paper. For description of the perfect codes, the following vectors will play a crucial role:

$$\begin{array}{rcl} \mathbf{b^1} &=& (\ \ s, \quad 1, \quad 1, \ \ldots, \ 1), \\ \mathbf{b^2} &=& (-1, \quad s, \quad 1, \ \ldots, \ 1), \\ \mathbf{b^3} &=& (-1, \ -1, \quad s, \ \ldots, \ 1), \\ \vdots & & \qquad\qquad\qquad \ddots \\ \mathbf{b^n} &=& (-1, \ -1, \ -1, \ \ldots, \ s). \end{array} \tag{1}$$

Let us call the vertices $\mathbf{b^1}, \ldots, \mathbf{b^n}$ *canonical local vertices for* $\mathbf{0}$.

Lemma 3.1. *Let $r \geqslant 1$, $n \geqslant 2$, and $G = \times_{i=1}^{n} C_{\ell_i}$, where each $\ell_i = k_i \ell$ for some $k_i \in \mathbb{Z}$ and $\ell = r^n + (r+1)^n$. If $k_1, \ldots, k_n$ are either all even or all odd integers, then there exist $\beta_i \in \mathbb{Z}$, $1 \leqslant i \leqslant n$, such that*

$$\sum_{i=1}^{n} \beta_i \mathbf{b^i} = (k_1 \ell, \ldots, k_n \ell).$$

Proof. For any $k = 1, \ldots, n$, let $\mathbf{e^k}$ be the vector defined with

$$\left(\mathbf{e^k}\right)_j = \begin{cases} 2\ell; & j = k, \\ 0; & \text{otherwise.} \end{cases}$$

We claim that any vector $\mathbf{e^k}$ can be expressed as:

$$\mathbf{e^k} = \sum_{i=1}^{n} \alpha_k^i \mathbf{b^i}. \tag{2}$$

The conditions

$$\left(\mathbf{e^k}\right)_j = \sum_{i=1}^{n} \left(\alpha_k^i \mathbf{b^i}\right)_j = 0, \quad j = 1, \ldots, n, \ j \neq k, \tag{3}$$

yield the following system of $n-1$ linear equations:

$$\begin{bmatrix} s & -1 & -1 & & -1 & -1 & & & \cdots & -1 \\ 1 & s & -1 & & -1 & -1 & & & & -1 \\ 1 & 1 & s & & -1 & -1 & & & & -1 \\ \vdots & & & \ddots & & & & & & \\ 1 & & & & s & -1 & & & & -1 \\ 1 & & & & & 1 & s & -1 & & -1 \\ & & & & & & 1 & s & -1 & \\ \vdots & & & & & \vdots & & & \ddots & \vdots \\ & & & & & & & & & -1 \\ 1 & & & & & 1 & & & 1 & s \end{bmatrix} \begin{bmatrix} \alpha_k^1 \\ \alpha_k^2 \\ \alpha_k^3 \\ \\ \vdots \\ \\ \vdots \\ \\ \alpha_k^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}.$$

Note that in the kth column of the above matrix we have $k-1$ consecutive -1's and $n-k$ consecutive 1's (and no s). Selecting α_k^n as an indefinite variable, the solutions of this system are given by

$$\alpha_k^i = \begin{cases} -\alpha_k^n \cdot \frac{r^i}{(r+1)^i}; & i = 1, \ldots, k-1, \\ -\alpha_k^n \cdot \frac{(r+1)^{n-1}+r^{n-1}}{r^{n-k-1}(r+1)^{k-1}}; & i = k, \\ \alpha_k^n \cdot \frac{(r+1)^{n-i}}{r^{n-i}}; & i = k+1, \ldots, n. \end{cases}$$

Setting $\alpha_k^n = -r^{n-k-1}(r+1)^{k-1}$ we get the following particular solution:

$$\alpha_k^i = \begin{cases} r^{n-k-1+i}(r+1)^{k-1-i}; & i = 1, \ldots, k-1, \\ (r+1)^{n-1} + r^{n-1}; & i = k, \\ -r^{-k-1+i}(r+1)^{n+k-1-i}; & i = k+1, \ldots, n. \end{cases}$$

Clearly, each α_k^i is an integer and

$$\begin{aligned} \sum_{i=1}^{n} \left(\alpha_k^i \mathbf{b^i}\right)_k &= s\alpha_k^k + \sum_{i=1}^{k-1} \alpha_k^i - \sum_{i=k+1}^{n} \alpha_k^i \\ &= (2r+1)\left((r+1)^{n-1} + r^{n-1}\right) + \sum_{i=2}^{n} (r+1)^{n-i} r^{i-2} \\ &= 2\left(r^n + (r+1)^n\right) = 2\ell. \end{aligned}$$

Since $\sum_{i=1}^{n}(\alpha_k^i \mathbf{b^i})_j = \mathbf{e^k})_j$ for $j = k$, by choice of the α_k^i, Eq. (2) follows. Now set

$$\delta_i = \frac{1}{2} \sum_{k=1}^{n} \alpha_k^i, \quad i = 1, \ldots, n.$$

Since exactly two summands in the above sum are odd we find that $\delta_i \in \mathbb{Z}$ for all i. Moreover, for $j = 1, \ldots, n$ we have

$$\sum_{i=1}^{n} \left(\delta_i \mathbf{b^i}\right)_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\alpha_k^i \mathbf{b^i}\right)_j = \ell,$$

and therefore

$$\sum_{i=1}^{n} \delta_i \mathbf{b^i} = (\ell, \ldots, \ell). \tag{4}$$

Applying (2) and (4) as many times as necessary, and having in mind that either all k_i's are even or all k_i's are odd, the assertion of the lemma easily follows. □

We next give a necessary technical detail needed for the proof of our main result.

Lemma 3.2. *Let* $\mathbf{b^1}, \ldots, \mathbf{b^n}$ *be the canonical local vertices for* $\mathbf{0}$ *given in* (1)*. Then for every nontrivial linear combination*

$$\mathbf{b} = \sum_{i=1}^{n} \alpha_i \mathbf{b^i}, \quad \alpha_i \in \mathbb{Z},$$

there is at least one component b_k *of* $\mathbf{b}$ *such that* $|b_k| > 2r$.

Proof. We first prove the lemma for the case $\alpha_i \neq 0$ for $i = 1, \ldots, n$. Suppose that $\alpha_i \cdot \alpha_{i+1} < 0$ for some i. In this case

$$\left|s\alpha_i - \alpha_{i+1} - (\alpha_i + s\alpha_{i+1})\right| \geqslant 4r + 2.$$

Since

$$b_i = \alpha_1 + \cdots + \alpha_{i-1} + s\alpha_i - \alpha_{i+1} - \alpha_{i+2} - \cdots - \alpha_n,$$
$$b_{i+1} = \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i + s\alpha_{i+1} - \alpha_{i+2} - \cdots - \alpha_n,$$

and the above sums differ only in the i-th and $(i+1)$-th summand, we find that

$$|b_i - b_{i+1}| \geqslant 4r + 2.$$

Therefore $|b_i| > 2r$ or $|b_{i+1}| > 2r$.

If $\alpha_i > 0$ for all i or $\alpha_i < 0$ for all i then $|b_n| > 2r$.

Finally, assume that $\alpha_{k_1} = \cdots = \alpha_{k_j} = 0$ and $\alpha_k \neq 0$ for $k \notin \{k_1, \ldots, k_j\}$. In this case consider only those components b_k of $\mathbf{b}$ for which $k \notin \{k_1, \ldots, k_j\}$. By the same argument as above we conclude that $|b_k| > 2r$ for some $k \notin \{k_1, \ldots, k_j\}$. □

Theorem 3.3. *Let $r \geqslant 1$, $n \geqslant 2$, and $G = \bigtimes_{i=1}^{n} C_{\ell_i}$, where each ℓ_i is a multiple of $r^n + (r+1)^n$. Then each connected component of G contains an r-perfect code.*

Proof. Let X be any connected component of G and $\mathbf{x}$ an arbitrary vertex of X. Set

$$Q = \left\{ \mathbf{x} + \sum_{i=1}^{n} \alpha_i \mathbf{b^i} \;\middle|\; \alpha_i \in \mathbb{Z} \right\},$$

where the computations in the i-th coordinate are done modulo ℓ_i. We claim Q is an r-perfect code.

We prove first that the r-balls centered at vertices from Q are pairwise disjoint. Assume on the contrary that there are vertices $\mathbf{u}, \mathbf{v} \in Q$ such that $d(\mathbf{u}, \mathbf{v}) < 2r + 1$. The set Q can be written as

$$Q = \left\{ \mathbf{u} + \sum_{i=1}^{n} \alpha_i \mathbf{b^i} \;\middle|\; \alpha_i \in \mathbb{Z} \right\},$$

hence $\mathbf{v} = \mathbf{u} + \sum_{i=1}^{n} \alpha_i \mathbf{b^i}$ for some $\alpha_i \in \mathbb{Z}$ (modulo the lengths of the cycles). Since the absolute value of every coordinate of $\mathbf{v} - \mathbf{u}$ is less than $2r + 1$, $\mathbf{v} - \mathbf{u}$ cannot be expressed as a linear combination $\sum_{i=1}^{n} \alpha_i \mathbf{b^i}$ (see Lemma 3.2). Therefore

$$\mathbf{v} - \mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{b^i} + (t_1\ell_1, \ldots, t_n\ell_n),$$

for some $t_1, \ldots, t_n \in \mathbb{Z}$ (not calculated modulo the lengths of the cycles). Since the coordinates of $\mathbf{v} - \mathbf{u}$ and the coordinates of the sum $\sum_{i=1}^{n} \alpha_i \mathbf{b^i}$ are either all odd or all even, we find that all the $t_i\ell_i$ are either all odd or all even. Setting $t_i\ell_i = k_i(r^n + (r+1)^n)$ for $i = 1, \ldots, n$, we find that the k_i's are either all odd or all even. Therefore Lemma 3.1 implies that

$$(t_1\ell_1, t_2\ell_2, \ldots, t_n\ell_n) = \sum_{i=1}^{n} \beta_i \mathbf{b^i}$$

for some $\beta_i \in \mathbb{Z}$, $i = 1, \ldots, n$. Hence

$$\mathbf{v} - \mathbf{u} = \sum_{i=1}^{n} (\alpha_i + \beta_i) \mathbf{b^i}$$

(not calculated modulo lengths of cycles), which brings us to a contradiction.

To complete the proof we must show that each vertex $\mathbf{u}$ of X belongs to an r-ball centered in a vertex from Q. Since X is connected there exists a path from $\mathbf{x}$ to $\mathbf{u}$ and we shall prove that this path is covered by $\bigcup_{\mathbf{q} \in Q} B(\mathbf{q}, r)$. It suffices to prove that for each vertex $\mathbf{q} \in Q$ and any $\mathbf{p} \in B(\mathbf{q}, r+1) \setminus B(\mathbf{q}, r)$, there is a vertex $\mathbf{t} \in Q$ such that $\mathbf{p} \in B(\mathbf{t}, r)$.

We may without loss of generality assume $\mathbf{q} = \mathbf{0}$. Since $|B(\mathbf{0}, r)| = r^n + (r+1)^n$ we have $|B(\mathbf{0}, r+1) \setminus B(\mathbf{0}, r)| = (r+2)^n - r^n$. Consider the following linear combinations of the canonical local vertices for $\mathbf{0}$:

$$\pm \sum_{i=1}^{m} (-1)^i \mathbf{b}^{\delta(\mathbf{i})} \tag{5}$$

where $\delta : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is a strictly increasing function and $m \leqslant n$ is odd. For a fixed m there are $2\binom{n}{m}$ vertices of the form (5) and each of them contains $(r+1)^{n-m}$ vertices of $B(\mathbf{0}, r+1) \setminus B(\mathbf{0}, r)$. Since

$$2 \sum_{i=1}^{\lceil n/2 \rceil} \binom{n}{2i-1} (r+1)^{n+1-2i} = (r+2)^n - r^n,$$

we find that the (pairwise disjoint) r-balls centered on the vertices of the form (5) cover $B(\mathbf{0}, r+1) \setminus B(\mathbf{0}, r)$. Therefore

$$\bigcup_{\mathbf{q} \in Q} B(\mathbf{q}, r) = V(X). \qquad \square$$

The local picture of a 2-dimensional code and the corresponding 2-balls are shown in Fig. 1. The 2-perfect code is generated by the vectors $\mathbf{b^1} = (5, 1)$ and $\mathbf{b^2} = (-1, 5)$ (the code vertices are

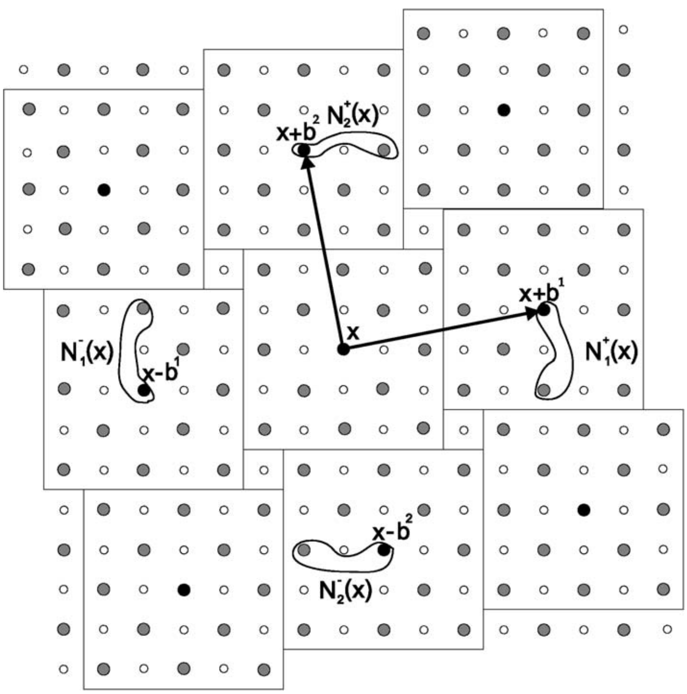

Fig. 1. A 2-dimensional 2-perfect code.

marked black). Note that each 2-ball covers only 13 among 25 encircled vertices (marked with gray color). The rest of the vertices, including the four isolated vertices, namely $(2,3)$, $(-3,2)$, $(-2,-3)$, and $(3,-2)$, are covered by 2-balls which can be reached by going once around a cycle or, if ℓ_1 and ℓ_2 are both even, these vertices lie in a component different from X. For further comments on Fig. 1 and the sets $N_1^{\pm}(\mathbf{x})$ and $N_2^{\pm}(\mathbf{x})$, see next section.

4. On the non-existence of other codes

In the previous section we constructed r-perfect codes in products $\bigtimes_{i=1}^{n} C_{\ell_i}$, where each ℓ_i is a multiple of $r^n + (r+1)^n$. In this section we establish a partial converse to this result. The converse suggests that the presented codes are probably the only r-perfect codes ($r \geqslant 2$) in direct products of cycles. More precisely, in this section we will show that an r-perfect code ($r \geqslant 2$) in $\bigtimes_{i=1}^{n} C_{\ell_i}$ is uniquely determined by n vertices. Moreover, if a code contains the vertex $\mathbf{0}$ and the canonical local vertices for $\mathbf{0}$, then any ℓ_i must be a multiple of $r^n + (r+1)^n$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an arbitrary vertex of $\bigtimes_{i=1}^{n} C_{\ell_i}$, let $r \geqslant 1$, and $s = 2r+1$. For $j = 1, \ldots, n$ we define

$$N_j^+(\mathbf{x}) = \bigl\{(x_1 + i_1, \ldots, x_n + i_n) \mid i_j = s, i_k \in \{-1, 1\} \text{ for } k \neq j\bigr\},$$
$$N_j^-(\mathbf{x}) = \bigl\{(x_1 + i_1, \ldots, x_n + i_n) \mid i_j = -s, i_k \in \{-1, 1\} \text{ for } k \neq j\bigr\}.$$

Note that for the perfect code from the previous section we have $\mathbf{x} + \mathbf{b^i} \in N_i^+(\mathbf{x})$ and $\mathbf{x} - \mathbf{b^i} \in N_i^-(\mathbf{x})$ for $i = 1, \ldots, n$. The sets $N_1^{\pm}(\mathbf{x})$ and $N_2^{\pm}(\mathbf{x})$ are shown in Fig. 1 for the product of two cycles. In addition, let

$$\mathcal{N}(\mathbf{x}) = \bigl\{N_j^+(\mathbf{x}), N_j^-(\mathbf{x}) \mid j = 1, \ldots, n\bigr\}.$$

Lemma 4.1. *Let $r \geqslant 1$ and let P be an r-perfect code of $\bigtimes_{i=1}^{n} C_{\ell_i}$, where $\ell_i \geqslant 2r+2$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in P$ then for every $M \in \mathcal{N}(\mathbf{x})$,*

$$|P \cap M| = 1.$$

Proof. We may without loss of generality consider $M = N_1^+(\mathbf{x})$.

Let r be an odd number, $s = 2r+1$, and consider the vertex

$$\mathbf{t} = (x_1 + r + 1, x_2, \ldots, x_n).$$

Since P is an r-perfect code and $\mathbf{t} \notin B(\mathbf{x}, r)$, there exists an r-ball $B(\mathbf{y}, r)$, where $\mathbf{y} \in P$, $\mathbf{t} \in B(\mathbf{y}, r)$, and $B(\mathbf{y}, r) \cap B(\mathbf{x}, r) = \emptyset$. Note that $d(\mathbf{x}, \mathbf{t}) = r+1$ and $d(\mathbf{t}, \mathbf{y}) = r$.

We claim that $y_1 = x_1 + s$. Since $\mathbf{t} \in B(\mathbf{y}, r)$ we have $x_1 + 1 \leqslant y_1 \leqslant x_1 + 2r + 1$ and $x_i - r \leqslant y_i \leqslant x_i + r$ for $i \geqslant 2$. As $|x_i - t_i|$ is even for $i \geqslant 1$ and $|t_i - y_i|$ is even/odd for $i \geqslant 1$ we have $|x_i - y_i|$ is even/odd for all $i \geqslant 1$. If $y_1 \neq x_1 + s$ then

$$\max_{1 \leqslant i \leqslant n} \bigl\{|x_i - y_i|\bigr\} \leqslant 2r,$$

but then by Lemma 2.3 $d(\mathbf{x}, \mathbf{y}) \leqslant 2r$, which is a contradiction. Therefore $y_1 = x_1 + s$. Since $\mathbf{t} \in B(\mathbf{y}, r)$ we have $x_i - r \leqslant y_i \leqslant x_i + r$ for $i = 2, \ldots, n$.

Suppose that $y_2 \geqslant x_2 + 1$. We want to prove that in this case $y_2 = x_2 + 1$. Assume on the contrary that $x_2 + 1 < y_2 \leqslant x_2 + r$ (note that in this case $r \geqslant 3$). Now consider the vertices

$$\begin{aligned}
\mathbf{z^1} &= (x_1 + r - 1, x_2 + r + 1, x_3, \ldots, x_n),\\
\mathbf{z^2} &= (x_1 - r - 1, x_2 + r - 1, x_3, \ldots, x_n),\\
\mathbf{z^3} &= (x_1 - r + 1, x_2 - r - 1, x_3, \ldots, x_n),\\
\mathbf{z^4} &= (x_1 + r + 1, x_2 - r + 1, x_3 \ldots, x_n).
\end{aligned}$$

Since $\mathbf{z^i} \notin B(\mathbf{x}, r) \cup B(\mathbf{y}, r)$, $1 \leqslant i \leqslant 4$, and since every r-ball containing two of them intersects $B(\mathbf{x}, r)$, there are disjoint r-balls $B(\mathbf{w^i}, r)$, where $\mathbf{w^i} \in P$, $\mathbf{z^i} \in B(\mathbf{w^i}, r)$, and $B(\mathbf{w^i}, r) \cap (B(\mathbf{x}, r) \cup B(\mathbf{y}, r)) = \emptyset$, $1 \leqslant i \leqslant 4$.

Because $B(\mathbf{w^1}, r) \cap (B(\mathbf{x}, r) \cup B(\mathbf{y}, r)) = \emptyset$ and $\mathbf{z^1} \in B(\mathbf{w^1}, r)$, we find that $(\mathbf{w^1})_1 = x_1 - 1$, $(\mathbf{w^1})_2 = x_2 + s$, and $x_i - r \leqslant (\mathbf{w^1})_i \leqslant x_i + r$ for $i \geqslant 3$. Using similar arguments for vertices $\mathbf{z^2}$, $\mathbf{z^3}$, and $\mathbf{z^4}$ we find that $(\mathbf{w^2})_1 = x_1 - s$, $(\mathbf{w^2})_2 = x_2 - 1$, $(\mathbf{w^3})_1 = x_1 + 1$, $(\mathbf{w^3})_2 = x_2 - s$, $(\mathbf{w^4})_1 = x_1 + s$, $(\mathbf{w^4})_2 = x_2 + 1$, and $x_i - r \leqslant (\mathbf{w^2})_i, (\mathbf{w^3})_i, (\mathbf{w^4})_i \leqslant x_i + r$ for $i \geqslant 3$. But then $\mathbf{y} \in B(\mathbf{w^4}, r)$ so we conclude that $y_2 = x_2 + 1$. (The projection of vertices and the corresponding r-balls onto the first and the second coordinate are schematically shown on Fig. 2.)

If $y_2 \leqslant x_2 - 1$ we consider the vertices $(x_1 + r - 1, x_2 - r - 1, x_3 \ldots, x_n)$, $(x_1 - r - 1, x_2 - r + 1, x_3, \ldots, x_n)$, $(x_1 - r + 1, x_2 + r + 1, x_3, \ldots, x_n)$, and $(x_1 + r + 1, x_2 + r - 1, x_3, \ldots, x_n)$. By arguments similar as above we find that $y_2 = x_2 - 1$. Since r is by our assumption odd, $y_2 \neq x_2$, thus $y_2 = x_2 + 1$ or $y_2 = x_2 - 1$. By symmetry we conclude $y_k = x_k + 1$ or $y_k = x_k - 1$ for

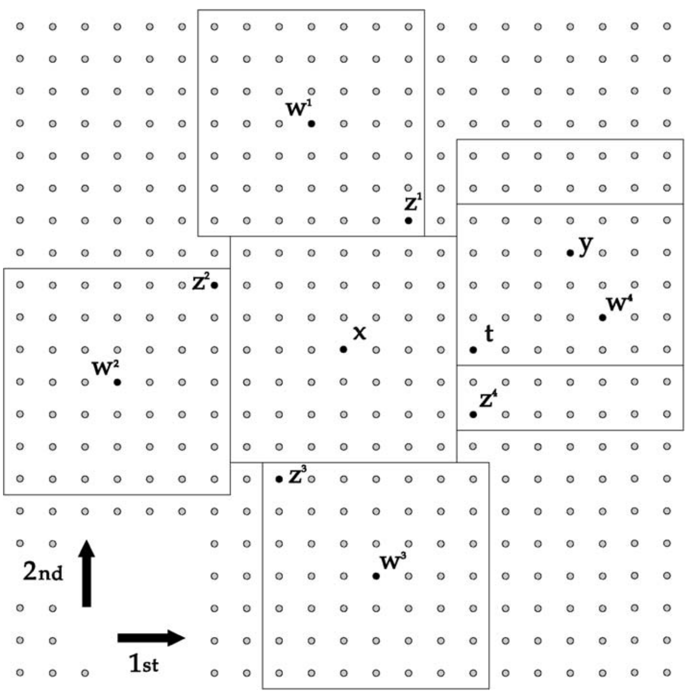

Fig. 2. Situation from the proof of Lemma 4.1. The first and the second coordinate are indicted in the lower left corner.

$k \geqslant 3$, hence $\mathbf{y} \in N_1^+(\mathbf{x})$. Because the vertices of $N_1^+(\mathbf{x})$ are pairwise at distance two, P contains exactly one of them, namely $\mathbf{y}$.

Arguments for r even are similar and left to the reader. □

We next introduce the so-called local structure of a code P. As we will see later, the local structure uniquely determines P. Let P be an r-perfect code of $\bigtimes_{i=1}^{n} C_{\ell_i}$, where $\ell_i \geqslant 2r+2$, and let $\mathbf{x} = (x_1, \ldots, x_n) \in P$. Then the set

$$\{\mathbf{x}\} \cup \left(P \cap \bigcup_{j=1}^{n-1} N_j^+(\mathbf{x}) \right)$$

is called *the* $\mathbf{x}$*-local structure of* P.

Note that Lemma 4.1 implies that a local structure of P contains n elements.

Lemma 4.2. *Let P be an r-perfect code of $\bigtimes_{i=1}^{n} C_{\ell_i}$, $\ell_i \geqslant 2r+2$, and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in P$. If $i \neq j$ and $\mathbf{y} \in N_j^+(\mathbf{x})$, $\mathbf{z} \in N_i^+(\mathbf{x})$ then*

(i) *If $z_j = x_j + 1$ then $y_i = x_i - 1$.*
(ii) *If $z_j = x_j - 1$ then $y_i = x_i + 1$.*

Moreover, there are exactly $2^{n(n-1)/2}$ possible $\mathbf{x}$*-local structures of P.*

Proof. Let $z_j = x_j + 1$ and assume on the contrary that $y_i = x_i + 1$. Without loss of generality assume that $x = (0, \ldots, 0)$, thus all coordinates of $\mathbf{y}$ and $\mathbf{z}$ are odd and

$$\max_{1 \leqslant i \leqslant n} \{|y_i - z_i|\} = 2r.$$

Hence, by Lemma 2.3, $d(\mathbf{y}, \mathbf{z}) \leqslant 2r$, which is a contradiction, because P is an r-perfect code.

Let now $z_j = x_j - 1$ and assume on the contrary that $y_i = x_i - 1$. We may again without loss of generality assume $x = (0, \ldots, 0)$. We claim that for $\mathbf{w} \in N_j^-(\mathbf{x}) \cap P$ we have $w_i = 1$. Suppose $w_i = -1$. Then if r is odd set $\mathbf{t} = (t_1, \ldots, t_n)$, where $t_i = 0$ for $i \neq j$ and $t_j = -r-1$, and if r is even set $t_i = 1$ for $i \neq j$ and $t_j = -r-1$. Then by straightforward case analysis, using Lemma 2.3 all the time, we find that for any vertex $\mathbf{u} \in B(\mathbf{t}, r)$, one of the distances $d(\mathbf{u}, \mathbf{w})$, $d(\mathbf{u}, \mathbf{y})$, and $d(\mathbf{u}, \mathbf{x})$ is less than $2r+1$, see Fig. 3. So $B(\mathbf{t}, r) \cap P = \emptyset$, a contradiction. It follows that $w_i = 1$, but then $d(\mathbf{w}, \mathbf{z}) \leqslant 2r$ which is not possible.

As $|N_1^+(\mathbf{x})| = 2^{n-1}$, there are 2^{n-1} possibilities to choose the vertex $\mathbf{a}$ from $N_1^+(\mathbf{x}) \cap P$ and since the first coordinate of the vertex $\mathbf{b}$ from $N_2^+(\mathbf{x}) \cap P$ is determined by the second coordinate of $\mathbf{a}$, there are 2^{n-2} possibilities how to choose $\mathbf{b}$. Continuing with this kind of reasoning we arrive at the number of possibilities for the $\mathbf{x}$-local structure of P, which is $2^{(n-1)+\cdots+2+1} = 2^{n(n-1)/2}$. □

Lemma 4.3. *Let P be an r-perfect code of a connected component of $\bigtimes_{i=1}^{n} C_{\ell_i}$, $\ell_i \geqslant 2r+2$, and let $\mathbf{x} \in P$. Then the set $\cup\mathcal{N}(\mathbf{x}) \cap P$ is uniquely determined by the* $\mathbf{x}$*-local structure of P.*

Proof. Let $\mathbf{y} \in N_n^+(\mathbf{x}) \cap P$, then by Lemma 4.2, $\mathbf{y}$ is determined by $\mathbf{x}$ and $P \cap \bigcup_{j=1}^{n-1} N_j^+(\mathbf{x})$, hence by the $\mathbf{x}$-local structure of P. Let $j \in \{1, \ldots, n\}$ and $\mathbf{u} \in N_j^+(\mathbf{x}) \cap P$, then we claim that

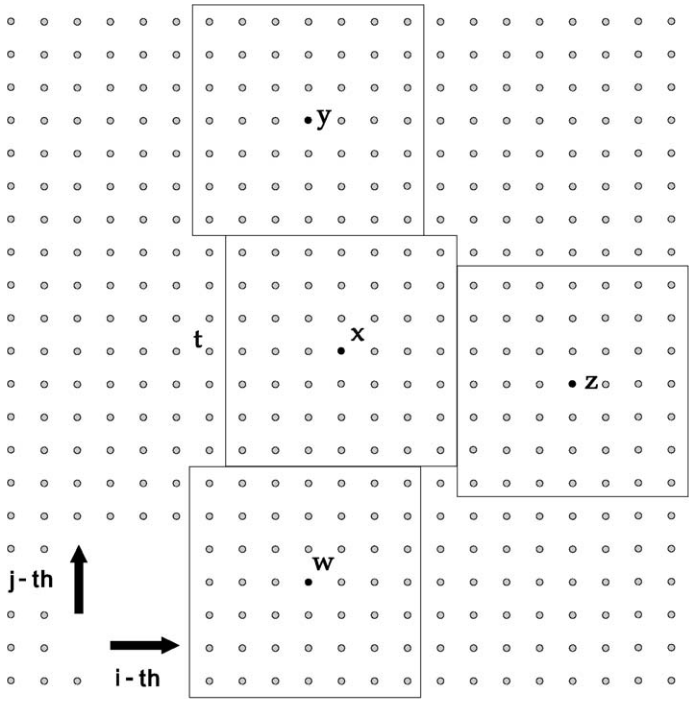

Fig. 3. Situation from the proof of Lemma 4.2.

for $\mathbf{y} \in N_j^-(\mathbf{x}) \cap P$ and for any $i \neq j$ we have $y_i = x_i - 1$ if $u_i = x_i + 1$ and $y_i = x_i + 1$ if $u_i = x_i - 1$. Suppose on the contrary, that for some $i \neq j$, $y_i = x_i + 1$ and $u_i = x_i + 1$. Then it is easy to see that for every vertex $\mathbf{z}$ from $N_i^+(\mathbf{x})$ either $d(\mathbf{z}, \mathbf{u}) \leqslant 2r$ or $d(\mathbf{z}, \mathbf{y}) \leqslant 2r$. This is a contradiction, since $\mathbf{z} \in P$ for some $\mathbf{z}$ in $N_i^+(\mathbf{x})$. $\square$

Let P be an r-perfect code of a connected component of $\bigtimes_{i=1}^n C_{\ell_i}$ and $\mathbf{x} \in P$. Then we set

$$B(\mathbf{x}) = \{\mathbf{a} - \mathbf{x} \mid \mathbf{a} \in \cup\mathcal{N}(\mathbf{x}) \cap P\}. \tag{6}$$

The set $B(\mathbf{x})$ is regarded as a set of vectors generating the perfect code. Note that the proof of the above lemma implies that $\mathbf{u} \in B(\mathbf{x})$ if and only if $-\mathbf{u} \in B(\mathbf{x})$. For the 2-perfect code depicted in Fig. 1 we have $B(\mathbf{x}) = \{\mathbf{b^1}, \mathbf{b^2}, -\mathbf{b^1}, -\mathbf{b^2}\}$, the vectors $\mathbf{b^1}$ and $\mathbf{b^2}$ are also marked. In the next lemma we give an explicit description of P via the set $B(\mathbf{x})$.

Lemma 4.4. *Let P be an r-perfect code of a connected component of $\bigtimes_{i=1}^n C_{\ell_i}$, where $r \geqslant 2$, and $\ell_i \geqslant 2r + 2$. Then for any vertex $\mathbf{x} \in P$,*

$$\left\{\mathbf{x} + \sum_{i=1}^n \alpha_i \mathbf{x^i} \;\middle|\; \alpha_i \in \mathbb{Z},\ \mathbf{x^i} \in B(\mathbf{x})\right\} \subseteq P.$$

Proof. Note that for $|\alpha_i| = 1$ and $\alpha_j = 0$ for $i \neq j$, the statement follows from the definition of $B(\mathbf{x})$.

We are going to prove that $B(\mathbf{x}) = B(\mathbf{y})$ holds for any $\mathbf{y} \in \mathcal{N}(\mathbf{x})$. The proof is then completed by a simple induction.

Let $\mathbf{y} \in N_j^+(\mathbf{x}) \cap P$ and $\mathbf{u} \in N_i^+(\mathbf{x}) \cap P$. We claim that for $\mathbf{z} \in N_i^+(\mathbf{y}) \cap P$ and for $k \neq i$ we have

$$z_k = y_k + 1 \quad \Longleftrightarrow \quad u_k = x_k + 1 \quad \text{and} \quad z_k = y_k - 1 \quad \Longleftrightarrow \quad u_k = x_k - 1, \tag{7}$$

and if $\mathbf{z} \in N_i^-(\mathbf{y}) \cap P$ and $\mathbf{u} \in N_i^-(\mathbf{x}) \cap P$, then for $k \neq i$ we have again

$$z_k = y_k + 1 \quad \Longleftrightarrow \quad u_k = x_k + 1 \quad \text{and} \quad z_k = y_k - 1 \quad \Longleftrightarrow \quad u_k = x_k - 1. \tag{8}$$

Assume $\mathbf{y} \in N_j^+(\mathbf{x}) \cap P$ and $\mathbf{u} \in N_i^+(\mathbf{x}) \cap P$ and suppose that the above equivalence does not hold; that is, for $\mathbf{z} \in N_i^+(\mathbf{y}) \cap P$, $z_k = y_k + 1$ and $u_k = x_k - 1$ for some $k \neq i$. Assume first that $k \neq j$ and $y_k = x_k + 1$. Then

$$z_i = x_i + s \pm 1, \qquad z_j = x_j + s \pm 1 \quad \text{and} \quad z_k = x_k + 2.$$

Consider the set $N_j^+(\mathbf{u})$ and the vertex $\mathbf{v} \in N_j^+(\mathbf{u}) \cap P$. Since $u_k = x_k - 1$ we have $v_k = x_k$ or $v_k = x_k - 2$, thus $\mathbf{v} \neq \mathbf{z}$ (since $z_k = x_k + 2$). Thus for $\mathbf{v}$ we have

$$v_i = x_i + s \pm 1, \qquad v_j = x_j + s \pm 1 \quad \text{and} \quad v_k = x_k \text{ or } v_k = x_k - 2.$$

Since for $\ell \neq i, j$, $z_\ell = x_\ell + a_\ell$ and $v_\ell = x_\ell + b_\ell$ for some $a_\ell, b_\ell \in \{-2, 0, 2\}$ we have (observe that all a_ℓ, b_ℓ are even)

$$d(\mathbf{v}, \mathbf{z}) = \max_{1 \leqslant \ell \leqslant n} |v_\ell - z_\ell| \leqslant 4 < 2r + 1,$$

so $\mathbf{v} = \mathbf{z}$, a contradiction.

Suppose that $k \neq j$ and $y_k = x_k - 1$. Then

$$z_i = x_i + s \pm 1, \qquad z_j = x_j + s \pm 1, \qquad z_k = x_k \quad \text{for some } k \neq i, j.$$

For $\mathbf{v} \in N_j^+(\mathbf{u}) \cap P$ we have again $v_k = x_k$ or $v_k = x_k - 2$. If $v_k = x_k - 2$ then $\mathbf{v} \neq \mathbf{z}$ and $d(\mathbf{v}, \mathbf{z}) \leqslant 4 < 2r + 1$. If $\mathbf{v} = \mathbf{z}$ then consider the vertex $\mathbf{m} \in N_i^-(\mathbf{x})$ and $\mathbf{o} \in N_i^-(\mathbf{y})$. Since $u_k = x_k - 1$ and $z_k = x_k$ we find that $m_k = x_k + 1$ and $o_k = x_k - 2$ (recall that $\mathbf{u} \in B(\mathbf{x}) \iff -\mathbf{u} \in B(\mathbf{x})$). Consider the vertex $\mathbf{p} \in N_j^+(\mathbf{m})$. Since $p_k = x_k$ or $p_k = x_k + 2$ we have $\mathbf{p} \neq \mathbf{o}$. Since $|o_\ell - p_\ell| \leqslant 4$ (and even) for $1 \leqslant \ell \leqslant n$ we have $d(\mathbf{o}, \mathbf{p}) \leqslant 4 < 2r + 1$.

The last remaining case is $k = j$ and in this case $z_j = y_j + 1$ and $u_j = x_j - 1$. Suppose that $y_i = x_i + 1$ and consider the vertex $\mathbf{v} \in N_i^-(\mathbf{z})$. For $\mathbf{v}$ we find that $v_j = x_j + s - 1$, $v_i = y_i - s = x_i - s + 1$ and for $\ell \neq i, j$, $v_\ell = x_\ell + i_\ell$ for some $i_\ell \in \{-2, 0, 2\}$. Hence $d(\mathbf{v}, \mathbf{x}) \leqslant 2r$, which is a contradiction, since P is an r-perfect code. Case $y_i = x_i - 1$ can be treated in a similar way, as can the statement for $\mathbf{z} \in N_i^-(\mathbf{y})$ and $\mathbf{u} \in N_i^-(\mathbf{x})$.

Thus by (7) and (8) for $\mathbf{y} \in \mathcal{N}(\mathbf{x})$ we find that $B(\mathbf{x}) = B(\mathbf{y})$. $\square$

Theorem 4.5. *Let P be an r-perfect code of a connected component of $\bigtimes_{i=1}^{n} C_{\ell_i}$, where $r \geqslant 2$, $n \geqslant 2$, and $\ell_i \geqslant 2r+2$. Suppose that P contains $\mathbf{0}$ and the canonical local vertices for $\mathbf{0}$. Then every ℓ_i is a multiple of $r^n + (r+1)^n$.*

Proof. By Lemma 4.4, for any scalars $\alpha_i \in \mathbb{Z}$ we have

$$\sum_{i=1}^{n} \alpha_i \mathbf{b^i} \in P. \tag{9}$$

We claim that for every fixed $k \in \{1, \ldots, n\}$, there exists a linear combination of the form (9) with the following properties

$$\sum_{i=1}^{n} \left(\alpha_k^i \mathbf{b^i}\right)_k = \omega \ell_k \quad \text{for some } \omega \in \{1, 2\}, \tag{10}$$

$$\sum_{i=1}^{n} \left(\alpha_k^i \mathbf{b^i}\right)_j = 0 \quad \text{for } j \neq k. \tag{11}$$

To see these, we will show that there is a sequence of vectors

$$\mathbf{u^j} = \sum_{i=1}^{n} \alpha_i^j \mathbf{b^i} \in P, \quad j = 0, 1, 2, \ldots$$

where

$$\mathbf{u^j} = \left(u_1^{(j)}, u_2^{(j)}, \ldots, u_n^{(j)}\right), \qquad \mathbf{u^0} = (0, \ldots, 0)$$

and $-2r \leqslant u_i^{(j)} \leqslant 2r$ for $i \neq k$ and $2r+2 \leqslant u_k^{(j+1)} - u_k^{(j)} \leqslant 4r+2$ and $u_k^{(j)}$ is even for all j, k.

The construction of this sequence is as follows:

$$\mathbf{u^1} = 2\mathbf{b^k}, \qquad \mathbf{u^2} = 4\mathbf{b^k}, \quad \ldots, \quad \mathbf{u^r} = 2r\mathbf{b^k}.$$

For $j \geqslant r$ take the following procedure. Let $i \geqslant k$ be the smallest coordinate such that $u_i^{(j)} = 2r$ and define

$$\mathbf{u^{j+1}} = \mathbf{u^j} - \mathbf{b^i} + \mathbf{b^k}.$$

If there is no coordinate of $\mathbf{u^j}$ equal to $2r$, then let $i \leqslant k$ be the smallest coordinate such that $u_i^{(j)} = -2r$ and define

$$\mathbf{u^{j+1}} = \mathbf{u^j} + \mathbf{b^i} + \mathbf{b^k}.$$

If there is no coordinate $u_i^{(j)}$ equal to $-2r$ or $2r$ then define

$$\mathbf{u^{j+1}} = \mathbf{u^j} + 2\mathbf{b^k}.$$

Case 1. Suppose ℓ_k is even and let $p \in \mathbb{N}$ be the smallest number such that

$$\ell_k - 2r \leqslant u_k^{(p)} \leqslant \ell_k + 2r.$$

Then $u_k^{(p)} - \ell_k$ is even and by Lemma 2.3,

$$d\big(\mathbf{u^p}, \mathbf{0}\big) \leqslant 2r.$$

Thus $\mathbf{u^p} = \mathbf{0}$ and $\omega = 1$.

Case 2. Suppose ℓ_k is odd and let $q \in \mathbb{N}$ be the smallest number such that

$$2\ell_k - 2r \leqslant u_k^{(q)} \leqslant 2\ell_k + 2r.$$

Then $u_k^{(q)} - 2\ell_k$ is even and by Lemma 2.3,

$$d\big(\mathbf{u^q}, \mathbf{0}\big) \leqslant 2r.$$

Thus $\mathbf{u^q} = \mathbf{0}$ and $\omega = 2$.

Hence Eq. (11) has at least one solution in $\mathbb{Z}^n$ and notably (11) is equivalent to Eq. (3) from the previous section. The solutions $(\alpha_k^1, \ldots, \alpha_k^n)$ of Eq. (3) are also given there. Since $\alpha_k^n \in \mathbb{Z}$ and the numbers r and $r+1$ don't have any common prime divisors, examination of the general solution shows that α_k^n is a multiple of $r^{n-k-1}(r+1)^{k-1}$. Suppose $\alpha_k^n = -tr^{n-k-1}(r+1)^{k-1}$, then

$$\alpha_k^i = \begin{cases} t \cdot r^{n-k-1+i}(r+1)^{k-1-i}; & i = 1, \ldots, k-1, \\ t\big((r+1)^{n-1} + r^{n-1}\big); & i = k, \\ -t \cdot r^{-k-1+i}(r+1)^{n+k-1-i}; & i = k+1, \ldots, n. \end{cases}$$

Now (10) yields

$$s\alpha_k^k + \sum_{i=1}^{k-1} \alpha_k^i - \sum_{i=k+1}^{n} \alpha_k^i = \omega\ell_k \quad \text{for some } \omega \in \{1, 2\},$$

thus

$$t(2r+1)\big((r+1)^{n-1} + r^{n-1}\big) + t\sum_{i=2}^{n}(r+1)^{n-i}r^{i-2} = \omega\ell_k.$$

Since the left side of the above equation is $2t(r^n + (r+1)^n)$ and $\omega \in \{1, 2\}$ we find that ℓ_k is a multiple of $r^n + (r+1)^n$. $\square$

The above theorem holds under much more general conditions. As we mentioned in the preliminaries, we have assumed that $\mathbf{0} \in P$, but we could start with any vertex $\mathbf{x}$ of P and consider the corresponding local vertices for $\mathbf{x}$. Moreover, let P be an r-perfect code of (a connected component of) $G = \bigtimes_{i=1}^{n} C_{\ell_i}$ with $\mathbf{0} \in P$. Then an isomorphism that maps $\mathbf{0}$ to $\mathbf{0}$ also maps the $\mathbf{0}$-local structure of P onto a $\mathbf{0}$-local structure. Hence any perfect code that can be obtained from the canonical one via some isomorphism that preserves $\mathbf{0}$ also fulfills Theorem 4.5.

Unfortunately, not all $\mathbf{0}$-local structures are isomorphic to the canonical $\mathbf{0}$-local structure. Consider, for instance, the $\mathbf{0}$-local structure containing $\mathbf{c^1} = (s, 1, 1, 1)$, $\mathbf{c^2} = (-1, s, -1, 1)$, $\mathbf{c^3} = (-1, 1, s, -1)$, and $\mathbf{c^4} = (-1, -1, 1, s)$. We claim that there is no r-perfect code containing these four vertices. (Consequently, this local structure is not isomorphic to the canonical one.) Consider the vertex $\mathbf{x} = (r-1, r+1, r+1, r+1)$ and the vertices $\mathbf{c^1}$ and $\mathbf{c^2} + \mathbf{c^3} + \mathbf{c^4} = (-3, s, s, s)$. Let $\mathbf{w} = (w_1, w_2, w_3, w_4)$ be the center of the r-ball containing $\mathbf{x}$. Then $w_1 \in \{s-2, s-4, \ldots, -1\}$ and $w_i \in \{s, s-2, \ldots, 1\}$ for $i = 2, 3, 4$. Hence $\mathbf{x}$ is not covered by an r-ball centered in $\mathbf{c^1}$ or $(-3, s, s, s)$. It is also straightforward to verify that any r-ball containing $\mathbf{x}$ intersects the r-ball centered in $\mathbf{c^1}$ or the r-ball centered in $(-3, s, s, s)$.

Since there are only 8 possible local structures in products of three cycles and 64 in products of four cycles, these two cases are reasonably small to check all possible local structures and isomorphisms between them. It is easily seen that if $n = 3$, then all local structures are isomorphic to the canonical one. In the case $n = 4$ we have exactly two non-isomorphic local structures, the canonical one and the one described above. Combining these arguments with Theorem 3.3 and Theorem 4.5 we have:

Theorem 4.6. *Let $r \geqslant 2$ and $3 \leqslant n \leqslant 4$. Then (a connected component of) $\bigtimes_{i=1}^{n} C_{\ell_i}$, $\ell_i \geqslant 2r+2$, contains an r-perfect code if and only if every ℓ_i is a multiple of $r^n + (r+1)^n$.*

We conclude the paper by conjecturing that Theorem 4.6 holds for $n \geqslant 5$ as well and asking whether there are other r-perfect codes besides the one constructed in this paper.

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