Vertex-, edge-, and total-colorings of Sierpiński-like graphs

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Abstract

Vertex-colorings, edge-colorings and total-colorings of the Sierpiński gasket graphs S_n, the Sierpiński graphs $S(n,k)$, graphs $S^+(n,k)$, and graphs $S^{++}(n,k)$ are considered. In particular, $\chi''(S_n)$, $\chi'(S(n,k))$, $\chi(S^+(n,k))$, $\chi(S^{++}(n,k))$, $\chi'(S^+(n,k))$, and $\chi'(S^{++}(n,k))$ are determined.

Key words: Sierpiński gasket graphs; Sierpiński graphs; chromatic number; chromatic index; total chromatic number

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1 Introduction

Graphs of "Sierpiński type" appear naturally in many different areas of mathematics as well as in several other scientific fields. One of the most important families of such graphs is formed by the Sierpiński gasket graphs—the graphs obtained after a finite number of iterations that in the limit give the Sierpiński gasket, see, for instance, [8]. These graphs were introduced already in 1944 by Scorer, Grundy and Smith [20]. Among others, the Sierpiński gasket graphs play an important role in dynamic systems and probability, cf. [7, 9], as well as in psychology, cf. [14].

Sierpiński gasket graphs are just a step from the Sierpiński graphs $S(n, 3)$. The graph S_n is obtained from $S(n, 3)$ by subdividing every edge of $S(n, 3)$ that lies in no triangle. This connection was already observed in psychological literature by Sydow back in 1970 [21]. One of the main features of the graphs $S(n, 3)$ is that they are precisely the graphs of the Tower of Hanoi puzzle with n discs. These graphs were quite extensively studied by now, see, for instance, [1, 4, 5, 12, 19].

In [11], the graphs $S(n, 3)$ were generalized to the Sierpiński graphs $S(n, k)$ for $k \geq 3$. The motivation for this generalization came from topological studies of the Lipscomb's space [15, 16]. (We note that the Sierpiński graphs independently appeared in [18].) As it turned out, the graphs $S(n, k)$ possess many appealing properties, as for instance several coding [3] and several metric properties [17]. The generalization of $S(n, 3)$ to $S(n, k)$ is done via a certain labeling technique (see Section 2) that in turn gives a new powerful tool for studying the classical Tower of Hanoi graphs $S(n, 3)$. The labeling technique has been fruitfully applied in [4, 19].

The graphs $S(n, k)$ are almost regular and there are at least two natural ways to extend them to regular graphs. In this spirit regularizations $S^+(n, k)$ and $S^{++}(n, k)$ were proposed in [13]. For these two families of graphs the exact crossing number can be determined (modulo the crossing number of complete graphs), thus they present the first known examples of graphs of "fractal" type for which this can be done [13].

Besides the mentioned properties, vertex and edge colorings of the graphs S_n and $S(n, k)$ were previously studied. Teguia and Godbole [22] showed that $\chi(S_n) = 3$. In fact, these colorings are unique [10]. In the latter paper it is also proved that for any $n \geq 2$, $\chi'(S_n) = 4$. Teguia and Godbole [22] asked what is the total chromatic number of Sierpiński gasket graphs. We answer their question in Section 3.

Parisse noticed that $\chi(S(n, k)) = k$ [17]. In Section 4 we determine the chromatic index of these graphs and the total chromatic number when k is odd. We also show that the famous Behzad-Vizing conjecture also holds when k is even.

In the last section we consider vertex-, edge-, and total-colorings of the graphs $S^+(n, k)$ and $S^{++}(n, k)$, and in particular determine their chromatic number and edge-chromatic number.

The results obtained in this paper together with the previously known results are collected in Table 1.

	χ	χ'	χ''
S_n	3 (uniquely) $n \geq 2$	4 $n \geq 2$	5 $n \geq 2$
$S(n,k)$	k $n \geq 1$, $k \geq 1$	k $n \geq 2$, $k \geq 2$	$k+1$ $n \geq 2$, $k \geq 3$, k odd
		3 (uniquely) $n \geq 1$, $k = 3$	3, $n \geq 2$, $k = 2$ 4, $n \geq 2$, $k = 4$
			$k+1 \leq \cdot \leq k+2$ $n \geq 2$, $k \geq 6$, k even
$S^+(n,k)$	k $n \geq 2$, $k \geq 3$	k $n \geq 2$, $k \geq 2$, k odd	$k+1 \leq \cdot \leq k+2$ $n \geq 2$, $k \geq 2$
		$k+1$ $n \geq 2$, $k \geq 2$, k even	
$S^{++}(n,k)$	k $n \geq 2$, $k \geq 2$	k $n \geq 2$, $k \geq 2$	$k+1$ $n \geq 2$, $k \geq 3$, k odd
			$k+1 \leq \cdot \leq k+2$ $n \geq 2$, $k \geq 2$, k even

Table 1: Summary of the results

2 Preliminaries

Let G be a graph. Recall that the chromatic number $\chi(G)$ (edge-chromatic number $\chi'(G)$) is the smallest number of colors needed for a proper vertex-coloring (edge-coloring) of G. Clearly, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the largest degree of G. Vizing's theorem asserts that in addition $\chi'(G) \leq \Delta(G) + 1$. G is called a graph of Type II if $\chi'(G) = \Delta(G) + 1$ holds, otherwise it is of Type I. It is well-known that the complete graph K_n is of Type I if and only if n is even. We will show that the graphs $S^+(n,k)$, $n \geq 2$, $k \geq 2$, k even, are of Type II.

The total chromatic number $\chi''(G)$ is the smallest number of colors needed for a proper coloring of both vertices and edges of G. Clearly, $\chi''(G) \geq \Delta(G) + 1$. Recall that $\chi''(K_n) = \Delta(K_n) + 1$ if n is odd and $\chi''(K_n) = \Delta(K_n) + 2$ if n is even, see [23]. Behzad-Vizing conjecture claims that $\chi''(G) \leq \Delta(G) + 2$. This conjecture has been verified for several classes of graphs, see [2, 23, 25] and references therein. All the graphs studied in this paper support the conjecture.

In the rest of this section we define the families of Sierpiński-like graphs considered in this paper.

We begin with the Sierpiński graphs $S(n,k)$ that are defined for $n \geq 1$ and $k \geq 1$ as follows. The vertex set of $S(n,k)$ consists of all n-tuples of integers $1,2,\ldots,k$, that is, $V(S(n,k)) = \{1,2,\ldots,k\}^n$. Two different vertices $u = (u_1,\ldots,u_n)$ and $v = (v_1,\ldots,v_n)$ are adjacent if and only if there exists an $h \in \{1,\ldots,n\}$ such that

(a) $u_t = v_t$, for $t = 1,\ldots,h-1$;
(b) $u_h \neq v_h$; and
(c) $u_t = v_h$ and $v_t = u_h$ for $t = h+1,\ldots,n$.

We will write $\langle u_1 u_2 \ldots u_n \rangle$ for $(u_1,u_2,\ldots,u_n)$ or even shorter $u_1 u_2 \ldots u_n$. See Fig. 1 for $S(3,4)$.

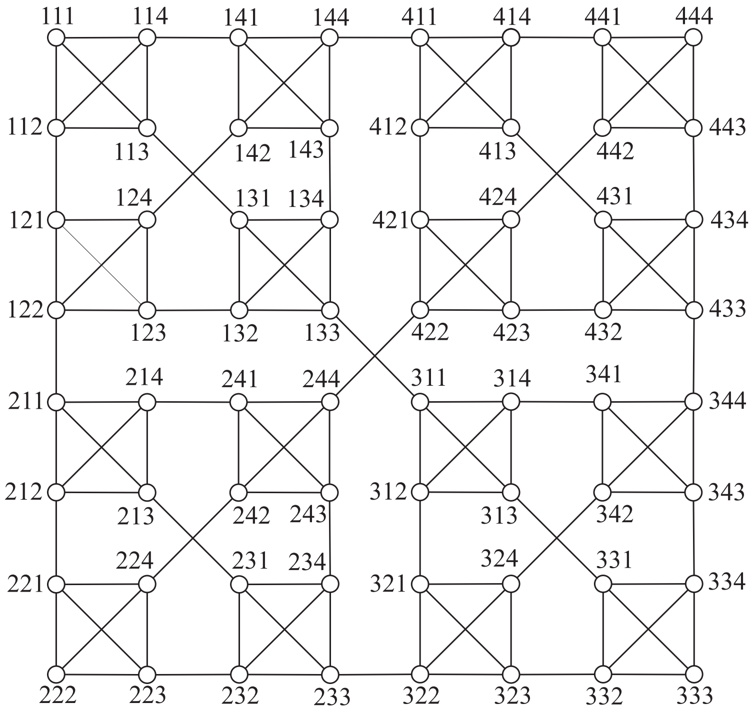

Figure 1: The Sierpiński graph $S(3,4)$

The vertices $\langle i \ldots i \rangle$, $i \in \{1,\ldots,k\}$, are called the *extreme vertices of* $S(n,k)$. For $i = 1,2,\ldots,k$ let $S_i(n+1,k)$ be the subgraph of $S(n+1,k)$ induced by the vertices of the form $\langle i \ldots \rangle$. Clearly, $S_i(n+1,k)$ is isomorphic to $S(n,k)$. Consequently, $S(n+1,k)$ contains k^n copies of the graph $S(1,k) = K_k$. The edges of $S(n,k)$ that lie in no induced K_k will be called *bridge edges*.

The *Sierpiński gasket graph* S_n, $n \geq 1$, is obtained from $S(n,3)$ by contracting all the edges of $S(n,3)$ that lie in no triangle, see Fig. 2 for S_4.

Following [10] we label the vertices of S_n as follows. Let $\langle u_1 \ldots u_r ij \ldots j \rangle$ and $\langle u_1 \ldots u_r ji \ldots i \rangle$ be endvertices of an edge of $S(n,3)$ that is contracted to a vertex

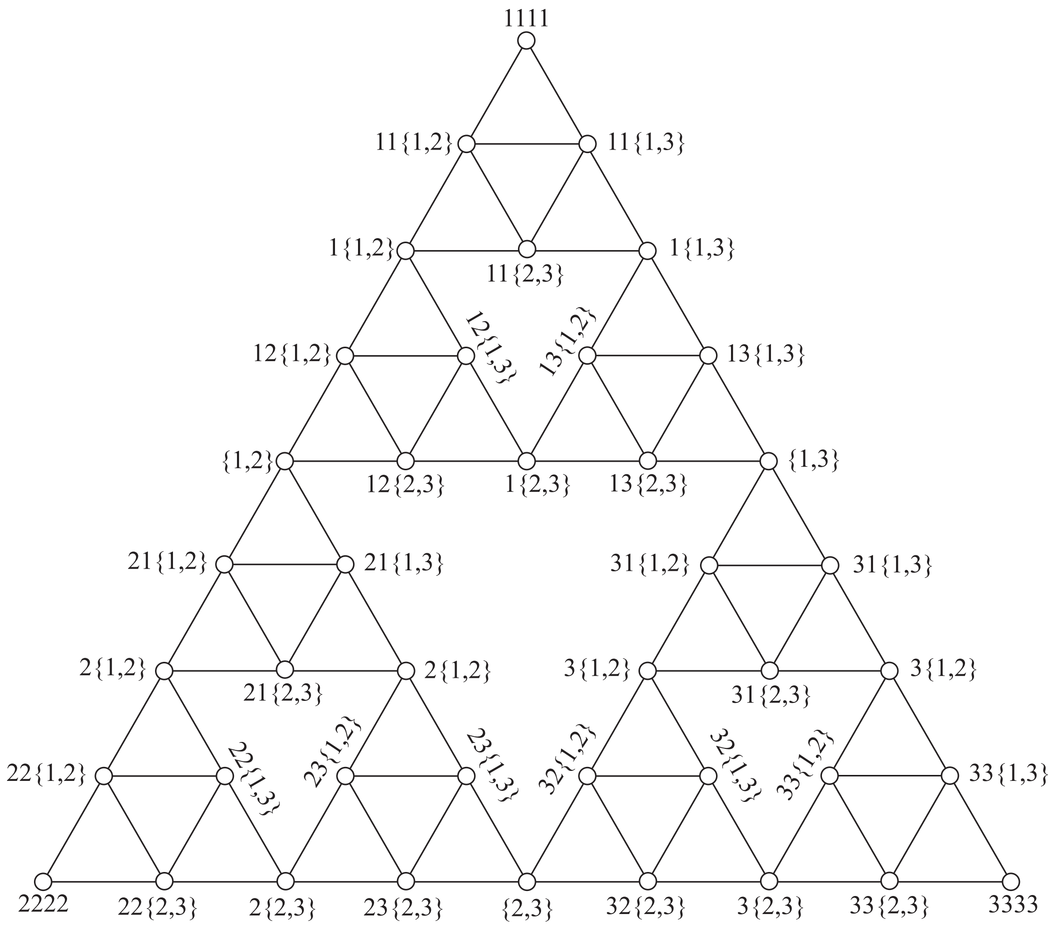

Figure 2: The Sierpiński gasket graph S_4

x of S_n. Then label x with $\langle u_1 \dots u_r \rangle \{i, j\}$, where $r \le n-2$. In this way S_n has three special vertices $\langle 1 \dots 1 \rangle$, $\langle 2 \dots 2 \rangle$, and $\langle 3 \dots 3 \rangle$, called *extreme vertices of* S_n, together with the vertices of the form

$$\langle u_1 \dots u_r \rangle \, \{i, j\} \, ,$$

where $0 \le r \le n-2$, and all the u_k's, i and j are from $\{1, 2, 3\}$. Note that S_{n+1} contains three isomorphic copies of S_n, a fact utmost useful for inductive arguments. We will denote these copies with $S_{n,i}$, $1 \le i \le 3$, where $S_{n,i}$ is the subgraph S_n of S_{n+1} containing $\langle i \dots i \rangle$.

The extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$ were introduced in [13] in the following way. The graph $S^+(n,k)$, $n \ge 1$, $k \ge 1$, is obtained from $S(n,k)$ by adding a new vertex w, called the *special vertex* of $S^+(n,k)$, and edges joining w with all extreme vertices of $S(n,k)$. These edges will be called the *additional edges* of $S^+(n,k)$. See Fig. 3 for $S^+(3,3)$.

The graphs $S^{++}(n,k)$, $n \geq 1$, $k \geq 1$, are defined as follows. For $n = 1$ we set $S^{++}(1,k) = K_{k+1}$. Suppose now that $n \geq 2$. Then $S^{++}(n,k)$ is the graph obtained from the disjoint union of $k+1$ copies of $S(n-1,k)$ in which the extreme vertices in distinct copies of $S(n-1,k)$ are connected as the complete graph K_{k+1}. With this the graphs $S^{++}(n,k)$ are well defined, see [13, Lemma 2.2]. See Fig. 3 for $S^{++}(3,3)$.

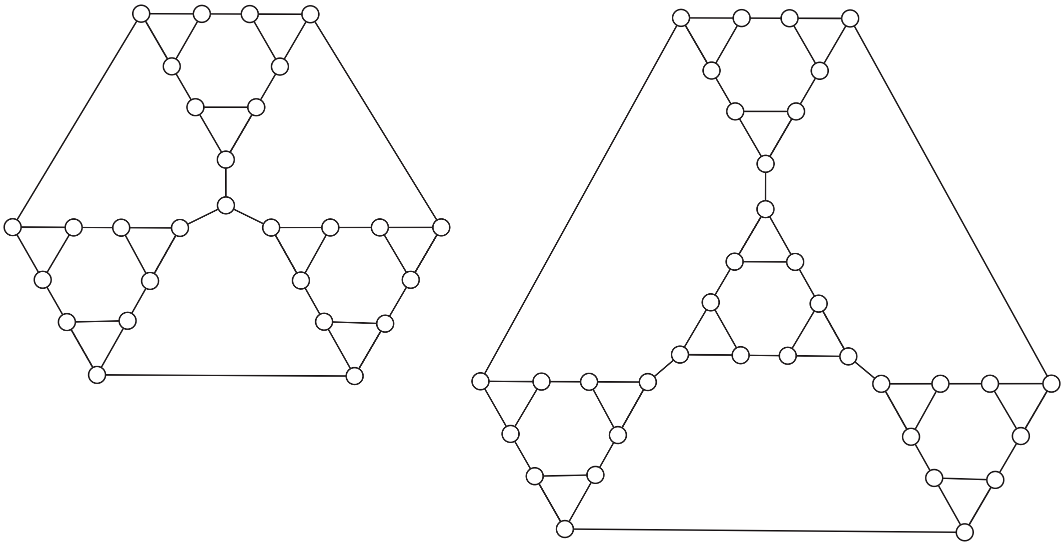

Figure 3: Graphs $S^+(3,3)$ and $S^{++}(3,3)$

3 Total colorings of S_n

In this section we answer a question from [22] with the following result.

Theorem 3.1 *For any $n \geq 2$, $\chi''(S_n) = 5$.*

Proof. As $\chi(S_n) \geq \Delta(S_n) + 1 = 5$, we only need to construct a total coloring with five colors. For a total coloring c of S_n we will use the following notation. Let $\{i,j,k\} = \{1,2,3\}$. Then if $c(\langle i \dots i \rangle) = x$, $c(\langle i \dots i \rangle \langle i \dots i\{i,j\} \rangle) = y$, and $c(\langle i \dots i \rangle \langle i \dots i\{i,k\} \rangle) = z$, we will write $C_i = \{x, \{y,z\}\}$.

First we construct a coloring of S_2 with $C_1 = \{3,\{1,2\}\}$, $C_2 = \{4,\{1,5\}\}$ and $C_3 = \{5,\{1,2\}\}$ as shown in Fig. 4. Then color S_3 as follows. Let c' be a coloring of $S_{3,1}$ such that $C_1' = \{3,\{1,2\}\}$, $C_2' = \{4,\{1,5\}\}$, and $C_3' = \{5,\{1,2\}\}$. Let c'' be a coloring of $S_{3,2}$ such that $C_1'' = \{4,\{2,3\}\}$, $C_2'' = \{1,\{3,5\}\}$, and $C_3'' = \{2,\{3,5\}\}$. Finally, let c''' be a coloring of $S_{3,3}$ with $C_1''' = \{5,\{3,4\}\}$, $C_2''' = \{2,\{1,4\}\}$, and $C_3''' = \{3,\{1,4\}\}$. Note that $c' = c$, and that c'' and c''' are obtained from c' by applying permutations $(1\,3)(2\,5)(4)$ and $(1\,4\,5\,3\,2)$, respectively. The coloring of S_3 is schematically shown on the right-hand side of Fig. 4.

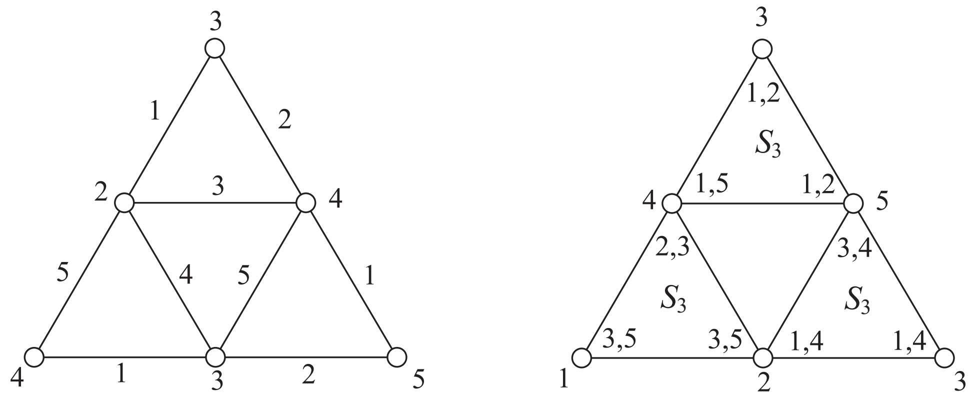

Figure 4: Colorings of S_2 and S_3

Let c be the constructed coloring of S_3, then $C_1 = \{3, \{1, 2\}\}$, $C_2 = \{1, \{3, 5\}\}$, and $C_3 = \{3, \{1, 4\}\}$. Next color S_4 as follows. Let $c' = c$ be a coloring of $S_{4,1}$, let c'' be a coloring of $S_{4,2}$ such that $C_1'' = \{3, \{4, 5\}\}$, $C_2'' = \{4, \{3, 1\}\}$, and $C_3'' = \{3, \{4, 2\}\}$, and let c''' be a coloring of $S_{4,3}$ with $C_1''' = \{3, \{2, 5\}\}$, $C_2''' = \{3, \{1, 5\}\}$, and $C_3''' = \{5, \{3, 4\}\}$. In this case, c'' and c''' are obtained from c' using permutations of colors $(1\,4\,2\,5)(3)$ and $(1\,5\,4)(2)(3)$, respectively. See the left coloring of Fig. 5.

For $n \geq 4$ we proceed by induction. Suppose that c is a total coloring of S_n with $C_1 = \{1, \{3, 5\}\}$, $C_2 = \{4, \{1, 3\}\}$, and $C_3 = \{5, \{3, 4\}\}$. Then let $c' = c$ be a coloring of $S_{n+1,1}$, let c'' be a coloring of $S_{n+1,2}$ with $C_1'' = \{4, \{2, 5\}\}$, $C_2'' = \{5, \{2, 3\}\}$, and $C_3'' = \{3, \{2, 4\}\}$. Finally, let c''' be a coloring of $S_{n+1,3}$ with $C_1''' = \{5, \{1, 2\}\}$, $C_2''' = \{3, \{1, 5\}\}$, and $C_3''' = \{2, \{1, 3\}\}$. Colorings c', c'', and c''' exist by induction. (Note that c'' and c''' are obtained from c' using permutations of colors $(1\,4\,3\,2)(5)$ and $(1\,3)(2\,4)(5)$, respectively.) See the right-hand side of Fig. 5.

Now, S_{n+1} is colored with a coloring c where $C_1 = \{1, \{3, 5\}\}$, $C_2 = \{5, \{2, 3\}\}$, and $C_3 = \{2, \{1, 3\}\}$. Finally, exchange the role of colors 2 and 4 in c and apply the induction. $\square$

4 Edge- and total-colorings of $S(n, k)$

In [10] it is shown that $S(n, 3)$ is uniquely 3-edge colorable. In this section we first extend this result by proving that for any k, $\chi'(S(n, k)) = k$.

Theorem 4.1 *For any $n \geq 2$ and any $k \geq 2$, $\chi'(S(n, k)) = k$.*

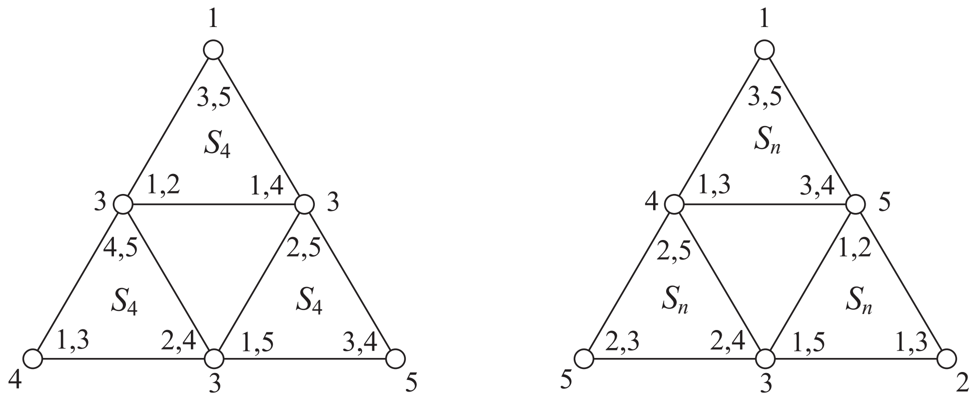

Figure 5: Colorings of S_4 and S_{n+1}

Proof. If k is even the conclusion is easy. Each subgraph K_k of $S(n,k)$ can be edge-colored with $k-1$ colors. Color the remaining edges, that is, the bridge edges of $S(n,k)$, with color k to obtain a desired coloring of $S(n,k)$.

Let now k be odd. For a vertex u of $S(n,k)$ and an edge-coloring c of it we will write C_u to denote the set of colors assigned to the edges incident with u. We will prove the following stronger claim.

Claim: For any $n \geq 1$ and any $k \geq 2$, $\chi'(S(n,k)) = k$. Moreover, for any $i,j \in \{1,\ldots,k\}$, $i \neq j$, $C_{i\ldots i} \neq C_{j\ldots j}$.

For $n = 1$, $S(1,k) = K_k$. It is well-known that K_k can be edge-colored with k colors such that $C_i \neq C_j$ for $i \neq j$. Assume the claim holds for $n \geq 1$. We wish to find an edge-coloring of $S(n+1,k)$. By the induction assumption, $S_h(n+1,k)$, $h \in \{1,\ldots,k\}$, can be colored with k colors where $C_{hi\ldots i} \neq C_{hj\ldots j}$, $i,j \in \{1,\ldots,k\}$, $i \neq j$. Let $\mathcal{M}$ be a mapping

$$\mathcal{M} : \{ij\ldots j \in V(S(n,k)) \mid i,j \in \{1,\ldots,k\}\} \to \{0,1,\ldots,k-1\}$$

defined as

$$\mathcal{M}(ij\ldots j) = i + j - 2 \pmod{k}.$$

Let $u = ij\ldots j$ and $v = il\ldots l$ be two different extreme vertices of $S_i(n,k)$. Then $\mathcal{M}(ij\ldots j) = i + j - 2 \pmod{k} \neq i + l - 2 \pmod{k} = \mathcal{M}(il\ldots l)$, because i is fixed and $j \neq l$. Since $S_i(n,k)$ is isomorphic to $S(n-1,k)$, by the induction assumption $\chi'(S_i(n,k)) = k$ and for any two different extreme vertices $ij\ldots j$ and $il\ldots l$, $C_{ij\ldots j} \neq C_{il\ldots l}$. The mapping $\mathcal{M}$ also assigns pairwise different numbers of the set $\{0,\ldots,k-1\}$ to the extreme vertices. Permute the colors of the proper edge coloring of the graph $S_i(n,k)$ in such a way that $C_{ij\ldots j} = \{0,\ldots,k-1\} \backslash \{\mathcal{M}(ij\ldots j)\}$. Consider the edges that connect subgraphs $S_i(n,k)$ and $S_j(n,k)$, for any $i,j \in \{1,\ldots,k\}$, $i \neq j$. Since $\mathcal{M}(ij\ldots j) = \mathcal{M}(ji\ldots i)$, the same color is missing at

$ij\dots j$ and $ji\dots i$. Hence the edge between these two vertices can be colored with $\mathcal{M}(ij\dots j)$ and we have constructed a proper k-edge-coloring of $S(n,k)$.

To complete the proof we need to prove that the extreme vertices receive pairwise different colors. For an extreme vertex $ii\dots i$ we have

$$\mathcal{M}(ii\dots i) = i + i - 2 \pmod{k} = 2(i-1) \pmod{k}\,.$$

Recall that k is odd. Hence, if $2(i-1) < k$, the extreme vertices receive pairwise different even numbers, while if $2(i-1) > k$, they receive pairwise different odd numbers. Finally, replace color 0 with k. □

In the rest of this section we consider total colorings of Sierpiński graphs. We first observe:

Proposition 4.2 *For any $n \geq 1$ and any $k \geq 1$, $\chi''(S(n,k)) \leq k+2$.*

Proof. Totally color every induced K_k of $S(n,k)$ with at most $k+1$ colors and color the remaining bridge edges with $k+2$. □

If k is odd, it is not difficult to give the exact value of the total chromatic number.

Proposition 4.3 *For any $n \geq 2$ and any odd $k \geq 3$, $\chi''(S(n,k)) = k+1$.*

Proof. Totally color every induced K_k of $S(n,k))$ with k colors and color the remaining edges with $k+1$. □

When k is even, the situation is more involved. Note first that $S(n,2)$ is the path on 2^n vertices, hence $\chi''(S(n,2)) = 3$. Next, for $k=4$ we have:

Proposition 4.4 *For any $n \geq 1$, $\chi''(S(n,4)) = 5$.*

Proof. Let $n \geq 2$ and let c be a total coloring of $S(n,4)$. For any $i,j \in \{1,2,3,4\}$ set $C_{ij\dots j} = (a,b)$, where $c(ij\dots j) = a$ and b is a color that is neither assigned to $ij\dots j$ nor any of its incident edges. Note that b is uniquely determined since $k=4$. For $n=1$ set $C_1 = (4,1)$, $C_2 = (1,2)$, $C_3 = (2,3)$, and $C_4 = (3,4)$, see Fig. 6.
The result will follow from the following stronger claim.

Claim: If n is odd, then we can color $S(n,4)$ such that $C_{i1\dots1} = (4,1)$, $C_{i2\dots2} = (1,2)$, $C_{i3\dots3} = (2,3)$, and $C_{i4\dots4} = (3,4)$. If n is even, we can color $S(n,4)$ such that $C_{i1\dots1} = (4,1)$, $C_{i2\dots2} = (4,3)$, $C_{i3\dots3} = (4,5)$, and $C_{i4\dots4} = (4,2)$.

Note that we can exchange the values of $C_{i2\dots2}$ and $C_{i4\dots4}$, if we mirror the coloring with respect to the diagonal between vertices $i1\dots1$ and $i4\dots4$. Let us

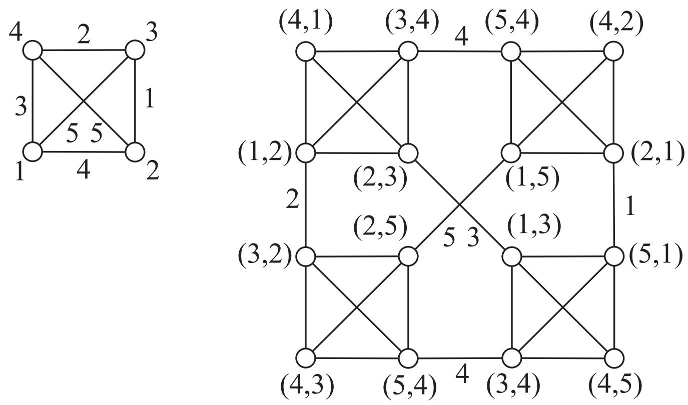

Figure 6: Cases $n = 1$ and $n = 2$

call the coloring from the claim the *standard coloring* and the derived one the *mirror coloring* of $S(n, 4)$.

For $n = 1$ and $n = 2$ the claim holds by Fig. 6.

Let $n \geq 2$ be even. We will construct four different total colorings c_i, $1 \leq i \leq 4$, of $S(n, 4)$ and combine them to a total coloring of $S(n + 1, 4)$.

Let c_1 be the standard coloring of $S(n, 4)$ such that $C_{11\ldots1} = (4, 1)$, $C_{12\ldots2} = (4, 3)$, $C_{13\ldots3} = (4, 5)$ and $C_{14\ldots4} = (4, 2)$. Let c_2 be the standard coloring such that $C_{21\ldots1} = (2, 3)$, $C_{22\ldots2} = (2, 4)$, $C_{23\ldots3} = (2, 1)$ and $C_{24\ldots4} = (2, 5)$. Note that c_2 is obtained from c_1 by the permutation $(1\,3\,4\,2\,5)$. Applying the permutation $(1\,5\,2\,4\,3)$ to c_1 we obtain the standard coloring c_3 for which $C_{31\ldots1} = (3, 5)$, $C_{32\ldots2} = (3, 1)$, $C_{33\ldots3} = (3, 2)$ and $C_{34\ldots4} = (3, 4)$. Finally, using $(1\,2\,3\,5\,4)$ the standard coloring c_4 is obtained for which $C_{41\ldots1} = (1, 2)$, $C_{42\ldots2} = (1, 5)$, $C_{43\ldots3} = (1, 4)$, and $C_{44\ldots4} = (1, 3)$. Now color $S(n + 1, 4)$ in such a way that $C_{11\ldots1} = (4, 1)$, $C_{22\ldots2} = (2, 4)$, $C_{33\ldots3} = (3, 2)$, and $C_{44\ldots4} = (1, 3)$.

Combine the colorings c_i to a coloring of $S(n+1, 4)$ as shown in the left-hand side of Fig. 7. ¿From this it is clear that the bridge edges of $S(n + 1, k)$ can be properly colored (with the missing colors between the corresponding vertices). To complete the even to odd case apply the mirror coloring to get $C_{11\ldots1} = (4, 1)$, $C_{22\ldots2} = (1, 3)$, $C_{33\ldots3} = (3, 2)$, and $C_{44\ldots4} = (2, 4)$. Finally, the exchange of colors 2 and 3 yields the desired coloring of $S(n + 1, 4)$, where $n + 1$ is odd.

Let $n \geq 3$ be odd. As in the previous case we first construct four different total colorings c_i, $1 \leq i \leq 4$, of $S(n, 4)$. Let c_1 be the standard coloring such that $C_{11\ldots1} = (4, 1)$, $C_{12\ldots2} = (1, 2)$, $C_{13\ldots3} = (2, 3)$, and $C_{14\ldots4} = (3, 4)$. Let c_2 be the mirror coloring of the coloring obtained from c_1 by permuting the colors as $(1\,2\,5)(3\,4)$. In this case, $C_{21\ldots1} = (3, 2)$, $C_{22\ldots2} = (4, 3)$, $C_{23\ldots3} = (5, 4)$, and $C_{24\ldots4} = (2, 5)$. Let c_3 be the standard coloring obtained from c_1 by means of $(1\,3\,5\,2\,4)$. Then $C_{31\ldots1} = (1, 3)$, $C_{32\ldots2} = (3, 4)$, $C_{33\ldots3} = (4, 5)$, and $C_{34\ldots4} = (5, 1)$.

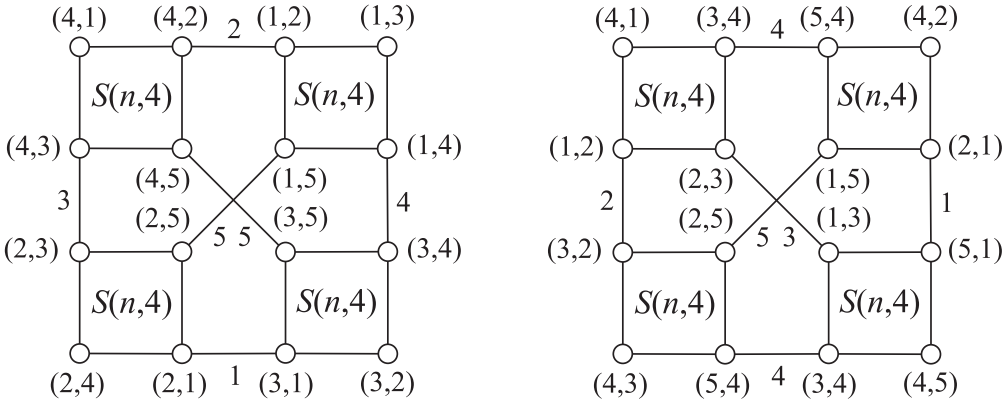

Figure 7: Even to odd and odd to even cases

The last coloring, c_4, is the mirror coloring of the coloring obtained from c_1 using $(1\,4\,5\,3)(2)$. Then $C_{41\ldots1} = (5,4)$, $C_{42\ldots2} = (1,5)$, $C_{43\ldots3} = (2,1)$ and $C_{44\ldots4} = (4,2)$. Now combine c_1, c_2, c_3, and c_4 into $S(n+1,4)$ as shown in the right-hand side of Fig. 7. Color every bridge edge with the missing color to obtain the desired total coloring. $\square$

For even $k \geq 6$ we were not able to decide whether $\chi''(S(n,k)) = k+1$ or $\chi''(S(n,k)) = k+2$. We do, however, suspect the following.

Conjecture 4.5 *For an even $k \geq 6$, $\chi''(S(n,k)) = k+2$.*

5 Colorings of $S^+(n,k)$ and $S^{++}(n,k)$

In this final section we consider the three types of colorings on the extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$.

We begin with the chromatic number for which the following natural coloring of $S(n,k)$ will be useful. Set $c(\langle u_1 \ldots u_n \rangle) = u_n$ for any vertex $\langle u_1 \ldots u_n \rangle$ of $S(n,k)$ to obtain a k-coloring of $S(n,k)$ [17]. We call this coloring the *canonical vertex-coloring* of $S(n,k)$.

Note that $S^+(n,2)$ is an odd cycle while $S^{++}(n,2)$ is an even cycle. For $k \geq 3$ we have:

Proposition 5.1 *For any $n \geq 2$ and any $k \geq 3$,*

$$\chi(S^+(n,k)) = \chi(S^{++}(n,k)) = k\,.$$

Proof. Let c be the canonical vertex-coloring of $S(n,k)$. Recall that $V(S^+(n,k)) = V(S(n,k)) \cup \{w\}$ and color the vertices of $S^+(n,k)$ as follows:

$$c'(u) = \begin{cases} 1; & u = k \dots kkk, \\ 2; & u = k \dots k1k; \\ k; & u \in \{w, k \dots kk1, k \dots k12\}; \\ c(u); & \text{otherwise}. \end{cases}$$

Since $k \geq 3$ it is straightforward to verify that c' is a proper coloring of $V(S^+(n,k))$.

Recall that $S^{++}(n,k)$ consists of $k+1$ copies of $S(n-1,k)$. Color $S(n,k)$ using the canonical vertex-coloring c. Let c'' be a coloring of the additional copy of $S(n-1,k)$ defined with

$$c''(u_1 \dots u_{n-1}) = \begin{cases} 1; & u_{n-1} = k, \\ u_{n-1} + 1; & \text{otherwise}. \end{cases}$$

Clearly, c'' is a proper k-coloring of $S(n-1,k)$. Since the corresponding extreme vertices of $S(n,k)$ and $S(n-1,k)$ are assigned different colors, c and c'' can be combined to a proper k-coloring of $S^{++}(n,k)$. $\square$

We continue with edge-colorings.

Proposition 5.2 *For any $n \geq 2$ and any $k \geq 2$,*

$$\chi'(S^+(n,k)) = \begin{cases} k; & k \text{ is odd}, \\ k+1; & k \text{ is even}. \end{cases}$$

Proof. Recall from the proof of Theorem 4.1 that when k is odd, there exists a k-edge-coloring of $S(n,k)$ such that $C_{ii\dots i} \neq C_{jj\dots j}$, $i,j \in \{1,\dots,k\}$, $i \neq j$, where $C_{ii\dots i}$ is the set of colors assigned to the edges incident to the vertex $ii\dots i$. Color each edge connecting the special vertex w with the vertex $i\dots i$, $i \in \{1,\dots,k\}$, with the color of the set $\{1,\dots,k\} \backslash C_{i\dots i}$. Hence $\chi'(S^+(n,k)) = k$ for odd k.

Let k be even. Graph $S^+(n,k)$ has k^n+1 vertices. Clearly, a given color can be used at most $(k^n+1)/2$ times. Since k is even, $(k^n+1)/2$ is not an integer, which implies that a given color can be used at most $k^n/2$ times. As $S^+(n,k)$ has $k(k^n+1)/2$ edges it follows that more than k colors are needed for a proper edge-coloring. By Vizing's theorem we conclude that $\chi'(S(n,k)) = k+1$. $\square$

Proposition 5.3 *For any $n \geq 2$ and any $k \geq 2$, $\chi'(S^{++}(n,k)) = k$.*

Proof. If k is even, color each induced K_k of $S^{++}(n,k)$ with $k-1$ colors and use color k on the remaining edges.

Recall again from the proof of Theorem 4.1 that for k odd, there exists a k-edge-coloring of $S(n,k)$ such that $C_{ii\ldots i} \neq C_{jj\ldots j}$, $i,j \in \{1,\ldots,k\}$, $i \neq j$. Apply the same theorem to color $S(n-1,k)$. Using the theorem twice, the corresponding extreme vertices miss the same color. Color the edges connecting $S(n,k)$ and $S(n-1,k)$ with the missing color to acquire a proper edge-coloring of $S^{++}(n,k)$. $\square$

It remains to consider the total-colorings.

Proposition 5.4 *For any $n \geq 2$ and $k \geq 2$, $\chi''(S^{+}(n,k)) \leq k+2$.*

Proof. Let k be odd. First totally color each induced K_k of $S^{+}(n,k)$ with k colors such that each extreme vertex receives a different color. Color the bridge edges with $k+1$. Next color the additional edges in $S^{+}(n,k)$ with the color of the extreme vertex to which the additional edge is adjacent and replace the extreme vertex's color with the color $k+1$. Finally, color the special vertex of $S^{+}(n,k)$ with $k+2$.

When k is even, we can totally color each complete graph in $S(n,k)$ with $k+1$ colors in such a way that $C_{ii\ldots i} \neq C_{jj\ldots j}$, $i,j \in \{1,\ldots,k\}$, $i \neq j$. Color the additional edges incident with $i\ldots i$, $i \in \{1,\ldots,k\}$, with this missing color. Finally, color the bridge edges and the remaining special vertex with $k+2$. $\square$

Proposition 5.5 *For any $n \geq 2$ and $k \geq 2$, $\chi''(S^{++}(n,k)) \leq k+2$.*

Proof. Totally color each complete subgraph K_k of $S^{++}(n,k)$ with at most $k+1$ colors and use color $k+2$ on the remaining edges. $\square$

Proposition 5.6 *For any $n \geq 2$ and any odd $k \geq 3$, $\chi''(S^{++}(n,k)) = k+1$.*

Proof. Totally color complete subgraphs K_k of $S^{++}(n,k)$ with k colors and color the bridge edges and the additional edges with $k+1$. $\square$

6 Concluding remarks

Theorem 4.1 has been independently obtained by Hinz and Parisse [6]. In the same paper they also determine the edge-chromatic number of the general Tower of Hanoi graphs, that is, the graphs of the Tower of Hanoi puzzle where more than 3 pegs are allowed. Surprisingly, it turned out that the difficult case to treat was when there are fewer discs than pegs in the corresponding Tower of Hanoi problem.

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