ELSEVIER

Discrete Mathematics 247 (2002) 235–242

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

# On the fractional chromatic number, the chromatic number, and graph products

Sandi Klavžar[a, *, 1], Hong-Gwa Yeh[b, 2]

[a] *Department of Mathematics, PEF, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia*

[b] *Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan*

Received 16 March 2000; revised 5 December 2000; accepted 2 April 2001

**Abstract**

It is shown that the difference between the chromatic number $\chi$ and the fractional chromatic number $\chi_f$ can be arbitrarily large in the class of uniquely colorable, vertex transitive graphs. For the lexicographic product $G \circ H$ it is shown that $\chi(G \circ H) \geqslant \chi_f(G)\,\chi(H)$. This bound has several consequences, in particular, it unifies and extends several known lower bounds. Lower bounds of Stahl (for general graphs) and of Bollobás and Thomason (for uniquely colorable graphs) are also proved in a simple, elementary way. 

*Keywords:* Chromatic number; Fractional chromatic number; Graph product; Uniquely colorable graph

## 1. Introduction

A *graph product* is defined on the Cartesian product of the vertex sets of the factors, while its edges are determined by the edge sets of the factors. There are 256 such products. The four most important of them — the Cartesian, the direct, the strong, and the lexicographic one — are called the *standard graph products* and are defined below. For algebraic and other reasons for the selection of these four products to be the standard ones, see the book [10].

For graphs $G$ and $H$, let $G \Box H$, $G \times H$, $G \boxtimes H$ and $G \circ H$ be the *Cartesian*, the *direct*, the *strong*, and the *lexicographic product* of $G$ and $H$, respectively. The vertex

* Corresponding author.

*E-mail addresses:* sandi.klavzar@uni-lj.si (S. Klavžar), hgyeh@nuk.edu.tw (H.-G. Yeh).

[1] Supported by the Ministry of Science and Technology of Slovenia under the Grant 101-504.

[2] Supported in part by the National Science Council under Grant NSC 89-2115-M008-008.



PII: S0012-365X(01)00312-0

set of any of these products is $V(G) \times V(H)$. Vertices $(a,x)$ and $(b,y)$ are adjacent in $G \square H$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$; are adjacent in $G \times H$ whenever $ab \in E(G)$ and $xy \in E(H)$; are adjacent in $G \boxtimes H$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $xy \in E(H)$; are adjacent in $G \circ H$ whenever $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. Another graph product that will be mentioned is the *disjunctive* product $G \vee H$ (called the *inclusive* product in [4,15] and the *Cartesian sum* in [3]) in which vertices $(a,x)$ and $(b,y)$ are adjacent whenever $ab \in E(G)$ or $xy \in E(H)$.

For the Cartesian product Sabidussi [18] showed that for any graphs $G$ and $H$, $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. For the direct product it is easy to see that $\chi(G \times H) \leqslant \min\{\chi(G), \chi(H)\}$. Hedetniemi [8] conjectured in 1966 that for any graphs $G$ and $H$, $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. Although the conjecture was widely approached, cf. recent survey [22], only special cases have been solved affirmatively. One of the reasons for the investigation of the chromatic number of the lexicographic product is its close relation to the fractional chromatic number, $\chi_f$, cf. [19], and to the concept of the $n$th chromatic number, cf. [20,21]. In addition, the chromatic number of lexicographic products form an important tool in the theory of approximation algorithms for the chromatic number of a graph [15], while in [4] the disjunctive product is used. For more results on the chromatic number of graph products see [12].

In Section 2 we show, using the graphs $(K_n \circ C_{2k+1}) \times K_{3n-1}$, that the difference between the chromatic number and the fractional chromatic number can be arbitrarily large in the class of uniquely colorable, vertex transitive graphs. Then we prove that for any graphs $G$ and $H$, $\chi(G \circ H) \geqslant \chi_f(G)\chi(H)$ and consider several consequences of this fact. In the final section we present two simple, elementary proofs of theorems of Stahl and of Bollobás and Thomason.

The graphs considered are finite and simple. As usually, $\chi(G)$ denotes the chromatic number of $G$ and $\alpha(G)$ its independence number.

We will mostly consider the chromatic number of the lexicographic product. Clearly, for any graphs $G$ and $H$ we have $\chi(G \circ H) \leqslant \chi(G)\chi(H)$. It is more difficult to obtain a good lower bound for $\chi(G \circ H)$. In the rest we will extensively use the following fundamental result due to Geller and Stahl.

**Theorem 1** (Geller and Stahl [6]). *If $\chi(H) = n$, then for any graph $G$, $\chi(G \circ H) = \chi(G \circ K_n)$.*

A graph $G$ is called *uniquely $n$-colorable* if any $n$-coloring of $G$ determines the same partition of $V(G)$ into color classes. We will apply the following result of Greenwell and Lovász.

**Theorem 2** (Greenwell and Lovász [7]). *If $G$ is a connected graph with $\chi(G) > n$, then $G \times K_n$ is uniquely $n$-colorable.*

The *fractional chromatic number* $\chi_f(G)$ of $G$ is defined as follows. Let $\mathscr{I}(G)$ be the set of independent sets of a graph $G$. A *fractional coloring* of $G$ is a mapping

$f : \mathscr{I}(G) \to [0,1]$ such that for each vertex $v$ of $G$ we have $\sum_{I \in \mathscr{I}, v \in I} f(I) \geqslant 1$. The *weight* $w(f)$ of the fractional coloring $f$ is defined as $w(f) = \sum_{I \in \mathscr{I}} f(I)$. Then $\chi_f(G)$ is the minimum of the weights of fractional colorings of $G$. Note that for any graph $G$, $\chi(G) \geqslant \chi_f(G)$. Gao and Zhu proved:

**Theorem 3** (Gao and Zhu [5]). *For any graphs $G$ and $H$, $\chi_f(G \circ H) = \chi_f(G)\chi_f(H)$.*

Analogous result for the disjunctive product is given in [4].

Finally, circulant graphs are defined as follows. Let $N$ be a set of nonzero elements of $\mathbb{Z}_k$ such that $N = -N$. The *circulant graph* $G(k,N)$ has vertices $0, 1, \ldots, k-1$ and $i$ is adjacent to $j$ if and only if $i - j \in N$, where the arithmetic is done $\bmod k$. Note that circulant graphs are vertex transitive.

## 2. Products and fractional chromatic number

In this section we treat the interplay between the fractional chromatic number and graph products. We first use the direct product and the lexicographic one to show that the difference between the chromatic number and the fractional chromatic number can be arbitrarily large in the class of uniquely colorable, vertex transitive graphs. Then we observe that for any graphs $G$ and $H$ we have $\chi(G \circ H) \geqslant \chi_f(G)\chi(H)$. We list some consequences of this bound and demonstrate that it extends previously known lower bounds.

We begin with the following well-known lemma, cf. [10].

**Lemma 4.** *For any graphs $G$ and $H$,*

(i) $\alpha(G \times H) \geqslant \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$, [11,17]
(ii) $\alpha(G \circ H) = \alpha(G)\alpha(H)$ [6].

In a uniquely colorable, vertex transitive graph $G$, all color classes of the unique coloring are of the same size. This observation might give a feeling that the chromatic number and the fractional chromatic number coincide on such graphs. (This is clearly not the case for graphs that are only vertex transitive, for instance, $\chi(C_{2k+1}) = 3$ and $\chi_f(C_{2k+1}) = 2 + 1/k$.) However, we have the following:

**Theorem 5.** *For any integer $n \geqslant 2$ there exists a uniquely colorable, vertex transitive graph $G$, such that $\chi(G) - \chi_f(G) > n - 2$.*

**Proof.** Let $n \geqslant 2$ and let $k$ be an arbitrary integer $> n$. Set

$$G = (K_n \circ C_{2k+1}) \times K_{3n-1}.$$

The lexicographic product and the direct product of vertex transitive graphs is vertex transitive, cf. [10], hence so is $G$. From Theorem 1 we infer that $\chi(K_n \circ C_{2k+1}) = 3n$. Therefore, $G$ is uniquely colorable by Theorem 2.

It is well-known that Hedetniemi's conjecture holds for complete graphs (see [2,22]), hence $\chi(G)=3n-1$. Moreover, it is known (and easy to prove) that for a vertex transitive graph $G$ we have $\chi_f(G)=|V(G)|/\alpha(G)$. Since $\alpha(G)\geqslant\alpha(K_n\circ C_{2k+1})(3n-1)$ by Lemma 4(i), we find out, using Lemma 4(ii), that $\alpha(G)\geqslant k(3n-1)$. Now,

$$\chi_f(G)\leqslant\frac{n(2k+1)(3n-1)}{k(3n-1)}=2n+\frac{n}{k}<2n+1.$$

Thus we conclude that $\chi(G)-\chi_f(G)>3n-1-(2n+1)=n-2$. $\square$

We now give a nonlinear lower bound for the chromatic number of the lexicographic product of graphs. Although its proof is quite short, it extends some previously known lower bounds.

**Theorem 6.** *For any graphs $G$ and $H$, $\chi(G\circ H)\geqslant\chi_f(G)\chi(H)$.*

**Proof.** Let $\chi(H)=n$. Then we have

$$\begin{aligned}\chi(G\circ H) &= \chi(G\circ K_n) \quad \text{(by Theorem 1)}\\ &\geqslant \chi_f(G\circ K_n)\\ &= \chi_f(G)\chi_f(K_n) \quad \text{(by Theorem 3)}\\ &= \chi_f(G)n \quad \text{(as } K_n \text{ is vertex transitive)}\\ &= \chi_f(G)\chi(H). \qquad \square\end{aligned}$$

Consider the circulant graphs $G_m=G(3m-1,\{1,4,\ldots,3m-2\})$. Then $\chi(G_m\circ K_n)\geqslant\chi_f(G_m)n=((3m-1)/m)n$. Therefore $\chi(G_m\circ K_n)\geqslant 3n-\lfloor n/m\rfloor$, which is the exact chromatic number of these graphs, cf. [13].

Since $G\circ H$ is a spanning subgraph of $G\vee H$, we infer that $\chi(G\vee H)\geqslant\chi(G\circ H)$. Thus Theorem 6 implies that $\chi(G\vee H)\geqslant\chi_f(G)\chi(H)$. This observation can be used to shorten some of the arguments from [3]. Moreover, we have:

**Corollary 7.** *For a graph $G$, the following conditions are equivalent*:

(i) $\chi_f(G)=\chi(G)$,
(ii) $\chi(G\vee H)=\chi(G)\chi(H)$, *for all graphs $H$*,
(iii) $\chi(G\circ H)=\chi(G)\chi(H)$, *for all graphs $H$*.

**Proof.** The equivalence between (i) and (ii) is proved in [19]. Since

$$\chi(G)\chi(H)\geqslant\chi(G\vee H)\geqslant\chi(G\circ H),$$

we infer that (iii) implies (ii). Finally (i) implies (iii) by Theorem 6. $\square$

It is not difficult to verify (cf. [13]) that

$$\chi_f(G) = \inf\{\chi(G \circ K_n)/n \mid n = 1, 2, \ldots\}.$$

Using this fact, we have another proof of Theorem 6. Indeed, let $\chi(H) = n$. Then $\chi_f(G) \leqslant \chi(G \circ K_n)/n = \chi(G \circ H)/\chi(H)$.

Let $G^{(n)} = G \circ G \circ G \circ \cdots \circ G$ and $G^{[n]} = G \vee G \vee G \vee \cdots \vee G$, $n$ times. Hell and Roberts showed in [9] that

$$\chi_f(G) = \inf_n \sqrt[n]{\chi(G^{[n]})} = \inf_n \sqrt[n]{\chi(G^{(n)})}.$$

They first proved $\chi_f(G) = \inf_n \sqrt[n]{\chi(G^{[n]})}$ and then claimed the second equality by using the duality theorem of linear programming. Using Theorem 6 we give an elementary proof of the second equality, that is, without using the duality theorem of LP.

By Theorem 6 we have $\chi(G^{(t)})/\chi(G^{(t-1)}) \geqslant \chi_f(G)$, for every $t \geqslant 1$. Hence

$$\sqrt[n]{\frac{\chi(G)}{1}\frac{\chi(G^{(2)})}{\chi(G)}\frac{\chi(G^{(3)})}{\chi(G^{(2)})}\cdots\frac{\chi(G^{(n)})}{\chi(G^{(n-1)})}} \geqslant \chi_f(G), \quad n = 1, 2, 3, \ldots$$

Therefore, $\inf_n \sqrt[n]{\chi(G^{(n)})} \geqslant \chi_f(G) = \inf_n \sqrt[n]{\chi(G^{[n]})}$, and so we conclude that $\inf_n \sqrt[n]{\chi(G^{(n)})} = \inf_n \sqrt[n]{\chi(G^{[n]})}$.

Note that the above argument is parallel to the proof of Theorem 1.6.2 on p. 13 of the book [19].

In the next two corollaries we show that Theorem 6 extends some known lower bounds. We first state:

**Corollary 8** (Stahl [20]). *Let $G$ be a nonbipartite graph. Then for any graph $H$,*

$$\chi(G \circ H) \geqslant 2\chi(H) + \left\lceil \frac{\chi(H)}{k} \right\rceil,$$

*where $2k+1$ is the length of a shortest odd cycle of $G$.*

**Proof.** We first observe that it suffices to prove the result for $G = C_{2k+1}$. As $C_{2k+1}$ is vertex transitive, $\chi_f(C_{2k+1}) = 2 + 1/k$. The result now follows by Theorem 6. □

Corollary 8 has been generalized in [5] to the so-called circular chromatic number of graphs.

Lovász [16] has shown that for any graph $G$ on $n$ vertices

$$\chi_f(G) \geqslant \frac{\chi(G)}{1 + \ln \alpha(G)}.$$

Therefore we also have:

**Corollary 9** (Lovász [16]). *For any graphs $G$ and $H$,*

$$\chi(G \circ H) \geqslant \frac{\chi(G)\chi(H)}{1 + \ln \alpha(G)}.$$

To see that the lower bound of Theorem 6 is in general better than those of Corollaries 8 and 9 consider the Hamming graphs $H_n = K_n \Box K_n$, $n \geqslant 2$. Let $G_n = H_n \circ K_n$. Corollary 8 gives $\chi(G_n) \geqslant 3n$. Since $\alpha(H_n) = n$, Corollary 9 implies that $\chi(G_n) \geqslant n^2/(1 + \ln n)$. Finally Theorem 6 asserts that $\chi(G_n) \geqslant n^2$.

To conclude the section we add that the inequality $\chi(G \circ H) \geqslant \chi(G)\chi_f(H)$ does not hold in general. For instance, $\chi(C_{2k+1} \circ K_n) = 2n + \lceil n/k \rceil$ while $\chi(C_{2k+1})\chi_f(K_n) = 3n$. This example is a nice illustration of the fact that the lexicographic product is not commutative.

## 3. Two simple and short proofs

In this section we present simple and elementary proofs of two further lower bounds. The main advantage of the presented proofs is that they are conceptually simpler than the existing ones.

The first result is due to Stahl [20], see also [14].

**Theorem 10** (Stahl [20]). *If $G$ has at least one edge, then for any graph $H$,*

$$\chi(G \circ H) \geqslant \chi(G) + 2\chi(H) - 2.$$

**Proof.** By Theorem 1 it suffices to prove the bound for $H = K_n$. Let $V(K_n) = \{1, 2, \dots, n\}$, $\chi(G \circ K_n) = l$, and let $c$ be an $l$-coloring of $G \circ K_n$. Set

$$U = \{v \in V(G); c(v, i) \leqslant 2n \text{ for } i = 1, \dots, n\}.$$

Then $U$ can be partitioned into independent sets $U' = \{v \in U; c(v, i) = 1 \text{ for some } i\}$ and $U \backslash U'$. Hence $U$ induces a bipartite subgraph of $G$.

For a vertex $u \in V(G) \backslash U$ let $i_u$ be a vertex of $K_n$ such that $c(u, i_u) \geqslant 2n+1$. Then the function $\phi : V(G) \backslash U \to \{2n + 1, \dots, l\}$ given by $\phi(u) = c(u, i_u)$ is an $(l - 2n)$-coloring of $G \backslash U$. For if $uu' \in E(G)$ then $(u, i_u)$ and $(u', i_{u'})$ are adjacent in $G \circ K_n$ and hence $c(u, i_u) \neq c(u', i_{u'})$. We conclude that $\chi(G) \leqslant (l - 2n) + 2$. $\square$

Bollobás and Thomason improved Theorem 10 for the case of uniquely colorable graphs:

**Theorem 11** (Bollobás and Thomason [1]). *If $G$ is uniquely $m$-colorable graph, $m > 2$, then for any graph $H$ with at least one edge,*

$$\chi(G \circ H) \geqslant \chi(G) + 2\chi(H) - 1.$$

**Proof.** It suffices to consider the case $\chi(G) = m$ and $H = K_n$, $n \geqslant 2$. Let $V(K_n) = \{1, 2, \dots, n\}$. Suppose on the contrary that $\chi(G \circ K_n) \leqslant \chi(G) + 2\chi(H) - 2$, and let $c$ be an $(m + 2n - 2)$-coloring of $G \circ K_n$. Let $U$, $U'$ and $\phi$ be defined as in the proof of

Theorem 10, where the vertices of $U'$ (resp. $U\backslash U'$) receive color 1 (resp. color 2). We claim that $c(u,i) \geqslant 2n+1$ for any $u \in V(G)\backslash U$ and any $i \in V(K_n)$. Assume on the contrary that $c(u,i) \leqslant 2n$. We may without loss of generality assume that $c(u,i) = 1$ (for otherwise we can redefine $U'$ accordingly). But then we can recolor $u$ with color 1 and still have an $m$-coloring of $G$, which is not possible since $G$ is uniquely colorable.

Since $|\{c(u,k)\colon u \in V(G)\backslash U \text{ and } k \in V(K_n)\}| \leqslant (m+2n-2)-2n = m-2$, $|V(G)\backslash U| \geqslant m-2$ and $n \geqslant 2$, there exist two distinct vertices $x,y \in V(G)\backslash U$ such that $c(x,i) = c(y,j)$ for some $i,j \in V(K_n)$. Then the function

$$\varphi(u) = \begin{cases} c(x,i), & u = x, \\ c(y,j), & u = y, \\ \phi(u) & \text{otherwise} \end{cases}$$

is an $m$-coloring of $G$. Now pick $r \in V(K_n)\backslash\{i\}$, then

$$\varphi'(u) = \begin{cases} c(x,r), & u = x, \\ c(y,j), & u = y, \\ \phi(u) & \text{otherwise} \end{cases}$$

is another $m$-coloring of $G$ different from $\varphi$, since $c(x,r) \neq c(y,j) = c(x,i)$. $\square$

## Acknowledgements

We are grateful to one of the referees for her/his relevant remarks. In particular, the present form of Theorem 6 is due to her/him.

## References

[1] B. Bollobás, A. Thomason, Set colourings of graphs, Discrete Math. 25 (1979) 21–26.
[2] S.A. Burr, P. Erdős, L. Lovász, On graphs of Ramsey type, Ars Combin. 1 (1976) 167–190.
[3] N. Čižek, S. Klavžar, On the chromatic number of the lexicographic product and the Cartesian sum of graphs, Discrete Math. 134 (1994) 17–24.
[4] U. Feige, Randomized graph products, chromatic numbers, and the Lovász $\vartheta$-function, Combinatorica 17 (1997) 79–90.
[5] G. Gao, X. Zhu, Star-extremal graphs and the lexicographic product of graphs, Discrete Math. 152 (1996) 147–156.
[6] D. Geller, S. Stahl, The chromatic number and other parameters of the lexicographic product, J. Combin. Theory Ser. B 19 (1975) 87–95.
[7] D. Greenwell, L. Lovász, Applications of product colouring, Acta Math. Acad. Sci. Hungar. 25 (1974) 335–340.
[8] S. Hedetniemi, Homomorphisms of graphs and automata, Technical Report 03105-44-T, University of Michigan, 1966.
[9] P. Hell, F.S. Roberts, Analogues of the Shannon capacity of a graph, Ann. Discrete Math. 12 (1982) 155–168.
[10] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, Wiley, New York, 2000.
[11] P.K. Jha, G. Slutzki, Independence numbers of product graphs, Appl. Math. Lett. 7 (4) (1994) 91–94.
[12] S. Klavžar, Coloring graph products—a survey, Discrete Math. 155 (1996) 135–145.

[13] S. Klavžar, On the fractional chromatic number and the lexicographic product of graphs, Discrete Math. 185 (1998) 259–263.
[14] S. Klavžar, U. Milutinović, Strong products of Kneser graphs, Discrete Math. 133 (1994) 297–300.
[15] N. Linial, U. Vazirani, Graph products and chromatic numbers, in: Proceedings of the 30th Annual IEEE Symposium on Foundation of Computer Science, IEEE Computer Soc. Press, Los Alamos, 1989, pp. 124–128.
[16] L. Lovász, On the ratio of optimal integral and fractional covers, Discrete Math. 13 (1975) 383–390.
[17] R.J. Nowakowski, D. Rall, Associative graph products and their independence, domination and coloring numbers, Discuss. Math. Graph Theory 16 (1996) 53–79.
[18] G. Sabidussi, Graphs with given group and given graph-theoretical properties, Canad. J. Math. 9 (1957) 515–525.
[19] E.R. Scheinerman, D.H. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs, Wiley, New York, 1997.
[20] S. Stahl, $n$-tuple colorings and associated graphs, J. Combin. Theory Ser. B 20 (1976) 185–203.
[21] S. Stahl, The multichromatic numbers of some Kneser graphs, Discrete Math. 185 (1998) 287–291.
[22] X. Zhu, A survey on Hedetniemi's conjecture, Taiwanese J. Math. 2 (1998) 1–24.