Maximal proper subgraphs of median graphs

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Abstract

For a median graph G and a vertex v of G that is not a cut-vertex we show that $G - v$ is a median graph precisely when v is not the center of a bipartite wheel, which is in turn equivalent with the existence of a certain edge elimination scheme for edges incident with v. This implies a characterization of vertex-critical (respectively, vertex-complete) median graphs, which are median graphs whose all vertex-deleted subgraphs are not median (respectively, are median). Moreover, two analogous characterizations for edge-deleted median graphs are given.
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1. Introduction and preliminaries

One of the frequent questions in graph theory is whether the deletion of a vertex or an edge preserves a certain property of a graph. In the language of graph classes one wants to know whether vertex- or edge-deleted subgraphs remain in the class or not. A well-known example where this is used are the vertex elimination schemes for chordal graphs and related classes, cf. [5]. In the famous Kelly–Ulam's reconstruction conjecture we are given the set of all vertex-deleted subgraphs, and the task is to reconstruct the original graph, cf. [4]. Results considering the conjecture for particular classes of graphs sometimes use the information which vertex-deleted subgraphs remain in the class. For instance, only vertex-deleted subgraphs of a tree which are also trees suffice to reconstruct the original tree [6].

Median graphs form a natural common generalization of trees and hypercubes [12,13]. They have been extensively studied, see the recent survey [10]. One of the motivations for the present work is due to an attempt to prove the reconstruction conjecture for the case of median graphs. Such a study might also lead to an insight on the reconstruction conjecture for the case of triangle-free graphs via a (surprising) connection between median graphs and triangle-free graphs [8]. Since median graphs are similar to trees (and include trees as well), it would be helpful to know which

vertex-deleted graphs are median. Nevertheless, we believe that the present investigation of vertex-deleted and edge-deleted median graphs could be of independent interest, and could find some other applications too.

In the next section we prove our main theorem. That is, given a median graph G, and a vertex v of G that is not a cut-vertex, two equivalent conditions to the condition that $G - v$ is a median graph are proved. Then, in the third section, we investigate four classes of median graphs with respect to the set of their maximal proper subgraphs. There are two natural types of such sets, the set of all vertex-deleted subgraphs and the set of all edge-deleted subgraphs of a given graph. These sets are sometimes called *vertex-deck* and *edge-deck* of a graph. We study two extreme properties of such decks, that is, either all graphs from the deck are median graphs, or all graphs from the deck are not median graphs. More precisely, we characterize the so-called edge-critical, vertex-critical, edge-complete, and vertex-complete median graphs. We conclude with some remarks concerning the algorithmic aspect of recognition of these classes.

For a graph G, the *distance $d_G(u, v)$* between vertices u and v is defined as the number of edges on a shortest u, v-path. The *interval $I(u, v)$* between vertices u and v consists of all vertices on shortest paths between u and v. A graph G is called a *median graph* if for every triple of vertices u, v and w of G we have $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$. The unique vertex of this intersection is called the *median* of the triple u, v, w. For instance, in the case of $K_{2,3}$, if u, v, w are vertices of degree 2, then $|I(u, v) \cap I(u, w) \cap I(v, w)| = 2$, hence any graph that contains (induced) $K_{2,3}$ is not median. It is also clear that median graphs are bipartite. The *k-dimensional hypercube* or the *k-cube Q_k* is the graph on 2^k vertices, which can be labeled by 0–1 words of length k, two vertices being adjacent if the corresponding labels differ in precisely one position. For instance, Q_1 is K_2, Q_2 is the square C_4, and Q_3 is the cube. Median graphs can be characterized as retracts of hypercubes [2], hence hypercubes are obviously median graphs.

A subgraph H of G is *convex*, if for any $u, v \in V(H)$, $I(u, v) \subseteq V(H)$. Since G is convex in G, and convex subgraphs are closed for intersections, we may speak of the *convex closure* of a subgraph $H \subset G$, defined as the smallest convex subgraph of G containing H. A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. The following useful characterization of median graphs is due to Bandelt [1].

Theorem 1.1. *A connected graph is median if and only if the convex closure of any isometric cycle is a hypercube.*

An edge of a graph G is a *square-edge* if it lies in exactly one 4-cycle of G. We will use the following results from [11], where Q_3^- is the graph obtained from the 3-cube Q_3 by removing a vertex.

Proposition 1.2. *Let e be an edge of a graph G which lies in at least two 4-cycles. Then not both G and $G - e$ are median graphs.*

Proposition 1.3. *Let e be a square-edge of a graph G.*

 (i) *If G is a median graph then so is $G - e$.*
(ii) *Suppose that $G - e$ is a median graph. Then G is a median graph if and only if e does not lie in some Q_3^-.*

In addition, we will also apply the following result of Soltan and Chepoi [15, Theorem 4.2.(6)], later independently obtained also by Škrekovski in [14].

Theorem 1.4. *Let $\alpha_i(G)$ denotes the number of induced i-cubes of a median graph G. Then*

$$\sum_{i \geqslant 0} (-1)^i \alpha_i(G) = 1.$$

2. Vertex-deleted median graphs

We need some more definitions before stating the main result. Let G be a median graph and let v be a vertex of G of degree k. Then v is called *square-dismantlable*, if the incident edges of v can be ordered as $e_1, e_2, \ldots, e_k$, such that e_1 is a square-edge of G and for $i = 2, \ldots, k - 1$ the edge e_i is a square-edge of $G - \{e_1, e_2, \ldots, e_{i-1}\}$. We will also say that the ordering of incident edges with v is square-dismantlable.

The *bipartite wheel BW_k, $k \geqslant 3$*, is the graph formed by the cycle C_{2k} and a vertex v adjacent to every second vertex of the cycle. The vertex v is called the *center* of BW_k. Note that BW_3 is isomorphic to the vertex deleted 3-cube Q_3^-.

Theorem 2.1. *Let* $G = (V, E)$ *be a median graph and* $v \in V$ *not a cut-vertex. Then the following statements are equivalent*:

(i) $G - v$ *is a median graph*;
(ii) v *is not the center of a bipartite wheel*;
(iii) v *is a square-dismantlable vertex.*

Proof. (i) $\Rightarrow$ (ii). Suppose that v is the center of the bipartite wheel $BW_k, k \geqslant 3$, which we call B. If v lies in an induced Q_3, then $G' = G - v$ is clearly not a median graph. So assume that v is not in an induced Q_3 (and hence also in no induced $Q_n, n \geqslant 4$). Therefore, recalling that $\alpha_i(G)$ denotes the number of induced i-cubes of G, we have

$$\alpha_0(G') = \alpha_0(G) - 1 \quad \text{and} \quad \alpha_i(G') = \alpha_i(G) \text{ for } i \geqslant 3.$$

Let j be the number of edges incident with v which are not edges of B. Suppose e_1 is such an edge. Since v is not a cut-vertex, there must be a cycle that contains e_1 and some edge e of B. Clearly there is also an isometric cycle that contains both of these edges, and since G is median, the convex closure of this cycle is a hypercube. It follows that e and e_1 lie in a 4-cycle. Suppose e_2 is another edge incident with v, and not in B. Since v is not a cut-vertex we again deduce that e_2 is in a 4-cycle with an edge of B or with e_1. We repeat this argument for all edges $e_3, \ldots, e_j$, and finally infer that there exist at least j squares which are incident with v and are not in B. Hence,

$$\alpha_1(G') = \alpha_1(G) - k - j, \quad \alpha_2(G') \leqslant \alpha_2(G) - k - j.$$

By Theorem 1.4, $\sum_{i \geqslant 0}(-1)^i \alpha_i(G) = 1$. Therefore,

$$\sum_{i \geqslant 0}(-1)^i \alpha_i(G') \leqslant (\alpha_0(G) - 1) - (\alpha_1(G) - k - j) + (\alpha_2(G) - k - j)$$
$$+ \sum_{i \geqslant 3}(-1)^i \alpha_i(G)$$
$$= \sum_{i \geqslant 0}(-1)^i \alpha_i(G) - 1$$
$$= 1 - 1 = 0.$$

Hence G' is not a median graph by Theorem 1.4.

(ii) $\Rightarrow$ (iii). Let $\{e_1, e_2, \ldots, e_k\}$ be the edges incident with v and suppose that no ordering of these edges is square-dismantlable. By Proposition 1.3(i) we may assume that none of the edges e_i, $1 \leqslant i \leqslant k$, is a square-edge. Since v is not a cut-vertex and G is $K_{2,3}$-free, we infer that $k \geqslant 3$ and every edge e_i is contained in at least two squares. For $i = 1, 2, \ldots, k$, let v_i be the endvertex of e_i different from v.

Consider an arbitrary square containing e_1. We may assume that it is of the form $S_1 = vv_1w_{12}v_2v$. As G is bipartite, $w_{12} \neq v_i$ for all i. Consider next $e_2 = vv_2$. It lies in at least one more square, say S_2. Note that S_2 cannot be of the form vv_2uv_1v, for otherwise G would contain a $K_{2,3}$. Therefore, we may assume without loss of generality that $S_2 = vv_2w_{23}v_3v$, where $w_{23} \neq v_i$ for all i. Moreover, $w_{23} \neq w_{12}$ for otherwise we have a $K_{2,3}$. Continue with the edge $e_3 = uv_3$ that lies in at least one more square besides S_2, say in S_3. Note that $S_3 = vv_3w_{12}v_1v$ yields a $K_{2,3}$ as well as $S_3 = vv_3w_{12}v_2v$. In addition, if $S_3 = vv_3uv_1v$, $u \neq w_{12}, w_{23}$, we have a bipartite wheel BW_3 and we are done. Therefore assume that $S_3 = vv_3v_{34}v_4v$. If we continue the above procedure we either get a bipartite wheel at some step or finally obtain the square $S_k = vv_kw_{k1}v_1v$, so that the squares $S_1, S_2, \ldots, S_k$ induce a BW_k. This contradiction completes the argument.

(iii) $\Rightarrow$ (i). Let $e_1, e_2, \ldots, e_k$ be a square-dismantlable ordering of the incident edges of v. Then $G_{k-1} = G - \{e_1, e_2, \ldots, e_{k-1}\}$ is a median graph by repetitive applications of Proposition 1.3(i). Finally, v is a pendant vertex of G_{k-1} and hence $G - v = G_{k-1} - v$ is a median graph. $\square$

We note that bipartite wheels play an important role in the theory of median graphs. For instance, Bandelt and Chepoi [3] proved that median graphs without convex bipartite wheels are precisely the graphs of acyclic cubical complexes.
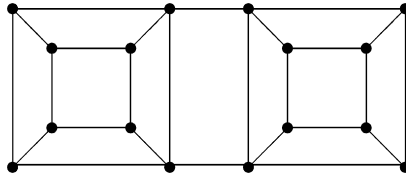
Fig. 1. A vertex-critical but not edge-critical median graph.

3. Vertex-, edge-, critical, complete median graphs

Let us call a median graph G *edge-critical* if for every edge e of G, $G - e$ is not a median graph, and *vertex-critical* if for every vertex v of G, $G - v$ is not a median graph. For instance, k-cubes with $k \geqslant 3$ are both edge-critical and vertex-critical. Similarly, we shall say that a median graph G is *edge-complete* if for every edge e of G, $G - e$ is a median graph, and *vertex-complete* if for every vertex v of G, $G - v$ is a median graph. For instance, the 4-cycle is both edge-complete and vertex-complete.

In the study of edge- and vertex-critical median graphs we can restrict ourselves to the 2-connected case. Using known results, edge-critical median graphs can be described as follows.

Corollary 3.1. *Let G be a 2-connected median graph. Then G is edge-critical if and only if every edge of G lies in at least two squares.*

Proof. Note that every edge of G lies in at least one square since G is 2-connected. Then the "if" part follows by Proposition 1.3(i), while the "only if" direction is implied by Proposition 1.2. □

For the following characterization of vertex-critical median graphs we need the main theorem of this paper.

Proposition 3.2. *Let G be a 2-connected median graph. Then G is vertex-critical if and only if any vertex of G that is incident with a square-edge is the center of a bipartite wheel.*

Proof. If G is vertex-critical, any vertex of G is the center of a bipartite wheel by Theorem 2.1.

Conversely, let v be an arbitrary vertex of G. If v is incident with a square-edge we are done. Otherwise there exists an edge vw that is contained in at least two squares, say $vwxyv$ and $vwx'y'v$. As G is $K_{2,3}$-free, the vertices x, y, x', y' are pairwise different. Suppose that $G - v$ is a median graph. Then the triple w, y, y' has a unique median z that is on distance one to the three vertices. Clearly, $z \neq v$. But then G contains a $K_{2,3}$ which is not possible since G is median. It follows that also in this case $G - v$ is not median. □

Combining Proposition 3.2 with Corollary 3.1 we get:

Corollary 3.3. *Let G be a 2-connected, edge-critical median graph. Then G is vertex-critical.*

The graph from Fig. 1 shows that the converse of Corollary 3.3 is not true in general.

Descriptions of vertex-complete and edge-complete median graphs are also not difficult. Note that in both cases the only maximal hypercubes of such graphs are 4-cycles (with one exception in the case of vertex-complete median graphs). Indeed, such graphs should obviously not possess a 3-cube as a subgraph. Furthermore, if they had an edge which is not in a 4-cycle, then this edge would be a bridge, and at least one of its endvertices would be a cut-vertex (except in the case of K_2), which makes such graphs not (edge- or vertex-) complete.

Proposition 3.4. *A median graph G is edge-complete if and only if every block* (*maximal 2-connected subgraph*) *of G is C_4.*

Proof. Let G be an edge-complete median graph. By Proposition 1.2 each edge of such graph must lie in only one 4-cycle. By Theorem 1.1 every edge of G which lies in a cycle, lies also in a hypercube, hence every block must either be a K_2 or a square. If it would be K_2, its edge would be a bridge, hence every block is a square.

For the converse assume that G is a graph of which every block is a square. It is clearly a median graph. Let $S = uvwzu$ be a square of G, and let us prove that $G - uv$ is median. Note that medians of the triples of $G - uv$ could be changed from those of G only in the case when the triples u', v', z' are of the following form: u' has u as the closest vertex from S, v' has v as the closest vertex from S, and z' has either w or z as the closest vertex of S (without loss of generality assume that z is the closest vertex to z' among the vertices of S). Since z is a cut-vertex, it is obvious that z is the unique median of u', v', z'. Hence $G - uv$ is also a median graph. $\quad\square$

The case of vertex-complete median graphs is somehow similar. First exclude the trivial case of K_2. As every vertex-complete median graph must be 2-connected, each 4-cycle must have an edge in common with some other 4-cycle, and by Theorem 2.1(ii) there should be no bipartite wheels in such graphs. Hence vertex-complete median graphs can be described as graphs that can be obtained from C_4 by successive attachments of 4-cycles along edges. To see that these graphs are indeed vertex-complete median graphs is again an easy exercise using induction on the number of 4-cycles. We leave the details to the reader.

Proposition 3.5. *A median graph G is vertex-complete if and only if G is either isomorphic to K_2 or G is one of the graphs defined by*:

(A) *C_4 is a vertex-complete median graph, and*
(B) *a graph obtained from a vertex-complete median graph G by identifying an edge of a 4-cycle with an arbitrary edge of G, is also vertex-complete.*

Median graphs admit isometric embeddings into hypercubes [12]. In other words, median graphs are partial cubes. Edge-critical partial cubes are studied in [9] but it seems that the problem of their characterization is much more involved than the corresponding problem for median graphs. The same also holds for vertex-critical partial cubes.

4. Concluding remarks

Let G be a median graph on n vertices with m edges. Then G contains at most $O(m \log n)$ squares that can be computed within the same time [7, Corollary 7.7]. Hence, using Corollary 3.1, the same time also suffices to determine whether a 2-connected median graph is edge-critical. To check whether G is vertex-critical we proceed as follows. We first determine all the squares and with every vertex v we remember the squares to which v belongs. Then we define the graph G_v as the graph whose vertices are the squares containing v, two vertices being adjacent if the corresponding squares share an edge. Then v is the center of a bipartite wheel if and only if G_v contains a cycle. If we repeat this check for every vertex of G, the time needed is proportional to the number of pairs of squares of G that is $O(m^2 \log^2 n) = O(n^2 \log^4 n)$. By Theorem 2.1 this complexity suffices to recognize vertex-critical median graphs. Note that this is slightly better than checking for any vertex v whether $G - v$ is median. Indeed, since the best known complexity for recognizing median graphs is $O((m \log n)^{1.41})$, the direct approach for recognizing vertex-critical median graphs would be of complexity $O(n(m \log n)^{1.41}) = O(n^{2.41} \log^{2.82} n)$. Nevertheless, we pose the question whether the complexity for recognizing vertex-critical graphs can be improved, for instance by a more careful analysis of the complexity of the above approach.

We conclude the paper with the following question: is there a natural, previously studied graph class for which all the four types of maximal proper subgraphs from Section 3 can be characterized nicely and nontrivially? For instance, in the case of planar graphs, we have the situation that all graphs are edge-complete and vertex-complete. The answer is also quite trivial for chordal graphs. All 2-connected chordal graphs are vertex-complete, and they are never edge-critical. A chordal graph is edge-complete precisely when every block is a complete graph.

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