On distance-balanced graphs

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Abstract

It is shown that the graphs for which the Szeged index equals $\frac{\|G\|\cdot|G|^2}{4}$ are precisely connected, bipartite, distance-balanced graphs. This enables to disprove a conjecture proposed in [Some new results on distance-based graph invariants, European J. Combin. 30 (2009) 1149–1163]. Infinite families of counterexamples are based on the Handa graph, the Folkman graph, and the Cartesian product of graph. Infinite families of distance balanced, non-regular graphs that are prime with respect to the Cartesian product are also constructed.

Key words: Szeged index; distance-balanced graphs; Folkman graph; Handa graph; graph products

1 Introduction

For an edge uv of a graph G let W_{uv} be the set of vertices closer to u than to v, that is

$$W_{uv} = \{w \in G \mid d(w,u) < d(w,v)\}.$$

Note that in bipartite graphs W_{uv} and W_{vu} form a partition of the vertex set of G for any edge uv. These sets play a prominent role in metric graph theory, see for instance [2]. More information on these sets and references can be found in [6].

The sets W_{uv} also appear in chemical graph theory as the key constituent of an important invariant, called the *Szeged index* of graph G (see [4] and [8] and references therein), defined by Gutman [7] as

$$Sz(G) = \sum_{uv \in E(G)} |W_{uv}| \cdot |W_{vu}|.$$

Denoting the number of vertices and edges of a graph G with $|G|$ and $\|G\|$, respectively, the following conjecture was proposed in [9]:

Conjecture 1.1 *For a connected graph G,*

$$Sz(G) = \frac{\|G\| \cdot |G|^2}{4}$$

if and only if G is bipartite and regular.

In the next section we show that the conditions are not necessary by constructing an infinite family of non-regular bipartite graphs for which the Szeged index is largest possible. The key observation for this construction is that bipartite, Szeged extremal graphs can be characterized as distance-balanced graphs. Moreover, we show that the conditions of the conjecture are also not sufficient by giving bipartite regular graphs that are not extremal with respect to the Szeged index. In the final section we propose constructions to obtain additional non-regular distance-balanced graphs.

2 Bipartite distance-balanced graphs

Let G be an arbitrary graph. From obvious inequality $2 \leq |W_{uv}| + |W_{vu}| \leq |G|$, using the arithmetic–geometric mean inequality we have

$$|W_{uv}| \cdot |W_{vu}| \leq \frac{|G|^2}{4}.$$

After summing for all edges uv from G, we derive that

$$Sz(G) \leq \frac{\|G\| \cdot |G|^2}{4}.$$

Conjecture 1.1 is thus asking for a characterization of graphs extremal with respect to the Szeged index. As noted in [9], the necessary conditions for achieving the equality are: G is bipartite, $|G|$ is even and G has no pendent vertices.

A graph G is *distance-balanced* if $|W_{uv}| = |W_{vu}|$ holds for any edge uv of G. These graphs were first studied by Handa [3] within the class of partial cubes and later for all graphs and named distance-balanced in [6]. An interesting observation from [1] asserts that they can be characterized as the graphs whose median sets are whole vertex sets. Our main observation is that in the bipartite case they characterize the extremal graphs with respect to the Szeged index.

Proposition 2.1 *A connected bipartite graph G is distance-balanced if and only if* $Sz(G) = \frac{\|G\|\cdot|G|^2}{4}$.

Proof. We have already observed that for any graph G, $Sz(G) \leq \frac{\|G\|\cdot|G|^2}{4}$. Suppose $Sz(G) = \frac{\|G\|\cdot|G|^2}{4}$ holds. Then

$$|W_{uv}| = |W_{vu}| = \frac{|G|}{2},$$

holds for all edges $uv \in E(G)$ and consecutively G is distance-balanced.

Conversely, suppose G is distance-balanced. Then for any edge uv, $|W_{uv}| = |W_{vu}|$ and since G is bipartite also $|W_{uv}|+|W_{vu}| = |G|$ holds. Therefore, $Sz(G) = \frac{\|G\|\cdot|G|^2}{4}$. ■

Let H be the *Handa graph*, see Fig. 2, that was constructed in [3] as an example of a bipartite distance-balanced graph (which is not “even”).

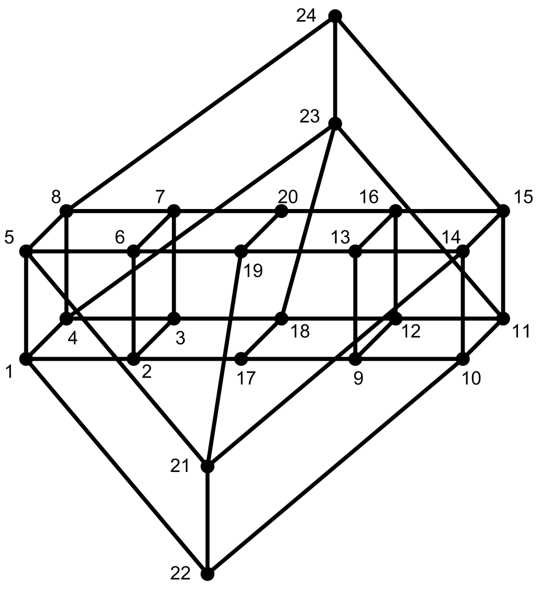

Figure 1: Handa graph

For $n \geq 1$ let H^n denote the Cartesian power of H, that is, the Cartesian product $H \,\square\, H \,\square\, \cdots \,\square\, H$ of n copies of H.

Theorem 2.2 *For any $n \geq 1$, H^n is a bipartite, non-regular, distance-balanced graph.*

Proof. We have already noted that H is bipartite and distance-balanced. Moreover, it is also not regular. For $n \geq 2$ the assertion follows because:

- $G \,\square\, G'$ is regular if and only if G and G' are regular;
- $G \,\square\, G'$ is bipartite if and only if G and G' are bipartite; and
- $G \,\square\, G'$ is distance-balanced if and only if G and G' are distance-balanced.

The first two assertions are well-known easy facts (cf. [5]), the last one is proved in [6]. ■

Thus a bipartite, distance-balanced graph need not be regular. On the other hand, Kutnar et al. [10] constructed an infinite family of semisymmetric graphs which are not distance-balanced. (A regular graph is called *semisymmetric* if it is edge-transitive but not vertex-transitive [11].) The first graph of their sequence is the well-known Folkman graph. Hence their construction gives an infinite family of regular bipartite graphs that are not distance-balanced. Thus the conditions of Conjecture 1.1 are not sufficient.

A simpler construction of such an infinite family can be done using the Cartesian product again. For any $n \geq 1$, F^n is regular bipartite graph that is not distance-balanced.

It would be interesting to have some other structural characterization of bipartite distance-balanced graphs. As already mentioned, they are of even order and have no pendant vertices. But more is true. Handa [3] proved that they are 2-connected and with the exception of even cycles have minimum degree at least 3. It is on open problem whether they (except even cycles) are 3-connected.

3 More non-regular distance-balanced graphs

To find more distance-balanced graphs that are not regular, we checked all graphs with ≤ 10 vertices with the help of Nauty [12]. The only graph that is distance-balanced and non-regular is depicted on Fig. 3 and has nine vertices. Note that it is not bipartite.

Based on this example, we can construct an infinite family of distance-balanced graphs that are non-regular and prime with respect to the Cartesian product. Let $k \geq 1$, then for $n = 6k+3$ consider the cycle C_{4k+2} with vertex labels $1, 2, \ldots, 4k+2$ and add additional vertices labeled $4k+3, 4k+4, \ldots, 6k+3$, such that vertex $4k+3$ is adjacent to $1, 2, 3$, vertex $4k+4$ is adjacent to $3, 4, 5$, ..., and finally vertex $6k+3$ is adjacent to $4k+1, 4k+2, 1$. The constructed graph, say G_k, is obviously non-regular. Because of the symmetry, we have to consider three types of edges:

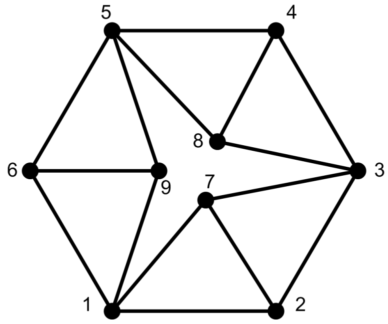

Figure 2: Non-regular distance-balanced graph

- the edges of the type $(i, i+1)$, $i = 1, 2, \ldots, 4k+2$;
- the edges of the type $(2i, 4k+2+i)$, $i = 1, 2, \ldots, 2k+1$;
- the edges of the type $(2i \pm 1, 4k+2+i)$, $i = 1, 2, \ldots, 2k+1$.

In all the cases it holds $|W_{uv}| = |W_{vu}| = 3k+1$, hence G_k is distance-balanced for any $k \geq 1$.

In [6] it is proved that the lexicographic product of two graphs is distance-balanced if and only if the first factor is distance-balanced and the second factor is regular. This gives another construction to get many additional (non-bipartite) non-regular distance-balanced graphs.

As noted in the previous section, there exist semisymmetric graphs that are not distance-balanced. On the other hand, the following holds for all edge-transitive graphs.

Proposition 3.1 *Let G be an edge-transitive graph. Then for any edge uv of G, the absolute value of $|W_{uv}| - |W_{vu}|$ is a constant (independent of uv).*

Proof. Let $e_1 = uv$ be an arbitrary edge of G and $c = \big|\,|W_{uv}| - |W_{vu}|\,\big|$. For an arbitrary edge $e_1' = u'v'$ there exists an automorphism γ of the line graph $L(G)$, such that $e_1' = \gamma(e_1)$. It follows that either u maps to u' and v maps to v', or u maps to v' and v maps to u'. In both cases, the automorphism preserve distances and the vertices that are closer to u are mapped to vertices closer to u' or v'. This means that $\big|\,|W_{u'v'}| - |W_{v'u'}|\,\big| = c$. ■

For instance, for the Folkman graph it holds $\big|\,|W_{uv}| - |W_{vu}|\,\big| = 6$ for any edge $uv \in E(G)$; for the Gray graph it holds $\big|\,|W_{uv}| - |W_{vu}|\,\big| = 8$ for any edge $uv \in E(G)$; and $\big|\,|W_{uv}| - |W_{vu}|\,\big| = |n - m|$ for the complete bipartite graph $K_{n,m}$.

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