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ScienceDirect
Discrete Mathematics 306 (2006) 2964 – 2967
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

Counting hypercubes in hypercubes☆

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Received 19 May 2004; accepted 3 October 2005
Available online 28 August 2006

Abstract

Subcubes of a hypercube are counted in three different ways, yielding a new graph theory interpretation of a known combinatorial identity. From this and the binomial inversion some additional combinatorial identities related to hypercubes are obtained.

Keywords: Hypercubes; Binomial identities; Enumeration

1. Introduction

Symbolic computation offers powerful methods for proving combinatorial identities, in particular the hypergeometric ones [9]. On the other hand, a (proven) identity increases its meaning if we can assign to it an (apparently unrelated) interpretation. Double counting is one of the main sources for obtaining combinatorial identities. Garbano et al. [5] counted the edges of a hypercube in two ways to obtain the following identity

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}. \tag{1}$$

This identity can also be obtained in some other ways (cf., for instance, [3]), nevertheless, the approach from [5] is interesting since it gives an attractive graph theory interpretation of (1). In this note, we show that the above idea can be extended to double counting of arbitrary d-dimensional hypercubes of a given n-dimensional hypercube, yielding the identity

$$\binom{n}{d} 2^{n-d} = \sum_{k=d}^{n} \binom{n}{k} \binom{k}{d}, \tag{2}$$

that holds for any $n \geqslant d \geqslant 0$, cf. [8, Review Exercise 3.8.8]. Note that for $d = 1$ the identity (2) becomes (1). Of course, (2) can be proved by other methods, for instance by differentiating (d times) the binomial theorem, or by using the

☆ Supported in part by the Ministry of Science of Slovenia under the Grant P1-0297-0101-01.
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0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.10.036

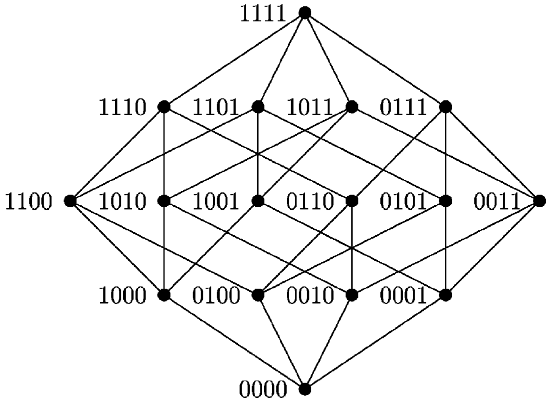

Fig. 1. The 4-cube Q_4.

identity

$$\binom{n}{k}\binom{k}{d}=\binom{n}{d}\binom{n-d}{k-d},$$

cf. [4, Theorem 8]. However, our approach yields a new combinatorial (graph theory) interpretation of (2).

In the rest of this section we formally introduce hypercubes and list those properties that are needed in our counting arguments. In the next section we count the number of induced d-dimensional hypercubes of a given n-dimensional hypercube in three different ways which in particular yields (2). In the final section we use the results of Section 2 combined with the binomial inversion to derive two additional identities for hypercubes.

For $d \geqslant 0$, the *d-dimension hypercube* Q_d is the graph on 2^d vertices representing all 0/1 tuples of length d, where two vertices are adjacent whenever the tuples differ in exactly one position. We shortly say that Q_d is the d-cube. By definition, $Q_0 = K_1$. Note also that the 1-cube is the complete graph on two vertices K_2 and that the 2-cube is the 4-cycle C_4.

A useful way of representing hypercubes is in terms of their distance levels. For $k = 0, 1, \ldots, n$ set $L_k = \{u \in V(Q_n) \mid u$ contains k ones$\}$. Then L_k is called the kth *distance level* of Q_n (with respect to $00\ldots0$) and the distance levels partition the vertex set of Q_n. Moreover, any edge of Q_n connects vertices of two consecutive distance levels. On Fig. 1 the 4-cube is drawn such that the vertices of a distance level lie on the same horizontal level.

Let u be a vertex of Q_n. Then the vertex v of Q_n that is obtained from u by interchanging zeros and ones is called the *antipodal vertex* of u (in Q_n). For instance, the antipodal vertex of 100111 is 011000. Note that the antipodal vertex of u in Q_n is the unique vertex of Q_n that is on the shortest-path distance n from u.

2. Counting sub-hypercubes

In this section, we count the number of induced d-cubes in Q_n ($n \geqslant d \geqslant 0$) in three different ways. For this sake let $\alpha_d(G)$ be the number of induced d-cubes of a graph G.

Symmetry approach: A graph G is called *vertex-transitive* is its automorphism group acts transitively on the vertex set of G. It is well-known and not difficult to see that hypercubes are vertex-transitive graphs, cf. [6, Lemma 3.1.1]. Let $n \geqslant d \geqslant 0$, and let u be an arbitrary vertex of Q_n. We may without loss of generality assume that $u = 00\ldots0$ and consider an arbitrary induced subgraph H of Q_n isomorphic to Q_d that contains u. Then H necessarily contains a vertex v that contains d one's, that is, v is the antipodal vertex of u in Q_d. Moreover, any vertex with d one's give rise to a unique d-cube that contains u. It follows that Q_n contains $\binom{n}{d}$ copies of Q_d that contain u. Since Q_n has 2^n vertices (and Q_d has 2^d of them) and is vertex transitive, we conclude that

$$\alpha_d(Q_n) = \frac{2^n\binom{n}{d}}{2^d} = \binom{n}{d}2^{n-d}. \tag{3}$$

Algebraic approach: The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ and $(a,x)(b,y)\in E(G\Box H)$ whenever either $ab\in E(G)$ and $x=y$, or $a=b$ and $xy\in E(H)$. This graph product is commutative and associative. The simplest example of Cartesian product graphs are just hypercubes: the n-cube Q_n is the Cartesian product of n copies of the complete graph on two vertices K_2.

The cube polynomial $c(G,x)$ of G is introduced in [1] as

$$c(G,x)=\sum_{d\geqslant 0}\alpha_d(G)x^i.$$

Moreover, in the same paper it is observed that for any graphs G and H,

$$c(G\Box H,x)=c(G,x)c(H,x). \tag{4}$$

Clearly, $c(K_2,x)=2+x$. Hence, having in mind that Q_n is the Cartesian product of n copies of K_2, equality (4) immediately implies that

$$\alpha_d(Q_n)=\binom{n}{d}2^{n-d}.$$

Metric approach: Let $1\leqslant k\leqslant n$, then a vertex u of the distance level L_k has k neighbors in L_{k-1}.

Let H be an induced subgraph of Q_n isomorphic to Q_d. Then by [7, Proposition 1.23 (iii)], H is uniquely determined by an arbitrary pair of its diametrical vertices. Select a vertex u of H with the largest number of one's. Then u is unique, and we call it the *top vertex of H*. Now, every vertex u of L_k gives rise to $\binom{k}{d}$ d-cubes for which u is the top vertex. Since in d neighbors in L_k there are $\binom{n}{k}$ vertices, they all together give $\binom{k}{d}\binom{n}{k}$ such d-cubes. Hence we conclude that

$$\alpha_d(Q_n)=\sum_{k=d}^{n}\binom{n}{k}\binom{k}{d}. \tag{5}$$

Hence (3) and (5) give (2).

3. More hypercube identities

The following identities also hold for hypercubes, see [4, Problem 59]:

$$1=\sum_{k\geqslant 0}(-1)^k\alpha_k(Q_n). \tag{6}$$

On the other hand, Soltan and Chepoi [11] proved that (6) holds for all median graphs, where median graphs form an important metrically defined class of graphs that contains hypercubes. For the definition and more information on this concept see [7,10]. In addition, Škrekovski [10] proved that for any median graph G:

$$t(G)=-\sum_{k\geqslant 0}(-1)^k\,k\,\alpha_k(G),$$

where $t(G)$ is the number of the equivalence classes of the so called Djoković-Winkler relation Θ (see [7]) defined on the edge set of G. Since $t(Q_n)=n$, and Q_n is a median graph, we get

$$n=-\sum_{k\geqslant 0}(-1)^k k\alpha_k(Q_n). \tag{7}$$

To conclude this note we show how (6) and (7) can be obtained from the counting results of Section 2. Let $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ be arbitrary real sequences. Then the *binomial inversion*, cf. [2, Corollary 5.2.4], asserts that

$$a_n=\sum_{k=0}^{n}\binom{n}{k}b_k \quad (\forall n\geqslant 0)$$

if and only if

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \quad (\forall n \geqslant 0).$$

Since we have $\binom{n}{d} 2^{n-d} = \sum_{k=d}^{n} \binom{n}{k}\binom{k}{d}$ and hence also $\binom{n}{d} 2^{n-d} = \sum_{k=0}^{n} \binom{n}{k}\binom{k}{d}$, the binomial inversions yields, for any $n \geqslant 0$ and any $d \geqslant 0$:

$$\binom{n}{d} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{k}{d} 2^{k-d} \sum_{k=d}^{n} (-1)^{n-k} \binom{n}{k}\binom{k}{d} 2^{k-d}. \tag{8}$$

For $d = 0$ the identity (8) gives, using (3),

$$1 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} = \sum_{k \geqslant 0} (-1)^k \alpha_k(Q_n),$$

and for $d = 1$ the identity (8) gives

$$n = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k 2^{k-1} = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k (n-k)\alpha_k(Q_n), \tag{9}$$

which is an alternative form of (7).

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