Coloring Sierpiński graphs and Sierpiński gasket graphs

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Abstract

Sierpiński graphs $S(n,3)$ are the graphs of the Tower of Hanoi puzzle with n disks, while Sierpiński gasket graphs S_n are the graphs naturally defined by the finite number of iterations that lead to the Sierpiński gasket. An explicit labeling of the vertices of S_n is introduced. It is proved that S_n is uniquely 3-colorable, that $S(n,3)$ is uniquely 3-edge-colorable, and that $\chi'(S_n) = 4$, thus answering a question from [15]. It is also shown that S_n contains a 1-perfect code only for $n = 1$ or $n = 3$ and that every $S(n,3)$ contains a unique Hamiltonian cycle.

Key words: Sierpiński graphs; Sierpiński gasket graphs; chromatic number; chromatic index; 1-perfect code, Hamiltonian cycle

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1 Introduction

Topological studies of the Lipscomb's space [11, 12] led in [8] to the definition of Sierpiński graphs $S(n,k)$. Another motivation for the introduction of these graphs is the fact that the graph $S(n,3)$ is isomorphic to the graph of the Tower of Hanoi puzzle with n disks [8], see also [5]. Sierpiński graphs were also independently studied in [14], where it is shown that they arise in a natural way from regular graphs.

The graphs $S(n,k)$ have many appealing properties and were studied from different points of view. They possess (essentially) unique 1-perfect codes [9], a result proved before for $S(n,3)$ in [2]. Alternative arguments for the uniqueness of 1-perfect codes in $S(n,k)$ were recently presented in [3] in order to determine

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their optimal $L(2,1)$-labelings. Moreover, (regularizations of) Sierpiński graphs are the first nontrivial families of graphs of “fractal” type for which the crossing number is known [10], while in [13] several metric invariants of these graphs are determined.

Hinz and Schief [7] used the connection between the graphs $S(n,3)$ with the Sierpiński gasket to compute the average distance of the latter, see also [1]. The graphs that are obtained in a natural way by n iterations of the process that leads to the Sierpiński gasket are called Sierpiński gasket graphs and denoted S_n. Teguia and Godbole [15] studied several properties of these graphs, in particular the chromatic number, the domination number, and the pebbling number.

In this paper we continue studies of the Sierpiński graphs $S(n,3)$ and the Sierpiński gasket graphs S_n. We first introduce an explicit labeling of the vertices of S_n that is obtained by “contracting” the Sierpiński labeling of $S(n,3)$. In the central part of the paper—Section 3—vertex- and edge-colorings of $S(n,3)$ and S_n are treated. It is in particular shown that S_n is uniquely 3-colorable (hence strengthening a result from [15]), that $S(n,3)$ is uniquely 3-edge-colorable, and that the chromatic index of S_n is 4 (hence answering a question from [15]). We conclude the paper by observing that S_n contains a 1-perfect code only for $n=1$ and $n=3$ and that every $S(n,3)$ contains a unique Hamiltonian cycle.

As usual, $\chi(G)$, $\chi'(G)$, and $\gamma(G)$ denote the chromatic number of G, the chromatic index of G, and the domination number of G, respectively. A 1-perfect code (also known as an efficient dominating set) in a graph G is a vertex subset of G such that the closed neighborhoods of its elements form a partition of $V(G)$. For any other graph theoretic concept not defined here we refer to [16].

2 Sierpiński graphs and Sierpiński gasket graphs

The *Sierpiński graphs* $S(n,3)$, $n \geq 1$, are defined in the following way:

$$V(S(n,3)) = \{1,2,3\}^n\,,$$

two different vertices $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots,v_n)$ being adjacent if and only if there exists an $h \in \{1,\ldots,n\}$ such that

(i) $u_t = v_t$, for $t = 1,\ldots,h-1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1,\ldots,n$.

We will shortly write $\langle u_1 \ldots u_n\rangle$ for $(u_1,\ldots,u_n)$. (On figures this convention is further shortened to $u_1 \ldots u_n$.) The graph $S(4,3)$ is shown in Fig. 1.

The vertices $\langle 1\ldots 1\rangle$, $\langle 2\ldots 2\rangle$, and $\langle 3\ldots 3\rangle$ are called the *extreme vertices of* $S(n,3)$. For $i=1,2,3$ let $S(n+1,3)_i$ be the subgraph of $S(n+1,3)$ induced by the vertices of the form $\langle i\ldots\rangle$. Clearly, $S(n+1,3)_i$ is isomorphic to $S(n,3)$.

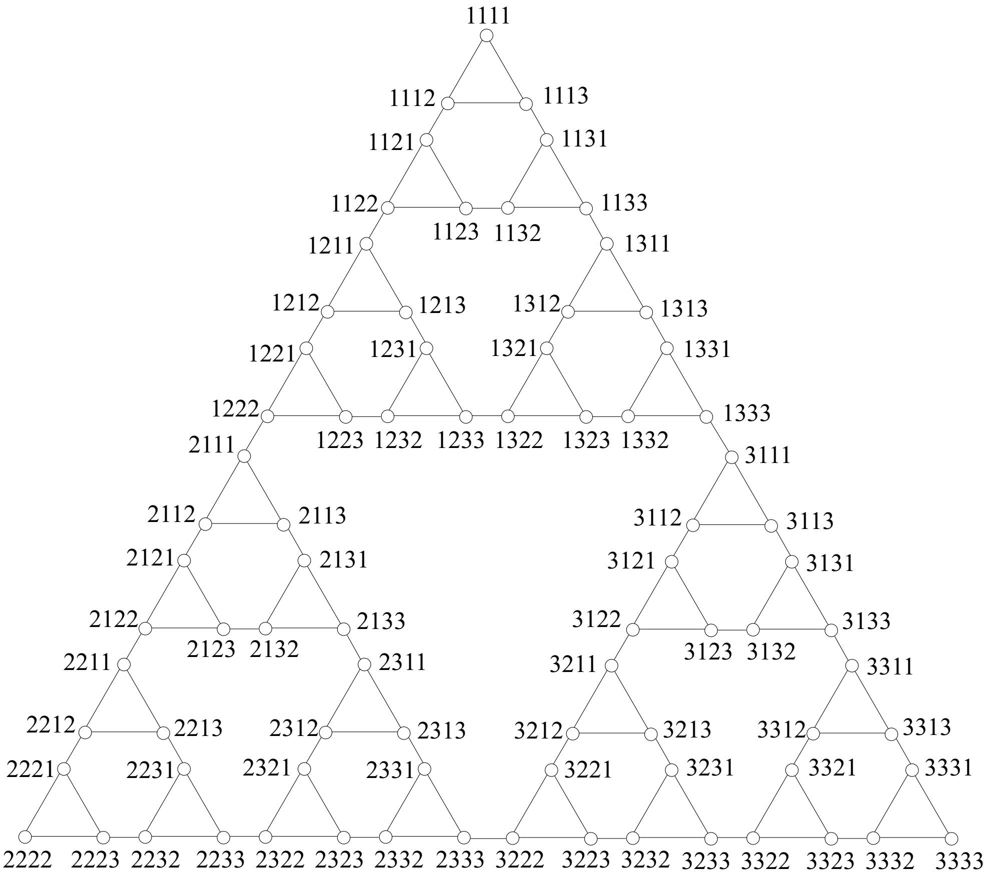

Figure 1: $S(4,3)$

The *Sierpiński gasket graphs* S_n, $n \geq 1$, are defined geometrically as the graphs whose vertices are the intersection points of the line segments of the finite Sierpiński gasket σ_n and line segments of the gasket as edges, see [15]. The Sierpiński gasket graph S_4 is shown in Fig. 2.

We now give an alternative description of the graphs S_n that yields an explicit labeling of the vertices of S_n. For this sake observe that S_n can be obtained from $S(n,3)$ by contracting all of its edges that lie in no triangle. Let $\langle u_1 \ldots u_r ij \ldots j\rangle$ and $\langle u_1 \ldots u_r ji \ldots i\rangle$ be endvertices of such an edge, then we will denote the corresponding vertex of S_n with $\langle u_1 \ldots u_r\rangle\{i,j\}$. Note that $r \leq n-2$ for such an edge. Then S_n is the graph with three special vertices $\langle 1 \ldots 1\rangle$, $\langle 2 \ldots 2\rangle$, and $\langle 3 \ldots 3\rangle$, called *extreme vertices of* S_n, together with the vertices of the form

$$\langle u_1 \ldots u_r\rangle\{i,j\}\,,$$

where $0 \leq r \leq n-2$, and all the u_k's, i and j are from $\{1,2,3\}$. Let us call this labeling the *quotient labeling* of S_n. (See Fig. 2 where the quotient labeling of S_4 is shown.) For a vertex $u = \langle u_1 \dots u_r \rangle \{i,j\}$ of S_n we will also say that $\langle u_1 \dots u_r \rangle$ is the *prefix* of u.

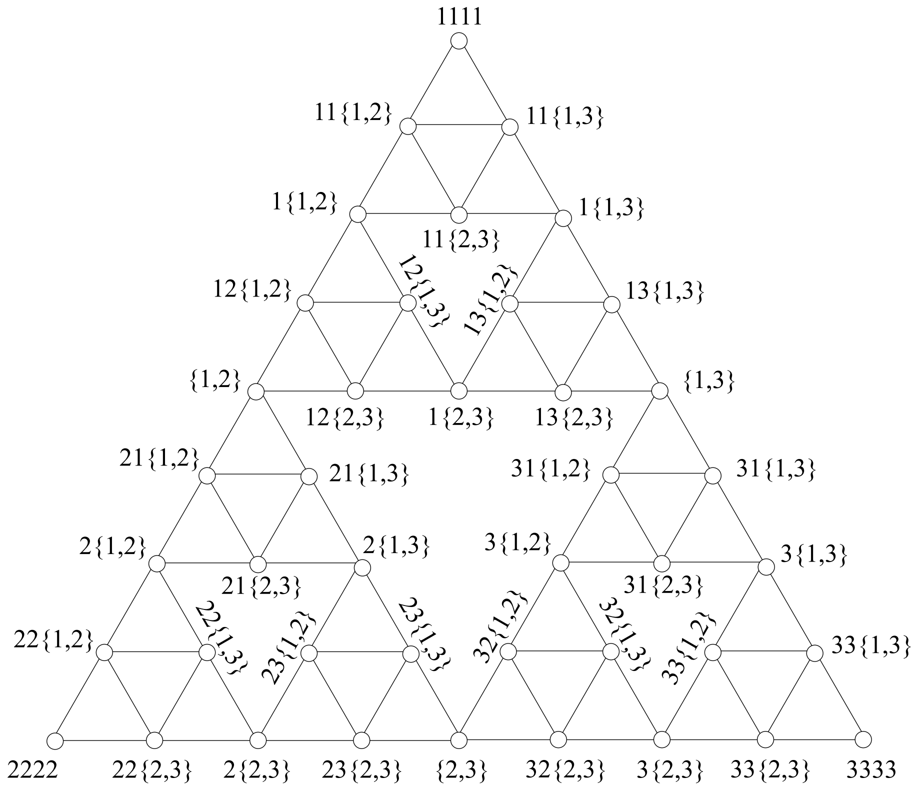

Figure 2: S_4 with its quotient labeling

For $i = 1,2,3$ let $S_{n,i}$ be the subgraph of S_{n+1} induced by $\langle i \dots i \rangle$, $\{i,j\}$, $\{i,k\}$, where $\{i,j,k\} = \{1,2,3\}$, and all the vertices whose prefix starts with i. Note that $S_{n,i}$ is isomorphic to S_n.

To explicitly describe adjacencies with respect to the quotient labeling of S_n, $n \geq 2$, note first that an extreme vertex $\langle i \dots i \rangle$ of S_n is adjacent to vertices $\langle i \dots i \rangle \{i,j\}$ and $\langle i \dots i \rangle \{i,k\}$, where $\{i,j,k\} = \{1,2,3\}$ and the prefixes are of length $n-2$. In particular, in S_2, an extreme vertex $\langle ii \rangle$ is adjacent to $\{i,j\}$ and $\{i,k\}$. To describe neighbors of other vertices we need the following notations. For a vertex $u = \langle u_1 \dots u_n \rangle$ of $S(n,3)$ and $s \leq n-2$ let $u_{(s)} = \langle u_1 \dots u_s \rangle$. Let $u = \langle u_1 \dots u_r \rangle \{i,j\}$ be a vertex of S_n. Then let

$$\overline{u} = \langle u_1 \dots u_r ij \dots j \rangle \quad \text{and} \quad \overline{\overline{u}} = \langle u_1 \dots u_r ji \dots i \rangle$$

be the endvertices of the edge of $S(n,3)$ contracted to u.

Proposition 2.1 *Let $n \geq 2$, let $u = \langle u_1 \dots u_r \rangle\{i,j\}$ be a vertex of S_n and let $\{i,j,k\} = \{1,2,3\}$.*

(i) If $0 \leq r \leq n-3$ then u is adjacent to

$$\overline{u}_{(n-2)}\{i,j\},\ \overline{u}_{(n-2)}\{j,k\},\ \overline{\overline{u}}_{(n-2)}\{i,j\},\ \text{and } \overline{\overline{u}}_{(n-2)}\{i,k\}\,.$$

(ii) If $r = n-2$ then u is adjacent to $\overline{u}_{(n-2)}\{i,k\}$, $\overline{\overline{u}}_{(n-2)}\{j,k\}$, to

$$\begin{cases} \overline{u}_{(t-1)}\{i,u_t\}, & t \text{ is the largest index with } u_t \neq i, 1 \leq t \leq n-2; \\ \langle i \dots i \rangle, & \text{no such } t \text{ exists}; \end{cases}$$

and to

$$\begin{cases} \overline{\overline{u}}_{(s-1)}\{j,u_s\}, & s \text{ is the largest index with } u_s \neq j, 1 \leq s \leq n-2; \\ \langle j \dots j \rangle, & \text{no such } s \text{ exists}; \end{cases}$$

Proof. (i) Since $r \leq n-3$, $\overline{u} = \langle u_1 \dots u_r ij \dots jj \rangle$ and $\overline{\overline{u}} = \langle u_1 \dots u_r ji \dots ii \rangle$, where $\overline{u}$ ends with at least two j's and $\overline{\overline{u}}$ ends with at least two i's. Hence $\overline{u}$ is in $S(n,3)$ adjacent to $\langle u_1 \dots u_r ij \dots ji \rangle$ and to $\langle u_1 \dots u_r ij \dots jk \rangle$. These two vertices contract to $\overline{u}_{(n-2)}\{i,j\}$ and $\overline{u}_{(n-2)}\{j,k\}$, respectively. Similarly, $\overline{\overline{u}}$ is adjacent to $\langle u_1 \dots u_r ji \dots ij \rangle$ and to $\langle u_1 \dots u_r ji \dots ik \rangle$ that contract $\overline{\overline{u}}_{(n-2)}\{i,j\}$ and $\overline{\overline{u}}_{(n-2)}\{i,k\}$, the two remaining vertices of S_n adjacent to u. (Note that the argument also holds for $r = 0$.)

(ii) Let $r = n-2$. Then $\overline{u} = \langle u_1 \dots u_{n-2} ij \rangle$ and $\overline{\overline{u}} = \langle u_1 \dots u_{n-2} ji \rangle$. The vertex $\overline{u}$ is in $S(n,3)$ adjacent to $\langle u_1 \dots u_{n-2} ik \rangle$ that contracts to $\overline{u}_{(n-2)}\{i,k\}$ and is also adjacent to $x = \langle u_1 \dots u_{n-2} ii \rangle$. If $u_1 = \cdots = u_{n-2} = i$, then $\overline{u}$ is adjacent to the extreme vertex $x = \langle i \dots i \rangle$ of $S(n,3)$, therefore u is adjacent to the extreme vertex $\langle i \dots i \rangle$ of S_n. Suppose not all u_i's are equal to i, and let t be the largest index such that $u_t \neq i$. Then $x = \langle u_1 \dots u_t i \dots i \rangle$, where $1 \leq t \leq n-2$. In this case u is also adjacent to $\overline{u}_{(t-1)}\{i,u_t\}$.

The other two neighbors of u that arise from the neighbors of $\overline{\overline{u}}$ are obtained analogously as the neighbors induced by $\overline{u}$. The details are left to the reader. □

We point out that in Proposition 2.1 the case $n = 2$ is treated in case (ii).

To conclude the section note that from the quotient labeling we can immediately infer that S_n contains $3 + \sum_{i=0}^{n-2} 3 \cdot 3^i = \frac{3}{2}(3^{n-1}+1)$ vertices.

3 Vertex- and edge-colorings

In this section we determine the chromatic number and the chromatic index of the Sierpiński graphs and the Sierpiński gasket graphs.

It is easy to see that for any $n \geq 1$, $\chi(S(n,3)) = 3$. As observed by Parisse [13], a natural 3-coloring of $S(n,3)$ can be obtained by setting

$$c(\langle u_1 \dots u_n \rangle) = u_n$$

for any vertex $\langle u_1 \dots u_n \rangle$ of $S(n,3)$. Teguia and Godbole [15, Proposition 2] showed an analogous result for the graphs S_n, namely $\chi(S_n) = 3$, $n \geq 1$. We can strengthen the latter result as follows.

Theorem 3.1 *S_n is uniquely 3-colorable for any $n \geq 1$.*

Proof. We prove the theorem by induction on n and pose the following stronger induction assumption: S_n is uniquely 3-colorable and in every 3-coloring the extreme vertices receive different colors.

The claim is clearly true for $S_1 = K_3$. Suppose it holds for S_n, $n \geq 2$, and consider an arbitrary 3-coloring c of S_{n+1}. By the induction assumption, $S_{n+1,1}$ is uniquely 3-colorable and we may without loss of generality assume that $c(\langle 1 \dots 1 \rangle) = 1$, $c(\{1,2\}) = 2$, and $c(\{1,3\}) = 3$. Then, considering $S_{n+1,2}$, the induction assumption implies $c(\{2,3\}) \neq 2$. Similarly because of $S_{n+1,3}$, we infer $c(\{2,3\}) \neq 3$. So $c(\{2,3\}) = 1$ and therefore $c(\langle 2 \dots 2 \rangle) = 3$ and $c(\langle 3 \dots 3 \rangle) = 2$. Induction completes the argument. □

In the rest of this section we consider edge-colorings. To show that Sierpiński graphs are uniquely 3-edge-colorable we prove the following.

Theorem 3.2 *Let $n \geq 1$. The 3-colorings of S_n are in a 1-1 correspondence with the 3-edge-colorings of $S(n,3)$.*

Proof. For $i, j \in \{1,2,3\}$, $i \neq j$, let $\overline{\{i,j\}} = \{1,2,3\} \setminus \{i,j\}$. In addition, for a vertex $u \in S(n,3)$ let $\widetilde{u}$ be the vertex of S_n to which u is mapped while contracted $S(n,3)$ to S_n.

Let c be a 3-coloring of S_n. Then for an arbitrary edge uv of $S(n,3)$ set

$$c'(uv) = \begin{cases} \overline{\{c(\widetilde{u}), c(\widetilde{v})\}}, & \widetilde{u} \neq \widetilde{v}; \\ c(\widetilde{u}), & \widetilde{u} = \widetilde{v}; \end{cases}$$

We claim that c' is an edge-coloring of $S(n,3)$. Let uv and vw be adjacent edges of $S(n,3)$. Suppose first that they belong to a common triangle. Then $\widetilde{u}$, $\widetilde{v}$, and $\widetilde{w}$ are pairwise different vertices of S_n which implies that $c'(uv) \neq c'(vw)$. The other case to consider is when, without loss of generality, vw belongs to a triangle while uv belongs to no triangle of $S(n,3)$. Then $c'(uv) = c(\widetilde{u}) = c(\widetilde{v})$, while $c(vw) = \overline{\{c(\widetilde{v}), c(\widetilde{w})\}} \neq c(\widetilde{u}) = c'(uv)$.

Let now c' be a 3-edge-coloring of $S(n,3)$. For a vertex $\langle u_1 \dots u_r\rangle\{i,j\}$ of S_n set

$$c(\langle u_1 \dots u_r\rangle\{i,j\}) = c'(\langle u_1 \dots u_r ij \dots j\rangle\langle u_1 \dots u_r ji \dots i\rangle)\,.$$

Suppose $c(\widetilde{u}) = c(\widetilde{v})$ holds for adjacent vertices $\widetilde{u}$ and $\widetilde{v}$ of S_n. Since $\widetilde{u}$ and $\widetilde{v}$ are adjacent they lie in a common triangle $\widetilde{T} = \widetilde{u}\widetilde{v}\widetilde{w}$ that in turn corresponds to a triangle $T = uvw$ of $S(n,3)$. Then $c(\widetilde{u}) = c(\widetilde{v})$ implies that the two edges of $S(n,3)$ that are adjacent to u and v, and do not lie in T, receive the same c'-color. But then none of the edges of T can be colored with this color, a contradiction.

We have thus shown that c properly colors the subgraph of S_n induced by all but the three extreme vertices. Clearly, c can be uniquely extended to a 3-coloring of S_n. □

Combining Theorems 3.1 and 3.2 we immediately get:

Corollary 3.3 *$S(n,3)$ is uniquely 3-edge-colorable for any $n \geq 1$.*

We next determine the chromatic index of the Sierpiński gasket graphs, a question posed in [15].

Clearly, $\chi'(S_n) \geq 4$ for $n \geq 2$. Consider a 4-edge coloring of S_2. The edges of its middle triangle receive different colors and their color classes contain at most two edges each. Since any color class contains at most 3 edges and S_2 has 9 edges, we infer that three color classes contain two edges while the remaining color class has 3 edges. But then the later edges alternate on the outer 6-cycle and we conclude that a 4-edge-coloring of S_2 is unique modulo permutations of the colors and the shift of the color class with three elements on the outer cycle. In particular, a 4-edge-coloring of S_2 is uniquely defined with the colors of its outer 6-cycle, hence in our figures we will color only such edges.

Note that $\chi'(S_1) = 3$, while for the other Sierpiński gasket graphs we have the following result.

Theorem 3.4 *For any $n \geq 2$, $\chi'(S_n) = 4$.*

Proof. It suffices to construct an edge-coloring with four colors for any $n \geq 2$.

Let c be an edge-coloring of S_n, then let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be the sets of colors assigned by c to the two edges incident with $\langle 1 \dots 1\rangle$, $\langle 2 \dots 2\rangle$, and $\langle 3 \dots 3\rangle$, respectively. To prove the theorem we pose the following stronger claim.

Claim: If n is even, then there exists a 4-edge-coloring c of S_n such that $\mathcal{C}_1 = \{1,2\}$, $\mathcal{C}_2 = \{1,3\}$, and $\mathcal{C}_3 = \{1,4\}$. If n is odd, then there exists a 4-edge-coloring c of S_n such that $\mathcal{C}_1 = \{1,2\}$, $\mathcal{C}_2 = \{1,3\}$, and $\mathcal{C}_3 = \{2,3\}$.

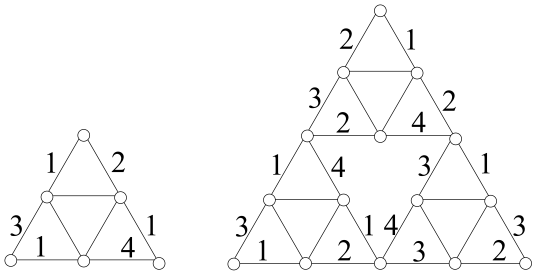

Figure 3: 4-edge-colorings of S_2 and S_3

The claim is true for $n = 2, 3$ as demonstrated in Fig. 3.

Let n be even. Then color S_{n+1} as follows. Let c' be a coloring of $S_{n+1,1}$ such that $\mathcal{C}'_1 = \{1, 2\}$, $\mathcal{C}'_2 = \{1, 3\}$, and $\mathcal{C}'_3 = \{1, 4\}$. Let c'' be a coloring of $S_{n+1,2}$ such that $\mathcal{C}''_1 = \{4, 2\}$, $\mathcal{C}''_2 = \{4, 1\}$, and $\mathcal{C}''_3 = \{4, 3\}$. Finally, let c''' be a coloring of $S_{n+1,3}$ such that $\mathcal{C}'''_1 = \{2, 3\}$, $\mathcal{C}'''_2 = \{2, 1\}$, and $\mathcal{C}'''_3 = \{2, 4\}$. See the left-hand side of Fig. 4. All there colorings exist by the induction assumption and by appropriate permutations of colors. Combine c', c'', and c''' to obtain a 4-edge-coloring of S_{n+1}. Finally exchange colors 3 and 4 to obtain a desired coloring.

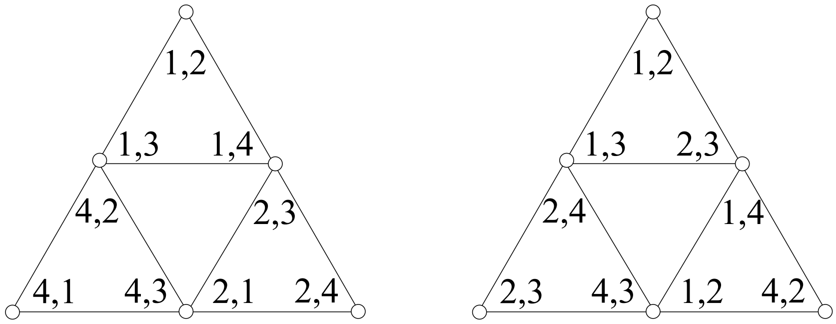

Figure 4: Even to odd case, and odd to even case

For n odd we proceed similarly. Let c' be a coloring of $S_{n+1,1}$ such that $\mathcal{C}'_1 = \{1, 2\}$, $\mathcal{C}'_2 = \{1, 3\}$, and $\mathcal{C}'_4 = \{2, 3\}$, let c'' be a coloring of $S_{n+1,2}$ such that $\mathcal{C}''_1 = \{2, 4\}$, $\mathcal{C}''_2 = \{2, 3\}$, and $\mathcal{C}''_3 = \{4, 3\}$, and c''' a coloring of $S_{n+1,3}$ such that $\mathcal{C}'''_1 = \{1, 4\}$, $\mathcal{C}'''_2 = \{1, 2\}$, and $\mathcal{C}'''_3 = \{4, 2\}$, see the right-hand side of Fig. 4. Again, there colorings exist by the induction assumption and by appropriate permutations of colors. Combine c', c'', and c''' to color S_{n+1} and exchange colors 1 and 2 to obtain a desired coloring of S_{n+1}. $\square$

4 On codes and Hamiltonicity

In the final section we present two additional aspects of Sierpiński (gasket) graphs.

It is proved in [15] that for every $n \geq 4$, $\gamma(S_n) = 3\gamma(S_{n-1})$. This enables us to quickly prove the following result that is in a strike contrast to the fact already mentioned in the introduction that every Sierpiński graph $S(n,k)$ contains essentially a unique 1-perfect code. Recall that if C is a 1-perfect code of a graph G, then $|C| = \gamma(G)$, see [4, Theorem 4.2].

Proposition 4.1 *S_n contains a 1-perfect code if and only if $n = 1$ or $n = 3$.*

Proof. It is straightforward to verify the result for $n \leq 3$.

Let $n \geq 4$ and suppose that C is a 1-perfect code of S_n. Assume $\langle 1 \dots 1 \rangle \in C$ and consider the vertex $\langle 1 \dots 1 \rangle \{2, 3\}$ with the prefix of length $n-2$. Then it can be dominated either with itself, with $\langle 1 \dots 1 \rangle \{1, 2\}$, or with $\langle 1 \dots 1 \rangle \{1, 3\}$, where the last two vertices have prefixes of length $n-3$. However, none of these vertices qualifies for C, so $\langle 1 \dots 1 \rangle \notin C$. Analogously $\langle 2 \dots 2 \rangle \notin C$ and $\langle 3 \dots 3 \rangle \notin C$. Hence every vertex of C is of degree 4 and as S_n contains $3(3^{n-1}+1)/2$ vertices, we infer that $|C| = 3(3^{n-1}+1)/10$. Since $|C| = \gamma(S_n)$, the above result of Teguia and Godbole implies that $\gamma(S_n) = |C| = 3^{n-2}$ for $n \geq 3$. But then $3(3^{n-1}+1)/10 = 3^{n-2}$, which reduces to $3^{n-2} = 3$ with $n = 3$ as the solution. □

We conclude the paper by proving that the Sierpiński graphs $S(n,3)$ contain unique Hamiltonian cycles. For this sake we first show:

Lemma 4.2 *Let $n \geq 1$ and let u, v be extreme vertices of $S(n,3)$. Then there exists a unique Hamiltonian u, v-path.*

Proof. The statement is clearly true for $n = 1$. Suppose it holds for $n \geq 2$ and without loss of generality consider the extreme vertices $\langle 1 \dots 1 \rangle$ and $\langle 2 \dots 2 \rangle$ of $S(n+1,3)$. By the induction assumption, there exists a unique Hamiltonian path P between $\langle 1 \dots 1 \rangle$ and $\langle 13 \dots 3 \rangle$ in $S(n+1,3)_1$, a unique Hamiltonian path Q between $\langle 31 \dots 1 \rangle$ and $\langle 32 \dots 2 \rangle$ in $S(n+1,3)_3$, and a unique Hamiltonian path S between $\langle 23 \dots 3 \rangle$ and $\langle 2 \dots 2 \rangle$ in $S(n+1,3)_2$. Then

$$\langle 1 \dots 1 \rangle P \langle 13 \dots 3 \rangle \langle 31 \dots 1 \rangle Q \langle 32 \dots 2 \rangle \langle 23 \dots 3 \rangle S \langle 2 \dots 2 \rangle$$

is a Hamiltonian path in $S(n+1,3)$. To see that it is unique, observe that $\langle 12 \dots 2 \rangle$ must appear before $\langle 13 \dots 3 \rangle$ on any Hamiltonian $\langle 1 \dots 1 \rangle, \langle 2 \dots 2 \rangle$-path. Indeed, suppose this is not the case. Then if we proceed from $\langle 13 \dots 3 \rangle$ to $\langle 31 \dots 1 \rangle$, the vertex $\langle 12 \dots 2 \rangle$ would appear on the Hamiltonian path just after $\langle 21 \dots 1 \rangle$ which is clearly not possible. And if we proceed from $\langle 13 \dots 3 \rangle$ to a vertex of $S(n+1,3)_1$, then the vertex $\langle 32 \dots 2 \rangle$ would appear on the Hamiltonian path just after $\langle 23\dots 3\rangle$, which is also not possible. Similarly, $\langle 3\dots 3\rangle$ must appear before $\langle 32\dots 2\rangle$ on any Hamiltonian $\langle 1\dots 1\rangle, \langle 2\dots 2\rangle$-path. Induction completes the argument. □

Theorem 4.3 *$S(n,3)$, $n \geq 1$, contains a unique Hamiltonian cycle.*

Proof. The case $n = 1$ is trivial. For $n > 1$ construct a Hamiltonian cycle of $S(n,3)$ by combining a Hamiltonian $\langle 12\dots 2\rangle, \langle 13\dots 3\rangle$-path in $S(n,3)_1$, a Hamiltonian $\langle 31\dots 1\rangle, \langle 32\dots 2\rangle$-path in $S(n,3)_3$, and a Hamiltonian $\langle 23\dots 3\rangle, \langle 21\dots 1\rangle$-path in $S(n,3)_2$. By Lemma 4.2, this Hamiltonian cycle is unique. □

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References

[1] T.-H. Chan, A statistical analysis of the Towers of Hanoi problem, Int. J. Comput. Math. 28 (1989) 57–65.

[2] P. Cull and I. Nelson, Error-correcting codes on the Towers of Hanoi graphs, Discrete Math. 208/209 (1999) 157–175.

[3] S. Gravier, S. Klavžar and M. Mollard, Codes and $L(2,1)$-labelings in Sierpiński graphs, Taiwan. J. Math. 9 (2005) 671–681.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[5] A. M. Hinz, Pascal's triangle and the Tower of Hanoi, Amer. Math. Monthly 99 (1992) 538–544.

[6] A. M. Hinz and D. Parisse, On the planarity of Hanoi graphs, Expo. Math. 20 (2002) 263–268.

[7] A. M. Hinz and A. Schief, The average distance on the Sierpiński gasket, Probab. Theory Related Fields 87 (1990) 129–138.

[8] S. Klavžar and U. Milutinović, Graphs $S(n,k)$ and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47(122) (1997) 95–104.

[9] S. Klavžar, U. Milutinović and C. Petr, 1-perfect codes in Sierpiński graphs, Bull. Austral. Math. Soc. 66 (2002) 369–384.

[10] S. Klavžar and B. Mohar, Crossing numbers of Sierpiński-like graphs, J. Graph Theory 50 (2005) 186–198.

[11] S. L. Lipscomb and J. C. Perry, Lipscomb's $L(A)$ space fractalized in Hilbert's $l^2(A)$ space, Proc. Amer. Math. Soc. 115 (1992) 1157–1165.

[12] U. Milutinović, Completeness of the Lipscomb space, Glas. Mat. Ser. III 27(47) (1992) 343–364.

[13] D. Parisse, On some metric properties of the Sierpiński graphs $S(n, k)$, Ars Combin., in press.

[14] T. Pisanski and T. W. Tucker, Growth in repeated truncations of maps, Atti Sem. Mat. Fis. Univ. Modena 49 (2001) 167–176.

[15] A. M. Teguia and A. P. Godbole, Sierpiński gasket graphs and some of their properties, Australas. J. Combin. 35 (2006) 181–192.

[16] D. B. West, Introduction to Graph Theory, Second Edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.