

A SURVEY ON PACKING COLORINGS

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Abstract

If $S = (a_1, a_2, \dots)$ is a non-decreasing sequence of positive integers, then an S -packing coloring of a graph G is a partition of $V(G)$ into sets X_1, X_2, \dots such that for each pair of distinct vertices in the set X_i , the distance between them is larger than a_i . If there exists an integer k such that $V(G) = X_1 \cup \dots \cup X_k$, then the partition is called an S -packing k -coloring. The S -packing chromatic number of G is the smallest k such that G admits an S -packing k -coloring. If $a_i = i$ for every i , then the terminology reduces to packing colorings and packing chromatic number. Since the introduction of these generalizations of the chromatic number in 2008 more than fifty papers followed. Here we survey the state of the art on the packing coloring, and its generalization, the S -packing coloring. We also list several conjectures and open problems.

Keywords: packing coloring, packing chromatic number, subcubic graph, S -packing chromatic number, computational complexity.

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1. INTRODUCTION

The packing chromatic number was first studied, under the name *broadcast chromatic number*, by Goddard, S.M. Hedetniemi, S.T. Hedetniemi, Harris, and Rall [35]. The concept was introduced because of potential applications in broadcast assignment problems. In all the papers following the seminal paper the terms

packing coloring and packing chromatic number were (and still are) used, the terms being coined by Brešar, Klavžar, and Rall [12]. Because of the numerous papers written on the packing chromatic number and since it is of continuing interest, we think now is the time to collect results and open problems in a survey paper.

In the next section we state some basic properties of the packing chromatic number; relate it with invariants α , β , χ , and ω ; give partial results on trees; and present known complexity results. In Section 3, we consider one of the most exciting topics related to the packing chromatic number, namely, how it behaves on subcubic graphs. In the subsequent section, the known effects on the packing chromatic number of local operations (vertex deletion, edge deletion, edge contraction, edge subdivision) are summarized. In Section 5 the packing chromatic number of specific classes of graphs is given with an emphasis on Cartesian products and Sierpiński-type graphs. The packing chromatic number has also been studied on infinite graphs, which is the focus of Section 6. The main roles here are played by several families of lattices and of distance graphs. Section 7 presents what is known about S -packing colorings (for $S \neq (1, 2, \dots)$), which is a wide generalization of packing colorings. We conclude with a list of conjectures and open problems. In the rest of the introduction we collect definitions and notation needed throughout the paper.

Given a graph $G = (V(G), E(G))$ and a positive integer i , an i -packing in G is a subset X of $V(G)$ such that the distance $d_G(u, v)$ between any two distinct vertices $u, v \in X$ is greater than i . (The *distance* $d_G(u, v)$ is the length of a shortest path between u and v in G .) The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k such that the vertex set of G can be partitioned into sets X_1, \dots, X_k , where X_i is an i -packing for each $i \in [k] = \{1, \dots, k\}$. Such a partition corresponds to a mapping $c : V(G) \rightarrow [k]$ such that $X_i = \{u \in V(G) : c(u) = i\}$. This mapping has the property that $c(u) = c(v) = i$ implies $d_G(u, v) > i$; c is called a *packing k -coloring*.

Along the way it will be useful to have the following generalization of the packing chromatic number in hand. Let $S = (a_1, a_2, \dots)$ be an infinite, non-decreasing sequence of positive integers. (All sequences of positive integers in this paper, whether finite or infinite, are assumed to be non-decreasing.) An *S -packing coloring* of G is a mapping $c : V(G) \rightarrow \mathbb{N}$ such that $c^{-1}(i)$ is an a_i -packing for each $i \in \mathbb{N}$. When such a mapping c exists for a graph G , we say G is *S -packing colorable*. If k is a positive integer and $c(u) \in [k]$ for each vertex u , then c is called an *S -packing k -coloring* or an (a_1, \dots, a_k) -*packing coloring* of G . The *S -packing chromatic number* of G , denoted $\chi_S(G)$, is the smallest k such that G has an S -packing k -coloring. If G does not admit an S -packing k -coloring for any $k \in \mathbb{N}$, then we set $\chi_S(G) = \infty$. For example, for any sequence S we have $\chi_S(K_{\mathbb{N}}) = \infty$, where $K_{\mathbb{N}}$ is the countably infinite complete graph. Note that

$((1)_{i \in \mathbb{N}})$ -packing colorings are the standard proper vertex colorings of a graph and that $((i)_{i \in \mathbb{N}})$ -packing colorings are just the packing colorings. Moreover, if ℓ is a positive integer, then $((\ell)_{i \in \mathbb{N}})$ -packing colorings are known in the literature as either ℓ -distance colorings or colorings of the ℓ^{th} power of a graph.

The study of distance colorings is a well developed area (see an early survey [49]), and covering all the results of it cannot fit in the scope of this survey. We mention in passing that the concepts of *facial packing edge-coloring* [17] and *facial packing vertex-coloring* [18] were introduced. They are defined on plane graphs as colorings of its edges (respectively vertices) such that every facial trail containing two edges (respectively vertices) with the same color i has length at least $i + 2$.

Unless stated otherwise, graphs considered will be finite. The order and the size of a (finite) graph G will be denoted with $n(G)$ and $m(G)$, respectively. If G is a graph, we will use the standard notation $\alpha(G)$ for the independence number, $\beta(G)$ for the vertex cover number, $\chi(G)$ for the chromatic number, $\omega(G)$ for the clique number, and $\Delta(G)$ for the largest degree of G . A graph G with $\Delta(G) \leq 3$ is *subcubic*. If $k \geq 1$ and G is a graph, then $\alpha_k(G)$ denotes the *k-independence number* of G , that is, the size of the largest union of k independent sets in $V(G)$. The diameter of G will be denoted by $\text{diam}(G)$. If $t \geq 1$ and G is a graph, then the t^{th} power of G is the graph with the same vertex set as G , two distinct vertices being adjacent if they are at distance at most t in G . Let $e = uv$ be an edge of G . The *edge subdivision* operation for e is executed as follows: first we remove the edge e from G and then add a new vertex x and edges ux, vx to G . A graph G with subdivided edge e is denoted by $S_e(G)$. The *subdivision graph* of a graph G , denoted by $S(G)$, is obtained from G by subdividing every edge of G . The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent in $G \square H$ if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. A *caterpillar* is a tree in which the subgraph of vertices that are not leaves induces a path. A graph is a *split graph* if its vertex set can be partitioned into two sets, one of which is independent and the other induces a complete graph. The *paw* is the graph obtained from $K_{1,3}$ by adding one edge.

2. BASIC PROPERTIES AND COMPLEXITY

In this section we first state some basic properties of the packing chromatic number. Then we compare this invariant with the vertex cover number, independence number, chromatic number, and clique number. At the end of the section we present some complexity results.

If an edge or a vertex is removed from a given graph G , then the distances

between the (remaining) vertices of G cannot decrease. Hence a packing coloring of G restricted to an arbitrary subgraph H is a packing coloring of H . This implies the following observation.

Observation 2.1. *If H is a subgraph of G , then $\chi_\rho(H) \leq \chi_\rho(G)$.*

In the seminal paper, Goddard *et al.* [35] determined the packing chromatic numbers of paths and cycles.

$$\begin{aligned} \text{If } n \geq 2, \text{ then } \chi_\rho(P_n) &= \begin{cases} 2, & n \in \{2, 3\}, \\ 3, & n \geq 4. \end{cases} \\ \text{If } n \geq 3, \text{ then } \chi_\rho(C_n) &= \begin{cases} 3, & n = 3 \text{ or } n \equiv 0 \pmod{4}, \\ 4, & \text{otherwise.} \end{cases} \end{aligned}$$

Also in [35] the authors provided characterizations of the graphs with packing chromatic number equal to 2 or 3, respectively.

Proposition 2.2 [35, Proposition 3.1]. *If G is a connected graph, then $\chi_\rho(G) = 2$ if and only if G is a star.*

The *join* of graphs G and H is obtained from the disjoint union of G and H by adding an edge between each vertex of G and each vertex of H .

Proposition 2.3 [35, Proposition 3.2]. *Let G be a 2-connected graph. Then $\chi_\rho(G) = 3$ if and only if G is either $S(H)$ for some bipartite multigraph H or the join of K_2 and an independent set.*

In order to characterize the graphs G with $\chi_\rho(G) = 3$, the authors of [35] used the following concept. If v is a vertex of a graph G , then a *T-add* to v is obtained by adding a new vertex w_v and a set X_v of independent vertices to G , adding the edge vw_v and some of the edges between the sets $\{v, w_v\}$ and X_v .

Proposition 2.4 [35, Proposition 3.3]. *Let G be a graph. Then $\chi_\rho(G) = 3$ if and only if G can be formed by taking some bipartite multigraph H with at least one edge and with bipartition (V_2, V_3) , subdividing every edge exactly once, adding leaves adjacent to some vertices in $V_2 \cup V_3$, and then performing a single T-add to some vertices in V_3 .*

2.1. Relations with $\alpha, \beta, \chi, \omega$

Independence number

Clearly, $\chi_\rho(G) \leq n(G)$, since there always exists a packing coloring of a given graph G such that the vertices of G receive pairwise distinct colors. A better general upper bound is the following one, which is proved by using a packing coloring in which the vertices of a largest independent set receive color 1 and each of the other vertices is assigned its own color.

Proposition 2.5. *If G is a graph, then*

$$\chi_\rho(G) \leq n(G) - \alpha(G) + 1,$$

with equality if $\text{diam}(G) = 2$.

There is also a general lower bound in terms of the independence number.

Proposition 2.6 [15, Proposition 4.1]. *If G is a graph, then*

$$\chi_\rho(G) \geq \Delta(G) - \alpha(G) + 2.$$

Equality is achieved if $\Delta(G) = n(G) - 1$.

Let \mathcal{C} be the class of graphs H defined as follows. Let $r \geq 3$ and $s \geq 2$ be positive integers. Let A be a K_r with three specified vertices a_1 , a_2 , and a , and let B be a K_s with specified vertices b and b_1 . A graph H is formed from the disjoint union of A and B together with a new vertex z and new edges between z and every vertex of $B \setminus \{b\}$. Next, identify the vertices a and b , and denote the new vertex by w . Finally, we can add any missing edge non-incident with z or a_2 , except a_1b_1 .

Theorem 2.7 [15, Theorem 4.2]. *If G is a graph with $\alpha(G) = 2$, then $\chi_\rho(G) = \Delta(G) - \alpha(G) + 2$ if and only if $\Delta(G) = n(G) - 1$ or $G \in \mathcal{C}$.*

Vertex cover number

Since $\alpha(G) + \beta(G) = n(G)$ holds for every graph G , Proposition 2.5 yields the following upper bound on $\chi_\rho(G)$ in terms of the vertex cover number of G .

Corollary 2.8 [35, Proposition 2.1]. *If G is a graph, then $\chi_\rho(G) \leq \beta(G) + 1$, with equality if $\text{diam}(G) = 2$.*

If a given graph G is a bipartite graph with diameter 3, then there is also a lower bound in terms of $\beta(G)$.

Proposition 2.9 [35, Proposition 2.2]. *If G is a bipartite graph of diameter 3, then $\beta(G) \leq \chi_\rho(G) \leq \beta(G) + 1$.*

Chromatic number and clique number

Trivial relationships between the packing chromatic number, the chromatic number, and the clique number of a graph G are given by

$$\omega(G) \leq \chi(G) \leq \chi_\rho(G).$$

Already in the seminal paper Goddard *et al.* [35] were interested in a characterization of graphs for which the packing chromatic number equals their chromatic number or their clique number. It was observed that a necessary condition for $\chi_\rho(G) = \omega(G)$ is that the neighbors of any maximum clique form an independent set, and at least one vertex of such a clique has no neighbors outside this clique (and thus can receive color 1). In the case of split graphs, this necessary condition is sufficient.

The fact that graphs with arbitrary clique number k and chromatic number of an arbitrary size greater than k can be constructed, prompts a question about the existence of graphs G with given $\omega(G)$, $\chi(G)$, and $\chi_\rho(G)$. In [35] it was proved that if $\chi_\rho(G) = \chi(G)$, then $\omega(G) \geq \chi(G) - 2$. This result strengthens as follows.

Theorem 2.10 [15, Theorem 3.3]. *If G is a graph with $\chi_\rho(G) = \chi(G)$, then $\omega(G) = \chi(G)$.*

A triple of integers (a, b, c) , where $2 \leq a \leq b \leq c$, is *realizable* if there exists a graph G such that $\omega(G) = a$, $\chi(G) = b$, and $\chi_\rho(G) = c$. It is easy to observe that the only graphs that realize the triple $(2, 2, 2)$ are stars. In [15] it was noted that the family of graphs G with $\Delta(G) = n(G) - 1$ from Theorem 2.7, which are obtained from K_n by removing the edges of a subgraph isomorphic to $K_{1,r}$, where $r + 1 < n$, have the property that $\chi_\rho(G) = \chi(G) = \omega(G) = n - 1$ for any $n \geq 3$.

The following result is important in investigating which triples are realizable.

Lemma 2.11 [15, Lemma 3.1]. *If (a, b, c) is realizable and $d > c$, then (a, b, d) is realizable.*

Lemma 2.11 implies that for a given a and b it is enough to determine the realizable triple (a, b, c) , where c is as small as possible. Thus, for given positive integers a and b define the function $m : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $m(a, b)$ is the smallest integer for which $(a, b, m(a, b))$ is realizable. Note that (a, b, c) is realizable if and only if $c \geq m(a, b)$. From Theorem 2.10 it follows that if G is a graph with $\omega(G) \neq \chi(G)$, then $\chi(G) \neq \chi_\rho(G)$. Hence if $b \geq 3$, then the triple $(2, b, b)$ is not realizable.

In the next result we collect what is known about the function m .

Theorem 2.12 [15]. *If $a \geq 2$, then the following hold.*

- (i) $m(a, a) = a$.
- (ii) $m(a, a + 1) = a + 2$.
- (iii) $a + 3 \leq m(a, a + 2) \leq 2a + 3$; in addition, $m(2, 4) = 7$.

It is well known that $\omega(M(K_k)) = k$ and $\chi(M(K_k)) = k + 1$, where $M(G)$ denotes the well-known Mycielskian of the graph G . Moreover, it was proved

in [15, Theorem 2.4] that $\chi_\rho(M(K_k)) = k + 2$. This yields the upper bound of Theorem 2.12(ii). The lower bounds of Theorem 2.12(ii) and (iii) follow from Theorem 2.10. Finally, the upper bound of (iii) is obtained by iterating the Mycielski construction on the complete graph K_a ; see [15] for the details as well as for other results on the packing chromatic number of Mycielski graphs.

For $a = 2$ we have the following non-realization result.

Theorem 2.13 [15, Theorem 3.6]. *If $b \geq 4$, then the triple $(2, b, b + 2)$ is not realizable. In other words, $m(2, b) \geq b + 3$.*

2.2. Partial results on trees

The packing chromatic number of a tree with diameter 2 (a star) is 2, and the packing chromatic number of a tree with diameter 3 is 3. If a tree has diameter 4, giving the packing chromatic number is more involved.

Proposition 2.14 [35, Proposition 5.1]. *Let T be a tree with diameter 4 and with central vertex v . For $i \in [3]$, let n_i denote the number of neighbors of v of degree i , and let L denote the number of neighbors of v of degree at least 4. If $L = 0$, then*

$$\chi_\rho(T) = \begin{cases} 4, & n_3 \geq 2 \text{ and } n_1 + n_2 + n_3 \geq 3, \\ 3, & \text{otherwise.} \end{cases}$$

If $L > 0$, then

$$\chi_\rho(T) = \begin{cases} L + 3, & n_3 \geq 1 \text{ and } n_1 + n_2 + n_3 \geq 2, \\ L + 1, & n_1 = n_2 = n_3 = 0, \\ L + 2, & \text{otherwise.} \end{cases}$$

Proposition 2.15 [35, Proposition 5.2]. *The minimum order of a tree with packing chromatic number 2 is 2. For 3 it is 4 and for 4 it is 8. Furthermore, P_4 is the unique tree on 4 vertices that requires three colors. The two trees on 8 vertices that require four colors are: (i) the diameter-4 tree with $n_3 = 2$, $n_1 = 1$ and $L = n_2 = 0$; and (ii) the diameter-5 tree where the two central vertices have degree 3 and for each central vertex its three neighbors have degrees 1, 2, and 3, respectively.*

The following gives an upper bound on the packing chromatic number of a tree in terms of its order.

Theorem 2.16 [35, Theorem 5.4]. *If T is a tree with $n(T) = n$, then $\chi_\rho(T) \leq (n + 7)/4$, except when $n = 4$ or $n = 8$, where the bound is $1/4$ more. The bounds are sharp.*

Sloper [62] studied the so-called eccentric colorings of trees. The *eccentricity*, $e(v)$, of a vertex v in a connected graph G is the largest integer r such that $d_G(v, w) = r$ for some $w \in V(G)$. If $c : V(G) \rightarrow [k]$ is a packing coloring of G with the added restriction that $c(u) \leq e(u)$ for each vertex u of G , then c is called an *eccentric k -coloring* of G . The *eccentric chromatic number* of G , denoted by $\chi_e(G)$, is the smallest k such that G has an eccentric k -coloring. Sloper proved the following result about eccentric colorings of binary trees. Recall that a *perfect binary tree* is a (rooted) tree in which all interior vertices have two children and all leaves have the same distance from the root.

Theorem 2.17 [62, Theorem 15]. *If T is a perfect binary tree of height at least 3, then $\chi_e(T) \leq 7$.*

Any tree with maximum degree at most 3 (rooted at an arbitrary vertex) is a subtree of a sufficiently large, complete binary tree. Hence, by disregarding the eccentricity restriction we get the following result about the packing-chromatic number of subcubic trees.

Theorem 2.18. *If T is a tree with $\Delta(T) \leq 3$, then $\chi_\rho(T) \leq 7$.*

Sloper also showed that the one-way infinite path admits a (2, 3, 4, 5, 6, 7)-packing coloring, which implies that $\chi_\rho(C) \leq 7$ for any caterpillar C . Simply color all leaves with 1.

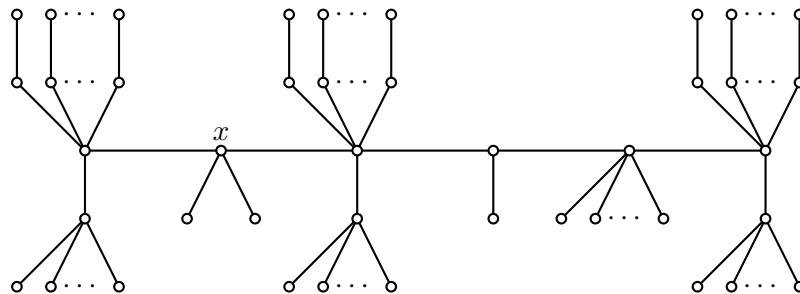


Figure 1. Tree T .

Consider the tree T as shown in Figure 1. The dotted lines indicate that there are at least four pendant paths of length 2 or pendant vertices present. It is not difficult to verify that $\chi_\rho(T) = 5$. On the other hand, as a surprise it was proved in [12, Proposition 10] that in every packing 5-coloring of T , the vertex x is colored 1. Hence the intuition asserting that for every tree T there exists a packing $\chi_\rho(T)$ -coloring such that every vertex adjacent to at least two leaves receives a color larger than 1 is wrong. These peculiar colorings led to the following concept [12].

For a coloring $c : V(G) \rightarrow [k]$, and a color $m \in [k]$, the number of vertices colored by m is denoted by c_m . A $\chi_\rho(G)$ -coloring is *monotone* if $c_m \geq c_{m+1}$ for all $m \in [k-1]$. It seems reasonable that a monotone χ_ρ -coloring exists for every graph. In this respect, the following holds.

Proposition 2.19 [12, Proposition 8]. *If G is a graph and $m \leq \lfloor \frac{\chi_\rho(G)}{2} \rfloor$, then there exists a $\chi_\rho(G)$ -coloring $c : V(G) \rightarrow [k]$ such that $c_m \geq c_n$ for all $n \geq 2m$.*

Contrasting the above intuition, a family of trees T with $\chi_\rho(T) = 3$ was constructed in [12] for which there does not exist a monotone packing 3-coloring.

With these peculiar properties of packing colorings of trees in mind we next move to the complexity of computing the packing chromatic number, in particular in the class of trees.

2.3. Complexity

We are interested in the following decision problem for the packing chromatic number.

PACKING COLORING

Input: A graph G and a positive integer k .

Question: Does G have a packing k -coloring?

First, since the VERTEX COVER PROBLEM is NP-complete for diameter 2 graphs, Corollary 2.8 implies that PACKING COLORING is NP-complete for general graphs. Note that the structure of graphs having packing chromatic number at most 3 is described by Proposition 2.4, and this result implies the existence of a polynomial algorithm to decide whether the packing chromatic number is at most 3. On the other hand, Goddard *et al.* proved that it is NP-hard to decide if an arbitrary graph has a $(1, 1, 2)$ -packing coloring. Using that result, they established the first NP-completeness theorem for PACKING COLORING.

Proposition 2.20 [35, Theorem 4.2]. *PACKING COLORING is NP-complete for $k = 4$, even when restricted to planar graphs.*

Along with the investigations from the end of Section 2.2 it was conjectured that PACKING COLORING might be difficult on the class of trees. This was confirmed by Fiala and Golovach with the following breakthrough result.

Theorem 2.21 [25, Theorem 1]. *PACKING COLORING is NP-complete for trees.*

Fiala and Golovach also initiated the study of the complexity of PACKING COLORING in chordal graphs with a given diameter. By Corollary 2.8, in chordal graphs with diameter 2 the packing chromatic number can be computed efficiently

as a result of the fact that the independence number in chordal graphs can be computed efficiently, see [34]. Recently, Kim, Lidický, Masařík, and Pfender made a further investigation and obtained the following dichotomy with the mentioned result on diameter 2 chordal graphs.

Theorem 2.22 [42, Theorem 5]. *PACKING COLORING is NP-complete even when restricted to chordal graphs with diameter at least 3.*

On the positive side, the packing chromatic number of an interval graph of order n and diameter d can be solved in $O(n^{d \ln(5d)})$ time [42, Theorem 6]. In addition, PACKING COLORING is solvable in polynomial time for graphs that have bounded tree-width and bounded diameter [25, Corollary 2]. Moreover, as noted in [35], the packing chromatic number of a split graph is equal to its clique number, hence PACKING COLORING is also polynomial on split graphs. Further, Argirollo, Nasini, and Torres proved in [1, Theorem 9] that for every fixed k , PACKING COLORING is solvable in polynomial time for the class of $(k, k - 4)$ graphs, which are the graphs in which no set of at most k vertices induces more than $k - 4$ distinct paths on four vertices. The same group of authors proved that $\chi_\rho(G)$ can be computed in polynomial time for the class of partner limited graphs [2, Theorem 4.14], and for an infinite subclass of lobster graphs, including caterpillars [2, Theorem 3.9].

3. SUBCUBIC GRAPHS

The question of whether the packing chromatic number of (sub)cubic graphs is bounded was open for several years. The answer is positive for several families of such graphs, for instance for (all subgraphs of) the infinite 3-regular tree (see Section 2.2), for (all subgraphs of) the infinite hexagonal lattice (see Section 6.1), for Sierpiński-type graphs (see Section 5.2), and for the so-called H -graphs [19, 52, 68]. Additional exact values or the upper bounds of the packing chromatic numbers of some additional (sub)cubic graphs can be found in [7, 35, 50, 52].

For a long time it was not known whether there are subcubic graphs with a large packing chromatic number. Gastineau and Togni [32, Proposition 6] provided an example of a cubic graph of order 38, whose packing chromatic number is 13 (and also an example of a cubic graph of order 24 and packing chromatic number 11). They also posed the question of whether there exists a (sub)cubic graph with larger packing chromatic number. The positive answer was derived by Brešar *et al.* in [13, Corollary 2.2], where a cubic graph on 78 vertices with packing chromatic number at least 14 was constructed. After these developments, using a probabilistic approach, Balogh, Kostochka, and Liu provided the following breakthrough.

Theorem 3.1 [3, Theorem 1]. *For each fixed integer $k \geq 12$, almost every cubic graph G of order n and girth at least $2k + 2$ satisfies $\chi_\rho(G) > k$.*

The proof of Theorem 3.1 is involved and uses the so-called configuration model technique, but does not give an explicit construction of a family of subcubic graphs with unbounded packing chromatic number. Such a family of graphs was independently from [3] provided by Brešar and Ferme [8]. Let us present their construction.

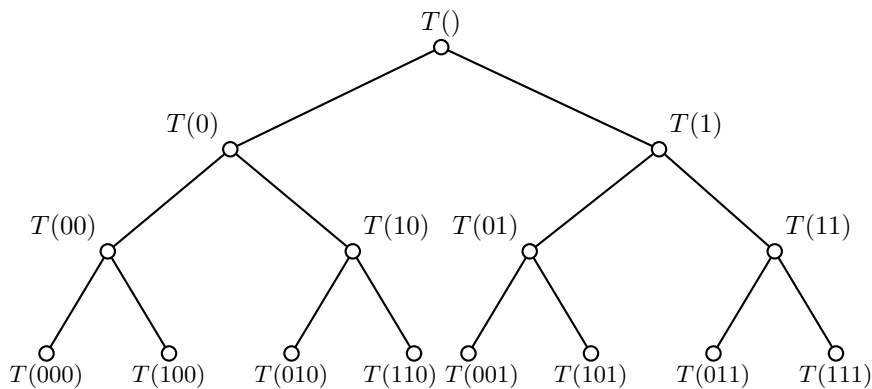


Figure 2. The labelling of the vertices of the perfect binary tree T of depth 3.

Given a perfect binary tree T , vertices are labelled as follows. The root gets the empty label, $T()$, while any other vertex gets the label as follows: if a vertex is the left (respectively, right) child, then 0 (respectively, 1) is added in front of the label of its parent. (For instance, the right child of $T(001)$ gets the label $T(1001)$.) The labelling of the perfect binary tree of depth 3 is shown in Figure 2.

Next, for a positive integer k , the graph G_k is constructed as follows. First, take 5 copies of the perfect binary tree of depth k (having 2^k leaves), and denote them by A, B, C, D and X . Further, for each binary k -tuple $\alpha_1 \cdots \alpha_k$, where $\alpha_i \in \{0, 1\}$, take a copy of the graph G_0 , shown in Figure 3, and denote it by $G_0(\alpha_1 \cdots \alpha_k)$. Then identify the vertex $a \in V(G_0(\alpha_1 \cdots \alpha_k))$ with the leaf $A(\alpha_1 \cdots \alpha_k)$, the vertex $b \in V(G_0(\alpha_1 \cdots \alpha_k))$ with the leaf $B(\alpha_1 \cdots \alpha_k)$, the vertex $c \in V(G_0(\alpha_1 \cdots \alpha_k))$ with the leaf $C(\alpha_1 \cdots \alpha_k)$, the vertex $d \in V(G_0(\alpha_1 \cdots \alpha_k))$ with the leaf $D(\alpha_1 \cdots \alpha_k)$, and the vertex $x \in V(G_0(\alpha_1 \cdots \alpha_k))$ with the leaf $X(\alpha_1 \cdots \alpha_k)$. Clearly, the resulting graph G_k is subcubic.

Using several properties involving distances between vertices in G_0 and between vertices in a perfect binary tree, the following result was proved in [8].

Theorem 3.2 [8, Theorem 6]. *If k is a positive integer, then $\text{diam}(G_k) \leq 2k + 6$ and $\chi_\rho(G_k) \geq 2k + 9$.*

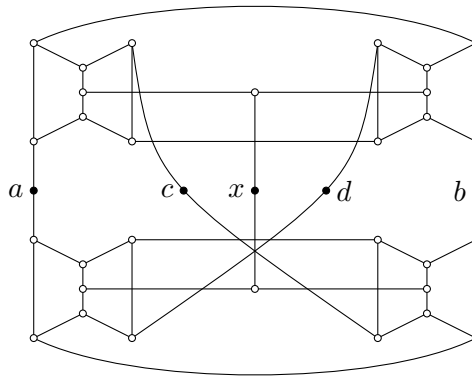


Figure 3. Graph G_0 .

Theorem 3.2 implies that for the family $\{G_k, k \geq 1\}$ there does not exist an integer N such that $\chi_\rho(G_k) \leq N$, for all $k \geq 1$.

In view of Theorem 3.1, Brešar and Ferme posed the question of whether the packing chromatic number is bounded in the class of subcubic planar graphs [8, Question 7]. The problem is not yet resolved even if one further restricts to subcubic outerplanar graphs. In the next theorem we collect several results concerning the packing chromatic number of subcubic outerplanar graphs, due to Gastineau, Holub, and Togni. Note that in a plane drawing of a 2-connected outerplanar graph G there exists a cycle C containing all vertices of G , with non-crossing chords of C dividing the interior of C into faces. A face is *internal* if it is bounded by more than two chords of C . (Several of these results give an affirmative answer to the problem of Brešar and Ferme in some specific subclasses.)

Theorem 3.3 [30, Theorems 3, 7-9, 11, Proposition 12].

- (i) *If G is a 2-connected subcubic outerplanar graph with no internal face, then $\chi_\rho(G) \leq 15$.*
- (ii) *If G is a 2-connected subcubic outerplanar graph with r internal faces, then $\chi_\rho(G) \leq 17 \cdot 6^{3r} - 2$.*
- (iii) *If G is a connected subcubic outerplanar graph with r (non external) faces, then $\chi_\rho(G) \leq 9 \cdot 6^r - 2$.*
- (iv) *If G is a 2-connected subcubic outerplanar graph with exactly one internal face, then $\chi_\rho(G) \leq 51$.*
- (v) *If G is a connected subcubic outerplanar graph with no internal face such that the block graph of G is a path, then $\chi_\rho(G) \leq 305$.*

(vi) *There exists an infinite family of 2-connected subcubic outerplanar graphs without internal faces and with packing chromatic number 5.*

For the special case of bipartite, 2-connected, subcubic outerplanar graphs, Brešar, Gastineau, and Togni proved:

Theorem 3.4 [10, Theorem 1]. *If G is a 2-connected, bipartite, subcubic outerplanar graph, then $\chi_\rho(G) \leq 7$. Moreover, the bound is sharp.*

4. OPERATIONS WITH VERTICES AND EDGES

In this section we survey the investigations concerning the effect on the packing chromatic number of a graph after local operations are performed: vertex deletion, edge deletion, edge contraction, or edge subdivision.

4.1. Vertex deletion

From Observation 2.1 we infer that $\chi_\rho(G - v) \leq \chi_\rho(G)$ holds for every vertex $v \in V(G)$. On the other hand, the difference $\chi_\rho(G) - \chi_\rho(G - v)$ can be arbitrarily large, see [13]. The situation is different for leaves.

Lemma 4.1 [44, Lemma 2.4]. *If x is a leaf of a graph G , then $\chi_\rho(G) - 1 \leq \chi_\rho(G - x) \leq \chi_\rho(G)$.*

A graph G is a *packing chromatic vertex-critical graph* (or shorter, a χ_ρ -vertex-critical graph) if for every vertex x of G we have $\chi_\rho(G - x) < \chi_\rho(G)$. In the case when $\chi_\rho(G) = k$, we say that G is k - χ_ρ -vertex-critical. This concept was introduced by Klavžar and Rall in [44]. They proved that also in the case of χ_ρ -vertex-critical graphs, the difference between $\chi_\rho(G)$ and $\chi_\rho(G - v)$ can be arbitrary. Even more, setting $\Delta_{\chi_\rho}(G) = \{\chi_\rho(G) - \chi_\rho(G - x) : x \in V(G)\}$, the following holds.

Theorem 4.2 [44, Theorem 3.1]. *Let $S = \{1, s_1, \dots, s_r\}$ be a set of positive integers, where $r \geq 1$. If for every $i \in [r]$ we have $\sum_{j=1, j \neq i}^r s_j \geq s_i - 1$, then there exists a χ_ρ -vertex-critical graph G such that $\Delta_{\chi_\rho}(G) = S$.*

Among numerous additional results on χ_ρ -vertex critical graphs from [44] we extract that (i) a graph G is 3- χ_ρ -vertex-critical if and only if $G \in \{C_3, P_4, C_4\}$ and that (ii) a k - χ_ρ -vertex-critical caterpillar exists if and only if $k \leq 7$ [44]. Moreover, Cartesian products of graphs are a rich source of χ_ρ -vertex-critical graphs as the next result indicates.

Theorem 4.3 [44, Theorem 6.3]. *If G and H are connected, vertex-transitive graphs on at least two vertices and $\text{diam}(G) + \text{diam}(H) \leq \chi_\rho(G)$, then $G \square H$ is χ_ρ -vertex-critical.*

4.2. Edge deletion

Applying Observation 2.1 once more we get that $\chi_\rho(G - e) \leq \chi_\rho(G)$ holds for every edge $e \in E(G)$. On the other hand we have:

Theorem 4.4 [9, Theorem 1]. *If e is an edge of a graph G , then $\chi_\rho(G - e) \geq \frac{\chi_\rho(G)+1}{2}$. Moreover, the bound is sharp.*

Brešar and Ferme also provided a realization result that shows that $\chi_\rho(G - e)$ can achieve any of the integers between the above two bounds.

Theorem 4.5 [9, Theorem 2]. *For an arbitrary integer $r \geq 3$ and for an arbitrary integer s , where $\frac{r+1}{2} \leq s \leq r$, there exists a graph G with an edge e such that $\chi_\rho(G) = r$ and $\chi_\rho(G - e) = s$.*

Theorem 4.5 directly implies the following result that was proved earlier in [13].

Proposition 4.6 [13, Proposition 3.1]. *For every positive integer t there exists a graph G with an edge e such that $\chi_\rho(G) - \chi_\rho(G - e) \geq t$.*

Here is a construction demonstrating Proposition 4.6. Let $k \geq 4$ and $n \geq 2k - 2$. Denote by A and B two copies of K_n and let $a, a' \in V(A)$, $b, b' \in V(B)$. Form the graph $G_{n,k}$ as follows. Take the disjoint union of A and B , add the edges ab and $a'b'$ and finally replace the edge $a'b'$ with a path of length $2k - 1$. The packing chromatic number of $G_{n,k}$ is at least $2n - 2$, but if we remove the edge ab from $G_{n,k}$, then the packing chromatic number of the obtained graph is at most $2(n - k) + 4$, which implies that the difference between $\chi_\rho(G_{n,k})$ and $\chi_\rho(G_{n,k} - ab)$ is arbitrary large, namely $\chi_\rho(G_{n,k}) - \chi_\rho(G_{n,k} - ab) \geq 2k - 6$.

In Section 4.1 packing chromatic vertex-critical graphs were defined with respect to vertex-deleted subgraphs. To consider all subgraphs, call a graph G *packing chromatic critical*, or shorter, χ_ρ -critical, if $\chi_\rho(H) < \chi_\rho(G)$ for every proper subgraph H of G . If G is χ_ρ -critical and $\chi_\rho(G) = k$ we also say that G is k - χ_ρ -critical. This concept was studied by Brešar and Ferme in [9], where it is proved among other results that K_2 is the only 2- χ_ρ -critical graph, that C_3 and P_4 are the only 3- χ_ρ -critical graphs, and that a k - χ_ρ -critical caterpillar T exists if and only if $k \leq 7$. In addition, they also characterized χ_ρ -critical graphs with diameter 2 as follows.

Theorem 4.7 [9, Theorem 13]. *If G is a graph with diameter 2, then G is χ_ρ -critical if and only if for each edge $e = u_1u_2 \in E(G)$ at least one of the following statements holds:*

- (i) $\alpha(G - e) > \alpha(G)$;
- (ii) *there exists a vertex $y \in N[u_i]$ such that $d_{G-e}(y, u_j) \geq 3$, where $\{i, j\} = \{1, 2\}$, and a maximum independent set A such that $A \cap \{y, u_j\} = \emptyset$.*

Next, consider block graphs. Recall that a *block* of a graph G is a maximal connected subgraph of G that has no cut vertices (or in other words, a maximal 2-connected subgraph or a K_2 whose edge is a cut-edge of G). A *block graph* is a graph in which each block is a complete graph. The next two results give characterizations of χ_ρ -critical block graphs with diameter 2 or 3.

Theorem 4.8 [9, Theorem 14]. *If G is a block graph with diameter 2, then G is χ_ρ -critical if and only if $\delta(G) \geq 2$.*

If a block graph G has diameter 3, then the central vertices of G induce a block, which we call the *central block*, while other blocks in G are *side blocks*.

Theorem 4.9 [9, Theorem 15]. *Let G be a block graph with diameter 3, and let B be the central block of G . Graph G is χ_ρ -critical if and only if one of the following three possibilities holds for the vertices of B .*

- (a) *All vertices in B have degree $n(B)$.*
- (b) *All vertices in B have degree $n(B) + 1$, and exactly $n(B) - 1$ vertices of B have two leaf neighbors.*
- (c) *For each vertex x of B , at least one of the following three properties holds:*
 - (c1) *x belongs to at least one side block of order at least 4, but does not have any leaf neighbor,*
 - (c2) *x belongs to at least two side blocks of order 3, but does not have any leaf neighbor,*
 - (c3) *x has degree $n(B) + 1$ and has two neighbors, which are both leaves, in addition, at least one vertex in B satisfies one of the properties (c1) or (c2).*

4.3. Edge contraction and edge subdivision

Let $G|e$ denote the graph obtained from G by contracting its edge e . Brešar *et al.* determined the lower and the upper bound for $\chi_\rho(G|e)$ as follows.

Theorem 4.10 [13, Theorem 3.2]. *If G is a graph and $e \in E(G)$, then*

$$\chi_\rho(G) - 1 \leq \chi_\rho(G|e) \leq 2\chi_\rho(G).$$

In [35] it was observed that $\chi_\rho(S(G)) = 3$ for every connected bipartite graph G with at least two edges. Brešar, Klavžar, and Rall further investigated the effect of edge subdivision on the packing chromatic number.

Proposition 4.11 [12, Lemma 6]. *If $n \geq 3$, then $\chi_\rho(S(K_n)) = n + 1$.*

Theorem 4.12 [12, Theorem 7]. *If G is a connected graph of order at least 3, then*

$$\omega(G) + 1 \leq \chi_\rho(S(G)) \leq \chi_\rho(G) + 1.$$

Moreover, the bounds are best possible.

For a single subdivided edge, Brešar, Klavžar, Rall, and Wash proved the following result.

Theorem 4.13 [13, Theorem 2.3]. *If G is a graph with $\chi_\rho(G) = j$, then*

$$\lfloor j/2 \rfloor + 1 \leq \chi_\rho(S_e(G)) \leq j + 1.$$

Moreover, for any $k \geq 2$ there exists a graph G with an edge e such that $k = \chi_\rho(G) = \chi_\rho(S_e(G)) - 1$.

Gastineau and Togni [32] asked whether the subdivision of any subcubic graph is packing 5-colorable, while Brešar, Klavžar, Rall, and Wash [14] turned the question into a conjecture.

Conjecture 4.14 [14, Conjecture 1.1]. *If G is a subcubic graph, then*

$$\chi_\rho(S(G)) \leq 5.$$

Since $\chi_\rho(S(G)) = 3$ for any connected bipartite graph G of order at least 3, and every subcubic graph is a subgraph of a cubic graph, it suffices to consider only cubic non-bipartite graphs. Moreover, since $\chi_\rho(S(K_4)) = 5$, it suffices to consider Conjecture 4.14 for 3-chromatic cubic graphs. Since $\chi_\rho(S(K_{n,n,n})) \rightarrow \infty$ as $n \rightarrow \infty$ (see [14, Proposition 2.2]), the conjecture does not hold for all 3-chromatic graphs.

The best general result on Conjecture 4.14 is due to Balogh, Kostochka, and Liu.

Theorem 4.15 [4, Theorem 8]. *For every connected subcubic graph G , the graph $S(G)$ has a packing 8-coloring such that color 8 is used at most once.*

Some additional support for Conjecture 4.14 will be given in Section 7.3.

4.4. Multiple and super subdivisions

Given a graph G and a positive integer i , the graph $S_i(G)$ is obtained from G by subdividing every edge of G precisely i times. In other words, $S_i(G)$ is obtained from G by replacing every edge of G with a path of length $i + 1$. Note that $S_1(G) = S(G)$.

Recall that $\chi_\rho(S_1(K_n)) = n + 1$, for $n \geq 3$. For multiple subdivisions of complete graphs we have the next result.

Proposition 4.16 [14, Proposition 5.1]. *If $n \geq 3$ and $i \geq 3$, then*

$$\chi_\rho(S_i(K_n)) = \begin{cases} 3, & i \equiv 3 \pmod{4}, \\ 4, & \text{otherwise.} \end{cases}$$

Moreover, $\chi_\rho(S_2(K_n)) \rightarrow \infty$ as $n \rightarrow \infty$.

Using the hereditary nature of the packing chromatic number, we get the following corollary.

Corollary 4.17 [14, Corollary 5.2]. *If G is a connected graph of order at least 3 and $i \geq 3$, then $3 \leq \chi_\rho(S_i(G)) \leq 4$.*

This result can be strengthened for trees by including the cases $i = 1$ and $i = 2$.

Theorem 4.18 [14, Theorem 5.3]. *If $i \geq 1$, then*

$$\max\{\chi_\rho(S_i(T)) : T \text{ is a tree}\} = \begin{cases} 3, & i \text{ odd,} \\ 4, & i \text{ even.} \end{cases}$$

Given a graph G and a positive integer m , the *super subdivision*, $SS_m(G)$, of G is the graph obtained from G by replacing each of its edges with a complete bipartite graph $K_{2,m}$. In other words, $SS_m(G)$ is obtained from G by first multiplying each of its edges m times and then making the subdivision graph from the resulting multigraph. We next present some results on the packing chromatic number of super subdivision graphs due to Lemdani, Abbas and Ferme.

Proposition 4.19 [53, Corollary 2.4]. *If $n \geq 3$ and $m \geq 1$, then $\chi_\rho(SS_m(K_n)) = n + 1$.*

Proposition 4.20 [53, Proposition 2.3]. *If $m \geq 1$ and G is a connected graph with $n(G) \geq 3$, then $\omega(G) + 1 \leq \chi_\rho(SS_m(G)) \leq \chi_\rho(G) + 1$.*

Proposition 4.21 [53, Proposition 2.6]. *If G is a graph and $m > \chi_\rho(G)/\delta(G)$, then $\chi_\rho(SS_m(G)) = \chi_\rho(SS_{m+1}(G))$.*

Proposition 4.22 [53, Proposition 2.7]. *If G is a bipartite graph with $n(G) \geq 3$ and $m \geq 1$, then $\chi_\rho(SS_m(G)) = 3$.*

Note that Proposition 4.19 presents a generalization of Proposition 4.11 and that the bounds of Proposition 4.20 generalize Theorem 4.12.

5. PACKING COLORINGS OF SPECIFIC CLASSES

In this section, we first present results about the packing chromatic number of Cartesian products of graphs, follow with results about Sierpiński graphs, and end by mentioning numerous results on the packing chromatic number of specific classes of graphs.

5.1. Cartesian products

Brešar, Klavžar and Rall proved the following general lower bound for the packing chromatic number of Cartesian products of graphs.

Theorem 5.1 [12, Theorem 1]. *If G and H are connected graphs of order at least 2, then*

$$\chi_\rho(G \square H) \geq (\chi_\rho(G) + 1)n(H) - \text{diam}(G \square H)(n(H) - 1) - 1.$$

Since the Cartesian product operation is commutative, the roles of G and H in the bound of Theorem 5.1 can be interchanged. Further, because $\text{diam}(G \square K_n) = \text{diam}(G) + 1$, Theorem 5.1 yields:

Corollary 5.2 [12, Corollary 2]. *If G is a graph of order at least 2 and $n \geq 2$, then*

$$\chi_\rho(G \square K_n) \geq n\chi_\rho(G) - (n - 1)\text{diam}(G).$$

Applying Proposition 2.5 we get $\chi_\rho(G \square H) \leq n(G)n(H) - \alpha(G \square H) + 1$. Since $\alpha(G \square H) \geq \alpha(G)\alpha(H) + \min\{n(G) - \alpha(G), n(H) - \alpha(H)\}$ holds [66], we get the following upper bound from [12]:

$$(1) \quad \chi_\rho(G \square H) \leq n(G)n(H) - \alpha(G)\alpha(H) - \min\{n(G) - \alpha(G), n(H) - \alpha(H)\} + 1.$$

In [38] it was proved that if G is a graph and H is a bipartite graph, then $\alpha(G \square H) \geq \alpha_2(G)n(H)/2$. This yields the following upper bound for a graph G and a bipartite graph H :

$$\chi_\rho(G \square H) \leq n(G)n(H) - \alpha_2(G)n(H)/2 + 1.$$

Products of complete graphs

From (1) we get that if $2 \leq m \leq n$, then $\chi_\rho(K_m \square K_n) \leq m(n - 1) + 1$. Theorem 5.1 applied on $K_m \square K_n$ shows that this bound is sharp; that is,

$$\chi_\rho(K_m \square K_n) = m(n - 1) + 1,$$

for $2 \leq m \leq n$ [12, p. 2305]. The special case of $\chi_\rho(K_m \square K_m)$ was later, independently obtained in [56, Theorem 2.1(i)]. Let $G^{k, \square}$ denote the k -th Cartesian power of G . Nasini, Severin, and Torres proved:

Theorem 5.3 [56, Theorem 2.1(ii)]. *If $n \geq 3$, then $\chi_\rho(K_n^{3, \square}) = n^3 - n^2 - n + 2$.*

Using Corollary 5.2, they derived the following bound.

Proposition 5.4 [56, Lemma 2.1]. *If n and k are positive integers larger than 1, then $\chi_\rho(K_n^{k, \square}) \geq n^k + k - 1 - \sum_{i=1}^{k-1} n^i$.*

Nasini *et al.* also made some computational experiments on products of complete graphs up to 10000 vertices, obtaining bounds in most of the cases.

Important examples of the Cartesian product of complete graphs are hypercubes. The k -dimensional hypercube Q_k is the Cartesian product of k copies of K_2 , that is, $Q_k = K_2^{k, \square}$. Exact values of $\chi_\rho(Q_k)$ are known up to dimension $k = 8$.

Theorem 5.5. $\chi_\rho(Q_1) = 2$, $\chi_\rho(Q_2) = 3$, $\chi_\rho(Q_3) = 5$, $\chi_\rho(Q_4) = 7$, $\chi_\rho(Q_5) = 15$, $\chi_\rho(Q_6) = 25$, $\chi_\rho(Q_7) = 49$, and $\chi_\rho(Q_8) = 95$.

The first five cases of Theorem 5.5 are from [35, Proposition 7.2], while the remaining three values were determined by Torres and Valencia-Pabon [65, Section 3]. Goddard *et al.* [35] proved the following asymptotic result.

Proposition 5.6 [35, Proposition 7.3]. $\chi_\rho(Q_k) \sim \left(\frac{1}{2} - O\left(\frac{1}{k}\right)\right) 2^k$.

Using the fact that $Q_k = Q_{k-1} \square K_2$ and applying Corollary 5.2, Torres and Valencia-Pabon [65] proved the following explicit lower bound and upper bound for the packing chromatic number of hypercubes, respectively.

Corollary 5.7 [65, Corollary 1.1]. *If $k \geq 2$, then, $\chi_\rho(Q_k) \geq 2\chi_\rho(Q_{k-1}) - (k-1)$.*

Theorem 5.8 [65, Theorem 3.1]. *If $k \geq 4$, then*

$$\chi_\rho(Q_k) \leq 3 + 2^k \left(\frac{1}{2} - \frac{1}{2^{\lceil \log_2 k \rceil}} \right) - 2 \left\lfloor \frac{k-4}{2} \right\rfloor.$$

Products of paths and cycles

In [35, Proposition 6.1], Goddard *et al.* determined $\chi_\rho(P_m \square P_n)$ for all $m \leq 5$ and $n \geq 2$. Besides the sporadic values for small n , the values are $\chi_\rho(P_2 \square P_n) = 5$ for $n \geq 6$, $\chi_\rho(P_3 \square P_n) = 7$ for $n \geq 12$, $\chi_\rho(P_4 \square P_n) = 8$ for $n \geq 10$, and $\chi_\rho(P_5 \square P_n) = 9$ for $n \geq 10$.

Shao and Vesel [60] (see also [61]) presented an integer linear programming model and a satisfiability test model for the packing coloring problem. Applying these models they gave some bounds and values for the packing chromatic number of Cartesian products of paths and cycles and the following exact result.

Proposition 5.9 [60, Proposition 1]. *If $11 \leq n \leq 17$, then $\chi_\rho(P_n \square C_6) = 12$.*

In [68, Theorem 8] a particular related result was proved asserting that if $n \geq 6$ and $n \equiv 0 \pmod{6}$, then $\chi_\rho(P_2 \square C_n) \leq 5$.

Jacobs, Jonck, and Joubert studied the packing chromatic number of the Cartesian product of cycles of specific length. More precisely, in [41, Theorem 2.1 and Theorem 3.4] they proved that if $t \geq 3$, then $9 \leq \chi_\rho(C_4 \square C_{4t}) \leq 11$. Moreover, if in addition t is divisible by 4, then $\chi_\rho(C_4 \square C_{4t}) = 9$ [41, Corollary 3.5].

5.2. Sierpiński-type graphs

If n and k are positive integers, then the *Sierpiński graph* S_k^n [43] has the vertex set

$$V(S_k^n) = \{u_1 \cdots u_n : u_i \in [k]\}.$$

Vertices $u = u_1 \cdots u_n \in V(S_k^n)$ and $v = v_1 \cdots v_n \in V(S_k^n)$ are adjacent if there exists $i \in [n]$ such that

- (i) $u_j = v_j$ if $j < i$,
- (ii) $u_i \neq v_i$, and
- (iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

Theorem 5.10 [11, Theorem 1], [19, Theorem 2]. *If $n \geq 1$, then*

$$\chi_\rho(S_3^n) = \begin{cases} 3, & n = 1, \\ 5, & n = 2, \\ 7, & n \in \{3, 4\}, \\ 8, & n \geq 5. \end{cases}$$

Theorem 5.10 was proved in [11, Theorem 1] up to the fact that if $n \geq 5$, then $8 \leq \chi_\rho(S_3^n) \leq 9$. The missing cases were then completed in [19, Theorem 2].

Brešar and Ferme [7] studied the packing chromatic numbers of Sierpiński graphs S_k^n for $k \geq 4$ and proved:

Theorem 5.11 [7, Theorem 1]. *If $k \geq 4$, then the sequence $(\chi_\rho(S_k^n))_{n \in \mathbb{N}}$ is unbounded.*

Generalized Sierpiński graphs

Let G be a graph on the vertex set $[k]$. Then the *generalized Sierpiński graph* S_G^n has the same vertex set as S_k^n , that is, $V(S_G^n) = [k]^n$. Vertices $u = u_1 \cdots u_n \in V(S_G^n)$ and $v = v_1 \cdots v_n \in V(S_G^n)$ are adjacent if there exists $i \in [n]$ such that

- (i) $u_j = v_j$ if $j < i$,
- (ii) $u_i \neq v_i$ and $\{u_i, v_i\} \in E(G)$, and
- (iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

Note that $S_{K_k}^n = S_k^n$. Theorem 5.11 immediately implies:

Corollary 5.12 [7, Corollary 2]. *If a graph G contains a subgraph isomorphic to K_k , where $k \geq 4$, then the sequence $(\chi_\rho(S_G^n))_{n \in \mathbb{N}}$ is unbounded.*

Hence $(\chi_\rho(S_{K_4}^n))_{n \in \mathbb{N}}$ is unbounded. On the other hand, Brešar and Ferme [7] and Korže and Vesel [47] determined the exact values for $\chi_\rho(S_G^n)$, where G is a connected graph on four vertices that is not a complete graph.

Theorem 5.13 [7, Theorem 5]. *If $n \geq 1$, then*

$$\chi_\rho(S_{C_4}^n) = \begin{cases} 3, & n = 1, \\ 4, & n = 2, \\ 5, & n \geq 3. \end{cases}$$

Theorem 5.14 [7, Theorem 6]. *If $n \geq 1$, then*

$$\chi_\rho(S_{P_4}^n) = \begin{cases} 3, & n = 1, \\ 4, & n = 2, \\ 5, & n \geq 3. \end{cases}$$

In [47, Theorem 3] Korže and Vesel extended Theorem 5.14 to all graphs $S_{P_k}^n$, $n \geq 2$, $k \geq 3$. They have also considered $S_{C_k}^n$ for $n \geq 2$ and $k \geq 3$ and determined their packing chromatic numbers in many cases, see [47, Theorem 4].

Theorem 5.15 [7, Theorem 7]. *If $n \geq 1$, then*

$$\chi_\rho(S_{K_{1,3}}^n) = \begin{cases} 2, & n = 1, \\ 3, & n \geq 2. \end{cases}$$

Theorem 5.16 [47, Theorem 1]. *If $n \geq 1$, then*

$$\chi_\rho(S_{K_4-e}^n) = \begin{cases} 3, & n = 1, \\ 6, & n = 2, \\ 8, & n = 3, \\ 9, & n = 4, \\ 10, & n \geq 5. \end{cases}$$

The values in Theorem 5.16 for $n \leq 3$ were first determined in [7, Theorem 9]. In addition, for $n \geq 4$ the values of $\chi_\rho(S_{K_4-e}^n)$ were shown to be bounded between 8 and 11.

Theorem 5.17 [47, Theorem 2]. *If $n \geq 1$, then*

$$\chi_\rho(S_{paw}^n) = \begin{cases} 3, & n = 1, \\ 5, & n = 2, \\ 7, & n = 3, \\ 8, & n \geq 4. \end{cases}$$

Sierpiński triangle graphs

If $n \geq 0$, then the *Sierpiński triangle graph* ST_3^n is the graph obtained from the Sierpiński graph S_3^{n+1} by contracting all the edges that lie in no triangle, see [39].

Theorem 5.18 [47, Theorem 5]. *If $n \geq 0$, then*

$$\chi_\rho(ST_3^n) = \begin{cases} 3, & n = 0, \\ 4, & n = 1, \\ 8, & n = 2, \\ 12, & n = 3. \end{cases}$$

Moreover, $12 \leq \chi_\rho(ST_3^4) \leq 15$, $12 \leq \chi_\rho(ST_3^5) \leq 19$, and if $n \geq 6$, then $12 \leq \chi_\rho(ST_3^n) \leq 20$.

For $n \leq 2$, Theorem 5.18 was earlier obtained by Brešar and Ferme [7, Theorem 11]. They also proved that $\chi_\rho(ST_3^n) \leq 31$ holds for every n .

The Sierpiński triangle graphs are subgraphs of the infinite triangular lattice, therefore in the limit they yield a subgraph of the infinite triangular lattice. It is interesting that the packing chromatic number of the triangular lattice is infinite, but as we can see, in the class of Sierpiński triangle graphs this number is bounded.

5.3. Corona graphs and specific classes

A *corona* of a graph G and a graph H is the graph $G \odot H$ obtained from G and $n(G)$ copies of H such that each vertex of G is adjacent to all vertices of its own copy of H . In particular, $G \odot K_1$ is the graph obtained from G by attaching exactly one leaf to each vertex of G . More generally, if $p \geq 1$, then the p -*corona* $G \odot pK_1$ is the graph obtained from G by adding p private leaves to every vertex of G .

Since every p -corona of a path is a caterpillar, from a result of Sloper [62] it follows that if $p \geq 1$ and $n \leq 34$, then $\chi_\rho(P_n \odot pK_1) \leq 6$; otherwise $\chi_\rho(P_n \odot pK_1) \leq 7$, see [50, pp. 667]. William and Roy [68, Proposition 6] proved that if $n \geq 8$, then $\chi_\rho(P_n \odot K_1) \leq 5$. Laïche, Bouchemakh, and Sopena extended this result as follows.

Theorem 5.19 [50, Theorem 4]. *If $n \geq 1$, then*

$$\chi_\rho(P_n \odot K_1) = \begin{cases} 2, & n = 1, \\ 3, & n \in \{2, 3\}, \\ 4, & 4 \leq n \leq 9, \\ 5, & n \geq 10. \end{cases}$$

In [67] it was proved that if $n \geq 6$ and $n \equiv 0, 2 \pmod{4}$, then $\chi_\rho(C_n \odot K_1) \leq 5$. A complete picture of these coronae is the following.

Theorem 5.20 [50, Theorem 5]. *If $n \geq 3$, then*

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4, & n \in \{3, 4\}, \\ 5, & n \geq 5. \end{cases}$$

In addition, the paper [50] reports the exact values of $\chi_\rho(P_n \odot pK_1)$ and of $\chi_\rho(C_n \odot pK_1)$ for every $p \geq 2$. We explicitly state only the values $\chi_\rho(C_n \odot 3K_1)$ because the result nicely reveals the intrinsic difficulty of the packing chromatic number.

Theorem 5.21 [50, Theorem 12]. *If $n \geq 3$, then*

$$\chi_\rho(C_n \odot 3K_1) = \begin{cases} 4, & n = 3, \\ 5, & n = 4, \\ 7, & n \in \{7, \dots, 13, 15, \dots, 22, 24, \dots, 27, 30, \dots, 36, 39, 40, 41\} \\ & \cup \{45, 47, \dots, 50, 53, 54, 55, 59, 62, 63, 64, 68, 77, 78, 91\}, \\ 6, & \text{otherwise.} \end{cases}$$

We next briefly mention some specific classes of graphs for which (or for some subfamilies of them) the packing chromatic number has been determined. The interested reader is invited to check the sources for definitions of the classes and the results.

William, Roy, and Rajasingh in [67] investigated, among others, kC_{2n} -linear graphs, gear graphs, barbell graphs, tadpole graphs, and lollipop graphs. William and Roy [68] further studied the packing chromatic number of windmill graphs and generalized theta graphs. Tight bounds for the packing chromatic number of the latter graphs were derived by Laïche, Bouchemakh and Sopena [51]. They also characterized undirected generalized theta graphs with packing chromatic number k for every $k \geq 3$. Moreover, they showed that the packing chromatic number of any oriented generalized theta graph lies between 2 and 5, and that both these bounds are tight. Roy [59] considered the packing chromatic numbers for certain fan and wheel related graphs. Rajalakshmi and Venkatachalam [58] determined the packing chromatic number of graphs derived from helm graphs, notably for the line graphs of helm graphs, while in [57] they considered double wheel graphs. The packing chromatic number of the so-called transformation graphs of paths, cycles, wheels, complete graphs, and stars was considered in [20].

Finally, the packing coloring concept can also be considered in directed graphs. So far this has been done for oriented graphs by Laïche *et al.* [50]. More precisely, the distance in an oriented graph considered in [50] between vertices u and v is the length of a shortest directed path from u to v or from v to u . With the distance defined in this manner, the packing chromatic number of an oriented graph is then defined in the natural way. Note that if \vec{G} is an orientation of a graph G , then $\chi_\rho(\vec{G}) \leq \chi_\rho(G)$. Among several interesting results from [50] we state the following one.

Theorem 5.22 [50, Theorem 21]. *If T is a tree and \vec{T} is an orientation of T , then $\chi_\rho(\vec{T}) \leq 3$.*

To end this section we mention two recent papers dealing with packing colorings on specific classes of graphs. In particular, some upper and lower bounds on the packing chromatic number of the lexicographic product of graphs were presented by Božović and Peterin [6]. Furthermore, the packing chromatic number of some Moore graphs was studied by Fresán-Figueroa, González-Moreno and Olsen [28].

6. INFINITE GRAPHS

With respect to infinite graphs, the packing chromatic number was mostly studied on different kinds of lattices (and their subgraphs) and on distance graphs. Denoting with P_∞ the *two-way infinite path*, Fiala, Klavžar, and Lidický obtained the following general result.

Proposition 6.1 [26, Corollary 2.2]. *If G is a finite graph, then $\chi_\rho(G \square P_\infty) < \infty$.*

6.1. Square, hexagonal and triangular lattices

Here we consider the square lattice $P_\infty^2 = P_\infty \square P_\infty$, its natural subgraphs, and related lattices $C_n \square P_\infty$.

For the square lattice, Goddard *et al.* [35] proved in their seminal paper that $9 \leq \chi_\rho(P_\infty^2) \leq 23$. In 2009, Fiala *et al.* [26, Theorem 3.11] improved the lower bound to 10. Ekstein, Fiala, Holub, and Lidický [21, Theorem 1] used a computer to further improve the lower bound to 12. The upper bound was lowered to 17 by Soukal and Holub [63, Theorem 1]. Using a SAT-solver, B. Martin, Raimondi, Chen, J. Martin [55] finally obtained the following state of the art on the packing chromatic number of the square lattice.

Theorem 6.2 [55]. $13 \leq \chi_\rho(P_\infty^2) \leq 15$.

From the proof of [35, Proposition 6.1] (by extending the presented color patterns in both directions) one can deduce that $\chi_\rho(P_2 \square P_\infty) = 5$, $\chi_\rho(P_3 \square P_\infty) = 7$, $\chi_\rho(P_4 \square P_\infty) = 8$, and $\chi_\rho(P_5 \square P_\infty) = 9$. Korže and Vesel [45, Propositions 3.1] extended these results to $\chi_\rho(P_6 \square P_\infty) = 10$ and gave the upper bounds in Table 1.

n	7	8	9	10	11	12	13
$\chi_\rho(P_n \square P_\infty) \leq \cdot$	11	12	13	14	14	15	15

Table 1. Upper bounds on $\chi_\rho(P_n \square P_\infty)$, $n \in \{7, 8, \dots, 13\}$.

Korže and Vesel [45, Propositions 4.1] also proved that $\chi_\rho(C_4 \square P_\infty) = 9$ and derived the upper bounds given in Table 2.

n	6	8	10	12
$\chi_\rho(C_n \square P_\infty) \leq \cdot$	13	15	22	17

Table 2. Upper bounds on $\chi_\rho(C_n \square P_\infty)$, $n \in \{6, 8, 10, 12\}$.

The 6-regular lattice $P_\infty^3 = P_\infty \square P_\infty \square P_\infty$ has also been studied. Finbow and Rall [27, Theorem 13] proved that P_∞^3 has infinite packing chromatic number. Moreover, if $m \geq 2$, then $\chi_\rho(P_m \square P_\infty^2) = \infty$; see [26, Theorem 3.8]. On the other hand, by Proposition 6.1 it follows that if $m, n \geq 1$, then $\chi_\rho(P_m \square P_n \square P_\infty) < \infty$. Korže and Vesel [45] studied the packing chromatic number for some infinite subgraphs of $P_2 \square P_\infty^2$. In particular, they proved that $\chi_\rho(P_2 \square P_3 \square P_\infty) \leq 18$ [45, Proposition 4.1].

Hexagonal lattice

The question of what is the maximum packing chromatic number of the infinite hexagonal lattice \mathcal{H} was initiated already in the seminal paper [35]. Firstly, Brešar, Klavžar and Rall bounded $\chi_\rho(\mathcal{H})$ as follows: $6 \leq \chi_\rho(\mathcal{H}) \leq 8$ [12, Theorem 4]. By exhibiting a packing coloring of \mathcal{H} that uses only 7 colors, Fiala, Klavžar, Lidický [26, Theorem 4.1] improved the upper bound to 7. Finally, Korže and Vesel [45] completed the investigation as follows.

Theorem 6.3 [45, Theorem 2.3]. *If \mathcal{H} is the infinite hexagonal lattice, then $\chi_\rho(\mathcal{H}) = 7$.*

A related is the following.

Theorem 6.4 [26, Theorem 4.5]. *If $m \geq 6$, then $\chi_\rho(P_m \square \mathcal{H}) = \infty$.*

Triangular lattice

If we add all edges of the form $[(i, j), (i + 1, j - 1)]$ to P_∞^2 , then the obtained graph is the *infinite planar triangular lattice*, denoted by \mathcal{T} .

Brešar, Klavžar and Rall proposed the following open question. Is it true that $\chi_\rho(\mathcal{T}) = \infty$ [12, Problem 5]? In 2010, Finbow and Rall [27, Theorem 6] gave an affirmative answer. On the other hand, using the following result, where \boxtimes denotes the strong product of graphs, Fiala *et al.* [26] proved that the packing chromatic number is finite for every finite strip of \mathcal{T} .

Theorem 6.5 [26, Theorem 2.1]. *If $n \geq 1$, then $\chi_\rho(K_n \boxtimes P_\infty) < 4^n$ and $\chi_\rho(K_n \boxtimes P_\infty) = \Omega(e^n)$.*

Note that Theorem 6.5 also implies Proposition 6.1.

6.2. Distance graphs

Let $D = \{d_1, \dots, d_k\}$, where each d_i is a positive integer. The *distance graph* $G(\mathbb{Z}, D)$ with the *distance set* D has the set \mathbb{Z} as the vertex set and two distinct vertices $i, j \in \mathbb{Z}$ are adjacent if and only if $|i - j| \in D$. To simplify the notation we will use $D(d_1, \dots, d_k)$ to denote $G(\mathbb{Z}, \{d_1, \dots, d_k\})$.

The study of packing colorings of distance graphs was initiated by Togni [64]. Among other results, he proved the following bounds.

Proposition 6.6 [64, Proposition 1]. *If t is a positive integer, then*

$$\chi_\rho(D(1, t)) \leq \begin{cases} 89, & t = 2q + 1, q \geq 35, \\ 40, & t = 2q + 1, q \geq 223, \\ 179, & t = 2q, q \geq 89, \\ 81, & t = 2q, q \geq 224, \\ 29, & t = 96q \pm 1, q \geq 1, \\ 59, & t = 96q + 1 \pm 1, q \geq 1. \end{cases}$$

To further simplify the notation, let $D^t = D(1, \dots, t)$ (note that D^t in fact coincides with the t^{th} power of P_∞). Togni proved that $\chi_\rho(D^2) = 8$ (see [64, Proposition 3]) and that $17 \leq \chi_\rho(D^3) \leq 23$ (see [64, Proposition 4]). Moreover, he obtained the following asymptotic result.

Proposition 6.7 [64, Proposition 2]. $\chi_\rho(D^t) = (1 + o(1))3^t$ and $\chi_\rho(D^t) = \Omega(e^t)$.

Note that Proposition 6.7 implies that for any finite $D \subset \mathbb{N}$, the packing chromatic number of $G(\mathbb{Z}, D)$ is finite.

In [64] Togni also bounded $\chi_\rho(G(\mathbb{Z}, D))$ for several small sets D . His results are in normal font presented in Table 3. Some of these bounds were improved by Ekstein, Holub, and Lidický [22]; the improved bounds are presented in the same table in bold font. Next, Ekstein, Holub, and Togni [23] further improved two values; these are the underlined items in the table. Finally, Shao and Vesel [60] improved three of the upper bounds; these improvements are marked with stars in the table. Note that the table asserts that $\chi_\rho(D(1, 2)) = 8$, $\chi_\rho(D(1, 3)) = 9$, $\chi_\rho(D(1, 4)) = 14$, $\chi_\rho(D(1, 5)) = 12$, and $\chi_\rho(D(2, 3)) = 13$.

Ekstein *et al.* [22] also investigated the packing chromatic number of $D(1, t)$ for large t . They observed that a distance graph $D(1, t)$, $t > 1$, can be drawn as an infinite spiral with t lines orthogonal to the spiral. This was the key observation in proving the next result.

Theorem 6.8 [22, Theorem 2]. *If $t \geq 575$ is an odd integer, then $\chi_\rho(D(1, t)) \leq 35$. If $t \geq 648$ is an even integer, then $\chi_\rho(D(1, t)) \leq 56$.*

Ekstein *et al.* [23] further improved the bounds on the packing chromatic number of some specific distance graphs as follows.

D	$\leq \chi_\rho(G(\mathbb{Z}, D))$	$\chi_\rho(G(\mathbb{Z}, D)) \leq$
$\{1, 2\}$	8	8
$\{1, 3\}$	9	9
$\{1, 4\}$	11, 14	16, 15 , 14*
$\{1, 5\}$	10, 12	12
$\{1, 6\}$	12, 15	23, 16*
$\{1, 7\}$	10, 14	15
$\{1, 8\}$	11, 15	25
$\{1, 9\}$	10, 13	18
$\{1, 2, 3\}$	17	23
$\{2, 3\}$	11, <u>13</u>	13
$\{2, 5\}$	14	23, <u>22</u> , 15*

Table 3. Bounds for the packing chromatic number of $G(\mathbb{Z}, D)$ for different sets D .

Theorem 6.9. *Let k and t be positive integers.*

- (i) [23, Theorem 1] *If k is odd, $t \geq 825$ is odd, and k and t are coprime, then $\chi_\rho(D(k, t)) \leq 30$.*
- (ii) [23, Theorem 3] *If k is odd, $t \geq 898$ is even, and k and t are coprime, then $\chi_\rho(D(k, t)) \leq 56$.*
- (iii) [23, Theorem 4] *If k is even, $t \geq 923$ is odd, and k and t are coprime, then $\chi_\rho(D(k, t)) \leq 56$.*

It is known that the distance graph $D(k, t)$ is connected if and only if k and t are coprime. For such distance graphs we have the following general lower bound.

Theorem 6.10 [23, Theorem 5]. *If $D(k, t)$ is connected and $t \geq 9$, then $\chi_\rho(D(k, t)) \geq 12$.*

The particular case when $k = 1$ of Theorem 6.10 was earlier deduced in [22, Corollary 4]. In order to determine the packing chromatic number of disconnected distance graphs $D(k, t)$, the following reduction can be useful.

Proposition 6.11 [23, Proposition 6]. *If $g = \gcd(k, t)$, then $\chi_\rho(D(k, t)) = \chi_\rho(D(\frac{k}{g}, \frac{t}{g}))$.*

Ekstein *et al.* also determined some new lower and upper bounds (or exact values) for the packing chromatic number of $D(k, t)$ for $2 \leq k < t \leq 10$; see [23].

7. S -PACKING COLORINGS

Throughout this section, $S = (a_1, a_2, \dots)$ will be a non-decreasing sequence of positive integers. Recall from Section 1 that $f : V(G) \rightarrow \mathbb{N}$ is an S -packing

coloring of a graph G if, whenever u and v are distinct vertices in G and $f(u) = f(v) = i$, then $d_G(u, v) > a_i$. If $S = (a_1, a_2, \dots)$ and $S' = (a'_1, a'_2, \dots)$ are non-decreasing sequences, then we write $S \geq S'$ if $a_i \geq a'_i$ for all i .

The concept of S -packing coloring was first mentioned in [35] where it was used to prove the NP-completeness of the natural decision problem related to χ_ρ . Later it was formally introduced by Goddard and Xu in [36]. They gave some general properties of S -packing colorings, provided some bounds, and determined χ_S for some known families of graphs. In particular, they made the following useful observations.

Observation 7.1 [36, Observation 1]. *Let $S_a = (a_1, a_2, \dots)$ and $S_b = (b_1, b_2, \dots)$. If $S_b \leq S_a$, then $\chi_{S_b}(G) \leq \chi_{S_a}(G)$.*

Observation 7.2 [36, Observation 2]. *If G_2 is a subgraph of G_1 , then $\chi_S(G_2) \leq \chi_S(G_1)$ for every sequence S .*

Observation 7.3 [36, Observation 3]. *If $S = (a_1, a_2, \dots)$ and G is a graph, then the following assertions hold.*

- (1) $1 \leq \chi_S(G) \leq n(G)$.
- (2) $\chi_S(G) = 1$ if and only if G has no edges.
- (3) $\chi_S(G) = n(G)$ if and only if G is connected and $a_1 \geq \text{diam}(G)$.

By considering all possibilities for the first two integers in S , Goddard and Xu gave a complete characterization of the graphs G such that $\chi_S(G) = 2$.

Proposition 7.4 [36, Proposition 4]. *Let $S = (a_1, a_2, \dots)$ and let G be a connected graph with $m(G) \geq 1$.*

- (1) *If $a_1 = a_2 = 1$, then $\chi_S(G) = 2$ if and only if G is bipartite.*
- (2) *If $a_1 = 1 < a_2$, then $\chi_S(G) = 2$ if and only if G is a star.*
- (3) *If $a_1 > 1$, then $\chi_S(G) = 2$ if and only if G is K_2 .*

Computing $\chi_S(G)$ is, of course, difficult for an arbitrary G and S . However, if $\text{diam}(G) = 2$ and some restrictions are placed on the integers in S , then more can be said as the next two results show. The first proposition is a direct generalization of Corollary 2.8.

Proposition 7.5 [36, Proposition 5]. *If $S = (1, a_2, a_3, \dots)$ and G is a finite graph without isolated vertices, then $\chi_S(G) \leq \beta(G) + 1$, with equality if $a_2 \geq \text{diam}(G)$.*

Proposition 7.6 [36, Proposition 6]. *Let $S = (a_1, a_2, \dots)$ and let G be a graph with $\text{diam}(G) = 2$. If exactly k of the integers a_i are 1, then $\chi_S(G) = n(G) - \alpha_k(G) + \min\{k, \chi(G)\}$.*

Applying Propositions 7.5 and 7.6 yields the S -packing chromatic number for all complete bipartite graphs.

Corollary 7.7 [36, Corollary 7]. *If $S = (a_1, a_2, \dots)$ and $m \leq n$, then*

$$\chi_S(K_{m,n}) = \begin{cases} 2, & a_1 = a_2 = 1, \\ m + 1, & a_1 = 1 < a_2, \\ m + n, & a_1 > 1. \end{cases}$$

Extending the definition of packing chromatic vertex-critical graphs, one says that a graph G is an S -packing chromatic vertex-critical graph (χ_S -critical for short) if $\chi_S(G - u) < \chi_S(G)$ for every $u \in V(G)$ [40]. If G is χ_S -critical and $\chi_S(G) = k$, then G is called k - χ_S -critical.

Setting $\Delta_{\chi_S}(G) = \{\chi_S(G) - \chi_S(G - u) : u \in V(G)\}$ we have the following result, where the notation b^k denotes the sequence of k elements b if k is a positive integer, or, the infinite sequence consisting only of elements b if $k = \infty$.

Theorem 7.8 [40, Theorem 3.1]. *If $S = (1^\ell, 2^\infty)$, $\ell \geq 1$, and $X = \{1, x_1, \dots, x_k\}$, $k \geq 1$, is a set of positive integers, then there exists a χ_S -critical graph G such that $\Delta_{\chi_S}(G) = X$.*

A variant of Theorem 4.2 for S -packing colorings is proved in [40, Theorem 3.2]. Among numerous additional results on χ_S -critical graphs from [40] we extract that (i) a graph G is 3- χ_S -critical if and only if $G \in \{C_3, P_4, C_4\}$ and that (ii) if $a_1 = a_2 = 2$ and $a_3 \geq 3$, then G is 4- χ_S -critical if and only if $G \in \{K_{1,3}, C_4, K_4 - e, K_4, P_6, C_6, paw\}$.

7.1. Infinite paths

Recall that P_∞ denotes the two-way infinite path. We will use P_∞^+ to denote the one-way infinite path. Goddard *et al.* established an upper bound for packing colorings of P_∞ if no color smaller than m is used.

Proposition 7.9 [35, Proposition 7.4]. *Let $S_m = (m, m + 1, m + 2, \dots)$. For all m sufficiently large, $\chi_{S_m}(P_\infty) \leq 2m$. Moreover, for all m , $\chi_{S_m}(P_\infty) \leq 2m + 3$.*

Examining the proof of [35, Proposition 7.4] it follows that if $k \geq 34$, then P_∞^+ admits a $(k, k + 1, \dots, 3k - 1)$ -packing coloring. The next two results were proved by Gastineau, Holub, and Togni [30] in their investigation of packing colorings of outerplanar graphs.

Lemma 7.10 [30, Lemma 4]. *There exists a $(5, 6, \dots, 15)$ -packing coloring of P_∞^+ such that the first vertex of P_∞^+ is at distance at least $\lceil (i - 5)/2 \rceil$ from any vertex of color i .*

Lemma 7.11 [30, Lemma 5]. *If k is a positive integer, then there exists a $(k, k + 1, \dots, 6k + 4)$ -packing coloring of the cycle C_n .*

Suppose c is an (a_1, a_2, \dots) -packing coloring of P_∞ and n is a positive integer. For any subpath, P , of P_∞ of order n , at most $\lceil \frac{n}{a_i+1} \rceil$ of the vertices of P can be colored i by c . This leads to the following result.

Proposition 7.12 [36, Proposition 10]. *Let $S = (a_1, a_2, \dots)$. If $\chi_S(P_\infty) \leq k$, then $\sum_{i=1}^k \frac{1}{a_i+1} \geq 1$.*

Proposition 7.12 implies the following corollary.

Corollary 7.13 [36, Corollary 11]. *If $S = (a_1, a_2, \dots)$ and $\sum_{i=1}^{\infty} \frac{1}{a_i+1} \leq 1$, then $\chi_S(P_\infty) = \infty$.*

Goddard and Xu used induction to prove the following result, which shows that the converse of Corollary 7.13 is false.

Proposition 7.14 [36, Proposition 18]. *If $S = (2^k)_{k \geq 0}$, then $\chi_S(P_\infty) = \infty$.*

By Proposition 7.4 it follows that $\chi_S(P_\infty) = 2$ if and only if $a_1 = a_2 = 1$. Goddard and Xu used Corollary 7.13 along with analyzing some special cases to prove the following result which gives a complete characterization of those sequences $S = (a_1, a_2, \dots)$ for which $\chi_S(P_\infty) = 3$.

Proposition 7.15 [36, Proposition 13]. *Let $S = (a_1, a_2, \dots)$. Then $\chi_S(P_\infty) = 3$ if and only if $(a_1, a_2, a_3) \in \{(1, 2, 3), (1, 3, 3), (2, 2, 2)\}$.*

For a positive integer n , let c be the coloring of P_∞ defined by the following rule: $c(k) = i$ if and only if $k \equiv i-1 \pmod{n+1}$, for each $i \in [n+1]$. If $c(r) = c(s)$ for $r \neq s$, then $|r-s| \geq n+1$. This implies that c is an (a_1, a_2, \dots, a_n) -packing coloring if the entries of the sequence $S = (a_1, a_2, \dots)$ are bounded by n . It follows that the sequence S must be unbounded if $\chi_S(P_\infty) = \infty$. It remains unknown what rate of growth of the sequence S will force $\chi_S(P_\infty) = \infty$. However, for sequences that are arithmetic progressions we have the following.

Proposition 7.16 [36, Proposition 16]. *If S is an arithmetic progression, then $\chi_S(P_\infty^+)$ is finite.*

7.2. S -packing colorings using 3 colors

In this subsection we summarize what is known about graphs that have (a_1, a_2, a_3) -packing colorings.

Let $a_1 \geq 2$. If a vertex u in a graph G has degree at least 3 and c is any (a_1, a_2, \dots) -packing coloring, then all the neighbors of u are assigned distinct colors by c . This implies that a connected graph G has a $(2, 2, 2)$ -packing coloring if and only if G is a path or G is a cycle of length a multiple of 3. See [35]. Similarly, the following is easy to verify.

Proposition 7.17 [36, Corollary 21]. *If $a_1 \geq 2$, $a_3 > 2$, and a connected graph G has an (a_1, a_2, a_3) -packing, then G has at most five vertices.*

Thus, we can restrict the discussion to graphs that admit a $(1, a_2, a_3)$ -packing coloring.

Proposition 7.18 [36, Proposition 22]. *Let $4 \leq a_2 \leq a_3$. A connected graph G has a $(1, a_2, a_3)$ -packing coloring if and only if $\beta(G) \leq 2$.*

Using the characterization of graphs having a $(1, 2, 3)$ -packing coloring given in Proposition 2.4, Goddard and Xu proved the following characterizations of graphs that admit a $(1, a_2, a_3)$ -packing coloring for $2 \leq a_2 \leq 3$ and $a_3 \geq 3$.

Proposition 7.19 [36, Proposition 23]. *Let G be a connected graph. Then*

- (1) *G has a $(1, 3, 3)$ -packing coloring if and only if $\beta(G) \leq 2$ or G is a subgraph of the subdivision of a bipartite multigraph.*
- (2) *For $a_3 \geq 4$, G has a $(1, 2, a_3)$ -packing coloring if and only if G is a subgraph of a subdivision or almost subdivision (subdivided all edges of G except one) of a bipartite multigraph with a dominating vertex.*
- (3) *For $a_3 \geq 4$, G has a $(1, 3, a_3)$ -packing coloring if and only if $\beta(G) \leq 2$ or G is a subgraph of the subdivision of a bipartite multigraph with a dominating vertex.*

7.3. Subcubic and subdivided graphs

Already in the seminal paper [35], Goddard *et al.* were interested in whether the packing chromatic number invariant is bounded on the class of cubic graphs. Even before that question was answered in the negative (see Section 3), there was some evidence that the subdivision graph of a cubic graph has packing chromatic number at most 5. Note that since any subcubic graph is a subgraph of a cubic graph, it follows that if all cubic graphs have (a_1, \dots, a_k) -packing colorings, then every subcubic graph admits an (a_1, \dots, a_k) -packing coloring.

Cranston and Kim [16, Theorem 1] proved that if G is a subcubic graph and G is not the Petersen graph, then G is $(2, 2, 2, 2, 2, 2, 2, 2)$ -packing colorable. Gastineau and Togni studied S -packing colorings of subdivisions of graphs [32]. The following result of theirs provides a connection between S -packing colorings of a graph G and those of the subdivision graph $S(G)$, and, as we will see, it suggests a method of attacking Conjecture 4.14.

Proposition 7.20 [32, Proposition 1]. *Let $S = (s_1, \dots, s_k)$. If G is S -packing colorable, then $S(G)$ is $(1, 2s_1 + 1, \dots, 2s_k + 1)$ -packing colorable.*

By Brooks' theorem every subcubic graph other than K_4 is $(1, 1, 1)$ -packing colorable. Gastineau and Togni proved the following corollary to Proposition 7.20 by giving a specific $(1, 3, 3, 3)$ -packing coloring of $S(K_4)$.

Corollary 7.21 [32, Corollary 1]. *If G is a subcubic graph, then $S(G)$ is $(1, 3, 3, 3)$ -packing colorable.*

Suppose $S = (1, a_2, \dots, a_k)$ such that $a_2 \geq 2$. If G is any graph that admits an S -packing coloring c and $c(v) = 1$ for some $v \in V(G)$, then $\deg(v) \leq k - 1$ since no pair of neighbors of v are assigned the same color under c . An immediate consequence of this is that no cubic graph is $(1, 2, 2)$ -packing colorable. On the other hand, there exist subcubic graphs that are $(1, 2, 2)$ -packing colorable (for example, any path), and Gastineau [29] proved that it is NP-complete to determine if a subcubic bipartite graph is $(1, 2, 2)$ -packing colorable. However, by allowing a partition of $V(G)$ into an independent set and at least six 2-packings we have the following result.

Theorem 7.22 [32, Theorem 1]. *Every subcubic graph is $(1, 2, 2, 2, 2, 2, 2)$ -packing colorable.*

The result is tight in the sense that the Petersen graph is a cubic graph that is not $(1, 2, 2, 2, 2, 2)$ -packing colorable. Based on some computational results it seems possible that the only subcubic graph that is not $(1, 2, 2, 2, 2, 2)$ -packing colorable is the Petersen graph. See [32, Table 1]. On the other hand, the following shows the sensitivity of Theorem 7.22.

Proposition 7.23 [32, Proposition 2]. *There exist bipartite cubic graphs that are not $(1, 2, 2, 2, 2, 3)$ -packing colorable.*

The next result shows that if some restrictions are put on the adjacencies within a subcubic graph, then the conclusions of Theorem 7.22 and Proposition 7.23 can be strengthened. We need the following definition. A graph G is called d -irregular if it has no adjacent vertices both of degree d .

Theorem 7.24 [32, Theorem 2]. *If G is a 3-irregular, subcubic graph, then G is $(1, 2, 2, 2)$ -packing colorable.*

The 5-cycle shows that three 2-packings are needed in Theorem 7.24, so in this sense the conclusion is best possible for 3-irregular subcubic graphs. However, there are 3-irregular subcubic graphs that possess a $(1, 2, 2, 2)$ -packing coloring c , such that $|c^{-1}(4)| = 1$. In such cases c can be altered to give a $(1, 2, 2, 3)$ -packing coloring [32, Figure 5].

By definition, every bipartite graph G is $(1, 1)$ -packing colorable. By Proposition 7.20, it follows that $S(G)$ is $(1, 3, 3)$ -packing colorable and hence $S(G)$ also admits a $(1, 2, 3)$ -packing coloring and a $(1, 2, 2)$ -packing coloring. Somewhat surprisingly, Gastineau and Togni showed that if a graph has minimum degree at least 3, then all of these are equivalent.

Proposition 7.25 [32, Corollary 2]. *The following statements are equivalent for every graph G with $\delta(G) \geq 3$:*

- $S(G)$ is $(1, 2, 2)$ -packing colorable,
- $S(G)$ is $(1, 2, 3)$ -packing colorable,
- $S(G)$ is $(1, 3, 3)$ -packing colorable,
- G is bipartite.

Goddard and Xu [36, Proposition 24] proved that the decision problem of whether G has a $(1, 1, a_3)$ -packing coloring is NP-hard for all a_3 . However, for subcubic graphs we have the following.

Theorem 7.26 [32, Theorem 3]. *Every subcubic graph is $(1, 1, 2, 2, 2)$ -packing colorable.*

The next proposition shows that because of the Petersen graph, the bound in Theorem 7.26 is in a certain sense best possible.

Proposition 7.27 [32, Proposition 4]. *If $a_3 \geq 2$, then the Petersen graph is not $(1, 1, a_3, a_4)$ -packing colorable.*

The following lemma proved useful in showing that several classes of cubic graphs admit $(1, 1, 2, 2)$ -packing colorings. The 2nd power of a graph G is the square of G .

Lemma 7.28 [14, Lemma 2.4]. *A graph G is $(1, 1, 2, 2)$ -packing colorable if and only if there is a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that V_2 and V_3 are independent sets in G and V_1 induces a bipartite graph in the square of G .*

A graph G of order $2n$ is a *generalized prism* of a cycle if G is obtained by adding a perfect matching M to the disjoint union of two cycles C and C' of order n such that each edge in M is incident to one vertex from each of C and C' . Brešar, Klavžar, Rall, and Wash [14] proved the following characterization of the Petersen graph related to Proposition 7.27.

Theorem 7.29 [14, Theorem 3.2]. *If G is a generalized prism of a cycle, then G is $(1, 1, 2, 2)$ -packing colorable if and only if G is not the Petersen graph.*

In [32, Proposition 5], a sporadic example of a cubic graph was found which is not $(1, 1, 3, 3, 3)$ -packing colorable. On the positive side, we have the following.

Theorem 7.30 [32, Theorem 4]. *Every 3-irregular subcubic graph is $(1, 1, 2)$ -packing colorable.*

In the rest of this subsection we give some applications of S -packing coloring to lend some insight into Conjecture 4.14. The next proposition shows the relationship between the conjecture and $(1, 1, 2, 2)$ -packing colorings. Its proof follows directly from Proposition 7.20.

Proposition 7.31 [14, Proposition 2.3]. *If G is $(1, 1, 2, 2)$ -packing colorable, then $\chi_\rho(S(G)) \leq 5$.*

Theorem 7.29 and Proposition 7.31 together imply that Conjecture 4.14 holds for all generalized prisms of cycles except for the Petersen graph. A packing 5-coloring of the subdivision of the Petersen graph was given in [14]. Thus we have the following corollary.

Corollary 7.32 [14, Corollary 3.3]. *If G is a generalized prism of a cycle, then $\chi_\rho(S(G)) \leq 5$.*

The next result extends Theorem 7.29.

Theorem 7.33 [14, Theorem 4.1]. *Let G be a connected, cubic graph of order $2n$ with a 2-factor \mathcal{F} and a perfect matching M . If \mathcal{F} contains a cycle C of length n where no edge of M has both vertices in C , and \mathcal{F} contains at most one 5-cycle, then G is $(1, 1, 2, 2)$ -packing colorable.*

The next result provides additional support for the truth of Conjecture 4.14, where $P(n, k)$ denotes a generalized Petersen graph.

Corollary 7.34 [14, Corollary 4.2]. *If n and k are positive integers such that $k < n/2$ and n is not a multiple of 5, then $P(n, k)$ has a $(1, 1, 2, 2)$ -packing coloring and hence $\chi_\rho(S(P(n, k))) \leq 5$.*

Intuitively, one can expect that any optimal packing coloring of $S(G)$ assigns color 1 to all the new vertices added in the subdivision. However, the following shows that this is not always the case.

Proposition 7.35 [14, Proposition 3.4]. *If G is not $(1, 1, 2, 2)$ -packing colorable and $\chi_\rho(S(G)) \leq 5$, then in every packing 5-coloring of $S(G)$ at least one of the vertices of $S(G)$ that does not belong to $V(G)$ receives a color larger than 1.*

A slightly better bound than the one from Theorem 4.15 can be obtained if a subcubic graph is 2-degenerate.

Theorem 7.36 [4, Theorem 5]. *If G is a subcubic graph in which every subgraph has a vertex of degree at most 2, then G has a $(1, 1, 2, 2, 3, 3)$ -packing coloring. In particular, $\chi_\rho(S(G)) \leq 7$.*

Liu, Liu, Rolek, and Yu [54] considered Conjecture 4.14 for subcubic graphs with bounded maximum average degree, which is defined as

$$\text{mad}(G) = \max \left\{ \frac{2m(H)}{n(H)} : H \text{ a subgraph of } G \right\}.$$

Theorem 7.37 [54, Theorem 1.3 and Corollary 1.5]. *If G is a subcubic graph with $\text{mad}(G) < 30/11$, then G has $(1, 1, 2, 2)$ -packing coloring and hence $\chi_\rho(S(G)) \leq 5$.*

For a planar graph with girth at least g , the maximum average degree is less than $2g/(g-2)$, see [5, Observation 1]. The following corollary follows immediately.

Corollary 7.38 [54, Corollary 1.4]. *If G is a subcubic planar graph with girth at least 8, then G has $(1, 1, 2, 2)$ -packing coloring and hence $\chi_\rho(S(G)) \leq 5$.*

Brešar, Gastineau, and Togni [10] provided some additional results about S -packing colorings of outerplanar graphs.

Theorem 7.39 [10, Theorem 2]. *Let $S = (1, 3, \dots, 3)$ be the sequence containing 3 exactly k times where $k \geq 3$. If G is a bipartite outerplanar graph such that $\Delta(G) \leq k$, then G is S -packing colorable.*

Brešar *et al.* provided an example showing that for $S = (1, 3, \dots, 3, 4)$, where 3 appears $\Delta(G) - 1$ times, there exists a bipartite outerplanar graph that does not admit an S -packing coloring. For subcubic outerplanar graphs that have no triangles, they proved the following result.

Theorem 7.40 [10, Theorem 3]. *If G is a subcubic, triangle-free, outerplanar graph, then G is $(1, 2, 2, 2)$ -packing colorable.*

Theorem 7.40 is complemented by two examples showing that the theorem is best possible for subcubic outerplanar graphs. The first example is a subcubic outerplanar graph having triangles, which is not $(1, 2, 2, 2)$ -packing colorable. The second example is a subcubic, triangle-free, outerplanar graph that does not admit a $(1, 2, 2, 3)$ -packing coloring.

Gastineau and Togni [33] considered the concept of edge-packing S -coloring of cubic graphs G , which coincides with S -packing coloring of the line graph $L(G)$ of G . Note that in this way results about the S -packing chromatic number of some families of 4-regular graphs are obtained. In particular, they proved [33, Theorems 3.3 and 3.7] that every line graph of a cubic graph which has a 2-factor is $(1, 1, 1, 3, 3)$ -packing colorable, as well as $(1, 1, 1, 4, 4, 4, 4, 4)$ -packing colorable. In [33, Theorem 3.1] they also noticed a result, attributed to Payan and to Fouquet and Vanherpe, asserting that the line graph of a cubic graph is $(1, 1, 1, 2)$ -packing colorable.

7.4. Infinite lattices

Results concerning the packing chromatic number of infinite lattices were presented in Section 6. In this subsection we summarize what is known about the S -packing chromatic number of these lattices when $S \neq (1, 2, 3, \dots)$. A standard way to prove that an infinite, locally finite graph G does not admit an (a_1, \dots, a_k) -packing coloring is through the use of density as defined by Fiala *et al.* [26].

For a positive integer r and a vertex u in G we denote by $N_r[u]$ the set of vertices in G whose distance from u is at most r . If $X \subseteq V(G)$, then the *density* of X in G is defined by

$$d_G(X) = \limsup_{r \rightarrow \infty} \max_{u \in V(G)} \left\{ \frac{|X \cap N_r[u]|}{|N_r[u]|} \right\}.$$

Lemma 7.41 [26, Lemma 3.5]. *If a graph G has an (a_1, \dots, a_k) -packing coloring c and $X_i = c^{-1}(i)$, then $\sum_{i=1}^k d_G(X_i) \geq 1$.*

Square lattice

It is clear that the vertex set of P_∞^2 can be partitioned into two independent sets. On the other hand, if $a_2 \geq 2$, then $\chi_S(P_\infty^2) \geq 5$ since the neighbors of any vertex colored 1 must receive pairwise distinct colors in any S -packing coloring. By exhibiting specific colorings and by using Lemma 7.41, Goddard and Xu [37] proved a sequence of results, which taken together give a complete characterization of the sequences S such that $\chi_S(P_\infty^2) \leq 6$.

Proposition 7.42 [37, Propositions 3–5]. *If $S = (a_1, a_2, \dots)$ is a non-decreasing sequence of positive integers, then*

- $\chi_S(P_\infty^2) = 2$ if and only if $a_1 = a_2 = 1$,
- there is no sequence S such that $\chi_S(P_\infty^2) = 3$ or $\chi_S(P_\infty^2) = 4$,
- $\chi_S(P_\infty^2) = 5$ if and only if either $a_1 = 1, a_2 \geq 2$, and $a_5 \leq 3$, or $a_1 = a_5 = 2$, and
- $\chi_S(P_\infty^2) = 6$ if and only if $(a_1, a_2, a_3, a_4, a_5, a_6)$ is the vector $(2, 2, 2, 2, 3, 3)$ or the vector $(1, 2, 2, 2, 4, 4)$.

In the language of the S -packing chromatic number, the results of Theorem 6.2 are stated as $13 \leq \chi_S(P_\infty) \leq 15$, where $S = (1, 2, 3, \dots)$. Using a density argument, Goddard and Xu [37] proved that $(1, 2, 3, \dots)$ is the only arithmetic progression that gives rise to a finite S -packing chromatic number for P_∞^2 .

Proposition 7.43 [37, Proposition 7]. *If S is any arithmetic progression other than $(1, 2, 3, \dots)$, then $\chi_S(P_\infty^2) = \infty$.*

Goddard and Xu investigated $\chi_S(P_\infty^2)$ for some sequences S , which are slight modifications of $(2, 3, 4, \dots)$. They proved that $\chi_S(P_\infty^2) = \infty$ for $S = (2, 2, 3, 4, 5, \dots)$ while $\chi_S(P_\infty^2) = 7$ for $S = (2, 2, 2, 2, 3, 4, 5, \dots)$. They were not able to determine $\chi_S(P_\infty^2)$ for the next natural sequence to consider, namely $S = (2, 2, 2, 3, 4, 5, \dots)$. Indeed, they remark that, “density arguments do not preclude” $\chi_S(P_\infty^2)$ from being finite, “and computer search takes too long.”

Back in 2003, in terms of distance colorings, Fertin, Godard, and Raspaud [24, Theorem 6] determined the (k, k, k, \dots) -packing chromatic number for the square lattice.

Theorem 7.44 [24, Theorem 6]. *If $S = (k, k, k, \dots)$, then*

$$\chi_S(P_\infty^2) = \begin{cases} \frac{(k+1)^2}{2}, & k \text{ is odd,} \\ \frac{k^2+2k+2}{2}, & k \text{ is even.} \end{cases}$$

Goddard and Xu [37] also investigated S -packing colorings of the subgraph $P_2 \square P_\infty$ of P_∞^2 and recorded some of their findings using the following idea. A finite non-decreasing sequence (a_1, \dots, a_k) is called a k -minimal packing chromatic sequence (k -MPCS) for a graph G if there exists an (a_1, \dots, a_k) -packing coloring of G , but G has no packing coloring for any sequence obtained from (a_1, \dots, a_k) by increasing any of its entries.

Proposition 7.45 [37, Proposition 9]. *If $k \leq 5$, then the k -MPCS for $P_2 \square P_\infty$ are $(1, 1)$, $(2, 2, 2, 2)$, $(1, 3, 3, 3)$, $(2, 2, 2, 3, 3)$, and $(1, 3, 3, 5, 5)$.*

Triangular and hexagonal lattices

Recall from Section 6.1 that Finbow and Rall [27, Theorem 6] showed that the packing chromatic number of the triangular lattice \mathcal{T} is infinite. This implies that $\chi_S(\mathcal{T}) = \infty$ if S is any strictly increasing infinite sequence. Goddard and Xu also determined the MPCS of length at most 6 for \mathcal{T} and at most 5 for \mathcal{H} .

Proposition 7.46 [37, Proposition 13]. *If $k \leq 6$, then the k -MPCS for \mathcal{T} are $(1, 1, 1)$, $(1, 1, 2, 2, 2)$, and $(1, 1, 3, 3, 3, 3)$.*

Proposition 7.47 [37, Proposition 14]. *If $k \leq 5$, then the k -MPCS for \mathcal{H} are $(1, 1)$, $(2, 2, 2, 2)$, $(1, 3, 3, 3)$, and $(2, 2, 2, 3, 3)$.*

Finally, in the language of S -packing colorings, Theorem 6.3 says $\chi_S(\mathcal{H}) = 7$ for $S = (1, 2, 3, \dots)$. Goddard and Xu proved that replacing S by any other nontrivial arithmetic sequence leads to the following result.

Proposition 7.48 [37, Proposition 16]. *If S is a nonconstant arithmetic progression other than $(1, 2, 3, \dots)$, then $\chi_S(\mathcal{H}) = \infty$.*

(d, n)-packing colorings

If d and n are positive integers, then a (d, n) -packing coloring of a graph G is an S -packing coloring of G , where $S = (a_1, a_2, \dots)$ with $a_i = d + \lfloor (i - 1)/n \rfloor$. That is, $a_1 = \dots = a_n = d$, $a_{n+1} = \dots = a_{2n} = d + 1$, and so on. These special S -packing colorings were introduced in [31] as a common generalization of packing colorings ($d = n = 1$) and d -distance colorings (where n is large enough with respect to $n(G)$). The study of (d, n) -packing colorings has until now been focused on infinite lattices. Here we will explicitly present results only on P_∞^2 ; see Table 4. Note that the range 13–15 for $(d, n) = (1, 1)$ is from Theorem 6.2, while the values for $(n, d) \in \{(2, 4), (2, 5)\}$ follow from Proposition 7.42. The remaining values in regular font are from [31], those in bold font are improvements derived in [46], and further improvements from [48] are marked with stars.

$d \setminus n$	1	2	3	4	5	6
1	13–15	2	2	2	2	2
2	∞	12–18*	8	6	5	5
3	∞	∞	16–22*	12–14*	11–12	10
4	∞	∞	44–97*	25–35*	20–26*	18–22*
5	∞	∞	199–?	50–108*	35–47*	29–37*
6	∞	∞	∞	118*–?	63*–92*	48*–65*

Table 4. Values and bounds on (d, n) -packing colorings of P_∞^2 .

Tables parallel to Table 4 for \mathcal{H} , \mathcal{T} , $P_\infty \boxtimes P_\infty$, $P_2 \square P_\infty$, and the octagonal grid were also produced by Gastineau, Kheddouci and Togni [31], and improved by Korže and Vesel [46].

7.5. Complexity

For a given finite or infinite sequence S , the S -packing coloring problem is:

<p>S-PACKING COLORING <i>Input:</i> A graph G. <i>Question:</i> Does G have an S-packing coloring ?</p>

In [25, Corollary 13] it was proven that S -PACKING COLORING for a non-decreasing sequence S with values bounded by a constant t can be solved polynomially for graphs of bounded treewidth.

Let $S = (a_1, a_2, a_3)$. If $a_1 \geq 2$, then it follows from Proposition 7.17 that S -PACKING COLORING is polynomial. Further, in Proposition 7.19 graphs that admit S -packing colorings where $S = (1, 3, a_3)$, $a_3 \geq 3$, or $S = (1, 2, a_3)$, $a_3 \geq 4$, are characterized in terms of classes of graphs that can be recognized in polynomial

time. It is well-known that the 3-coloring problem, that is, the $(1, 1, 1)$ -PACKING COLORING problem, is NP-complete. From Proposition 2.4 we also know that the $(1, 2, 3)$ -PACKING COLORING problem is polynomial. The remaining cases when $S = (1, 1, a_3)$, $a_3 \geq 2$, and when $S = (1, 2, 2)$ are proved to be difficult in [36, Proposition 24] and in [36, Proposition 25], respectively. Summarizing this discussion, we have the following result.

Theorem 7.49 [36]. *If $k \geq 1$, then $(1, 1, k)$ -PACKING COLORING and $(1, 2, 2)$ -PACKING COLORING are both NP-complete. Otherwise, S -PACKING COLORING is polynomial-time solvable for $|S| = 3$.*

Gastineau [29] strengthened Theorem 7.49 by considering subcubic graphs, cubic graphs, bipartite graphs, and trees. His findings are summarized in Table 5, where S -PACKING COLORING is abbreviated as S -PC.

class of graphs	$(1, 1, 1)$ -PC	$(1, 2, 2)$ -PC	$(1, 1, k)$ -PC, $k \geq 2$
subcubic	polynomial	NP-complete	NP-complete
cubic	polynomial	polynomial	NP-complete
bipartite	polynomial	NP-complete	polynomial
trees	polynomial	polynomial	polynomial

Table 5. Complexity of $(1, a_2, a_3)$ -PACKING COLORING for different classes.

Gastineau proved the following appealing dichotomy result.

Theorem 7.50 [29, Theorem 3.1]. *Let $S = (a_1, a_2, a_3, a_4)$. If $S \geq S'$ where S' is one of the sequences $(2, 3, 3, 3)$, $(2, 2, 3, 4)$, $(1, 4, 4, 4)$, and $(1, 2, 5, 6)$, then S -PACKING COLORING is polynomial-time solvable. Otherwise, S -PACKING COLORING is NP-complete.*

8. CONJECTURES AND PROBLEMS

To conclude the paper, we collect conjectures and open problems as possible directions for future research on the packing chromatic number.

Problems from Section 2

Recall that a triple of integers (a, b, c) is realizable if there exists a graph G such that $\omega(G) = a$, $\chi(G) = b$, and $\chi_\rho(G) = c$, and that $m(a, b)$ is the smallest integer for which $(a, b, m(a, b))$ is realizable. Based on the results about $m(a, b)$ summarized in Section 2.1, Brešar, Klavžar, Rall and Wash [15] posed:

Conjecture 1 [15, Conjecture 3.7]. *There does not exist a graph G with $\omega(G) = 3$, $\chi(G) = 5$ and $\chi_\rho(G) = 6$ (that is, the triple $(3, 5, 6)$ is not realizable).*

In fact, they suspected that $(k, k + 2, k + 3)$ is not realizable for any $k \geq 3$. Hence we state:

Problem 2. Is it true that $(k, k + 2, k + 3)$ is not realizable for any $k \geq 3$?

Recall that a $\chi_\rho(G)$ -coloring is monotone if $c_m \geq c_{m+1}$ for all $m \in [\chi_\rho(G) - 1]$, where c_m is the number of vertices colored by m .

Problem 3. Find conditions for a graph, which ensure that it admits a monotone χ_ρ -coloring.

Problems from Section 3

Related to the existence result from Theorem 3.1 and the corresponding explicit construction from [8], Brešar and Ferme posed the following:

Problem 4 [8, Question 7]. Is the packing chromatic number bounded on the class of subcubic planar graphs?

Theorems 3.3 and 3.4 indicate that the question of boundedness of the packing chromatic number could be challenging and not easy even in subcubic outerplanar graphs and subcubic planar bipartite graphs. Therefore it makes sense to restrict the previous problem to two subclasses of subcubic planar graphs.

Problem 5. Is the packing chromatic number bounded on the class of subcubic planar bipartite graphs?

Problem 6. Is the packing chromatic number bounded on the class of subcubic outerplanar graphs?

Problems from Section 4

With respect to packing chromatic vertex-critical graphs we select the following two problems. Similar problems are also open for the S -packing chromatic vertex-critical graphs.

Problem 7 [44]. (i) Characterize 4 - χ_ρ -vertex-critical graphs.

(ii) Characterize vertex-transitive χ_ρ -vertex-critical graphs.

Even in the more restrictive class of 4 - χ_ρ -critical graphs a similar characterization could be difficult.

Problem 8 [9]. (i) Characterize 4 - χ_ρ -critical graphs.

(ii) Characterize χ_ρ -critical graphs of radius 2.

Concerning the operation of subdividing a single edge e in a graph G , the following two natural questions arise from Theorem 4.13.

- Problem 9** [13]. (i) Given an integer $j \geq 3$ is there a graph G with $\chi_\rho(G) = j$ having an edge e such that $\chi_\rho(S_e(G)) = \lfloor j/2 \rfloor + 1$?
- (ii) Is there a graph G such that $\chi_\rho(S_e(G)) < \chi_\rho(G) - 1$, for every edge e of G ?

For the sake of completeness, we restate Conjecture 4.14 here.

Conjecture 10 [14, Conjecture 1.1]. *If G is a subcubic graph, then $\chi_\rho(S(G)) \leq 5$.*

Problems from Section 5

If one of the factor graphs in a Cartesian product is complete, then the answer to the following problem is positive.

Problem 11 [12, Problem 3]. If G and H are graphs, is then

$$\chi_\rho(G \square H) \leq \max\{\chi_\rho(G)n(H), \chi_\rho(H)n(G)\}?$$

Korže and Vesel [45] conjectured that if a packing k -coloring of $P_m \square P_n$ exists, then there exists also a packing k -coloring such that roughly half of the vertices receive color 1. More precisely:

Conjecture 12 [45, Conjecture 3.2]. *Let $n \geq 4$ and let $\chi_\rho(P_m \square P_n) = k$. There exists a packing k -coloring of $P_m \square P_n$ with $|X_1| = \alpha(P_m \square P_n) = \lceil \frac{nm}{2} \rceil$, where X_1 is the set of vertices colored with 1.*

Problems from Section 6

There are many problems concerning the packing chromatic number of infinite graphs; in particular, improving the bounds involving lattices and distance graphs from Section 6. The most attractive among them seems to be the following.

Problem 13. Determine the exact value of $\chi_\rho(P_\infty^2)$.

Note that by Theorem 6.2, $\chi_\rho(P_\infty^2) \in \{13, 14, 15\}$.

Problems from Section 7

Gastineau and Togni asked in [32] whether every subcubic graph except the Petersen graph is $(1, 1, 2, 3)$ -packing colorable. Theorem 7.29 proves that among generalized prisms of a cycle only the Petersen graph fails to admit a $(1, 1, 2, 2)$ -packing coloring. The following weaker version of the question of Gastineau and Togni seems natural. (A positive answer to the below question proves Conjecture 10.)

Problem 14. Is every subcubic graph except the Petersen graph $(1, 1, 2, 2)$ -packing colorable?

By Theorem 7.26, every subcubic graph is $(1, 1, 2, 2, 2)$ -packing colorable. The following related question is open.

Problem 15 [32]. Is every subcubic graph except the Petersen graph $(1, 2, 2, 2, 2)$ -packing colorable?

Several intriguing problems arise in connection with S -packing colorings of infinite lattices.

Problem 16 [37]. (i) Determine the sequences S for which $\chi_S(P_\infty^2) < \infty$. In particular, is $\chi_S(P_\infty^2)$ finite when $S = (2, 2, 2, 3, 4, 5, \dots)$?

(ii) Determine $\chi_S(P_2 \square P_\infty)$ for $S = (2, 3, 4, \dots)$.

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