The degree sequence of Fibonacci and Lucas cubes

Sandi Klavžar

Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
and
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
e-mail: `sandi.klavzar@fmf.uni-lj.si`

Michel Mollard

CNRS Université Joseph Fourier
Institut Fourier, BP 74
100 rue des Maths, 38402 St Martin d'Hères Cedex, France
e-mail: `michel.mollard@ujf-grenoble.fr`

Marko Petkovšek

Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
e-mail: `marko.petkovsek@fmf.uni-lj.si`

Abstract

The Fibonacci cube Γ_n is the subgraph of the n-cube induced by the binary strings that contain no two consecutive 1's. The Lucas cube Λ_n is obtained from Γ_n by removing vertices that start and end with 1. It is proved that the number of vertices of degree k in Γ_n and Λ_n is $\sum_{i=0}^{k} \binom{n-2i}{k-i}\binom{i+1}{n-k-i+1}$ and $\sum_{i=0}^{k} \left[2\binom{i}{2i+k-n}\binom{n-2i-1}{k-i} + \binom{i-1}{2i+k-n}\binom{n-2i}{k-i} \right]$, respectively. Both results are obtained in two ways, since each of the approaches yields additional results on the degree sequences of these cubes. In particular, the number of vertices of high resp. low degree in Γ_n is expressed as a sum of few terms, and the generating functions are given from which the moments of the degree sequences of Γ_n and Λ_n are easily computed.

Key words: Fibonacci cube; Lucas cube; degree sequence; generating function

AMS subject classifications: 05C07, 05A15

1 Introduction

A *Fibonacci string* is a binary string that contains no two consecutive 1's. The *Fibonacci cube* Γ_n, $n \geq 0$, is defined as follows. Its vertices are all Fibonacci strings

of length n, two vertices are adjacent if they differ in precisely one bit. In particular, $\Gamma_0 = K_1$, $\Gamma_1 = K_2$, and Γ_2 is the path on three vertices. Alternatively, Γ_n can be defined as the so-called simplex graph of the complement of the path on n vertices, cf. [1].

Fibonacci cubes were introduced as a model for interconnection networks [7] and received a lot of attention afterwards. For different studies of their structure we refer to [2, 3, 6, 8, 12, 13]. These cubes also found an application in theoretical chemistry. There, perfect matchings in hexagonal graphs reflect the stability of the corresponding benzenoid molecules and the so-called resonance graphs capture the structure of the perfect matching. It is appealing that Fibonacci cubes are precisely the resonance graphs of a special class of hexagonal graphs called fibonaccenes, the result proved in [10]. We also mention that Fibonacci cubes led to the concept of the Fibonacci dimension of a graph [1] and that they can be recognized in $O(|E(G)|\log|V(G)|)$ time [15].

Lucas cubes form a class of graphs closely related to Fibonacci cubes. The Lucas cube Λ_n, $n \geq 0$, is the subgraph of the n-cube induced by Fibonacci strings $b_1 \ldots b_n$ such that not both b_1 and b_n are 1. In particular, $\Lambda_0 = \Lambda_1 = K_1$ and $\Lambda_2 = \Gamma_2$ is the path on three vertices. For different aspects of Lucas cubes see [2, 6, 8, 9, 11, 16].

In this paper we are interested in the degree sequence of Fibonacci and Lucas cubes. One of our motivations is the fact that several partial results were previously obtained in order to attack different problems on Fibonacci cubes. In the seminal paper [7, Lemma 6] it was observed that the degrees of Γ_n are at least $\lfloor (n+2)/3 \rfloor$ and (obviously) not more than n. More than ten years later, a recursive formula for computing the degree of any vertex of Γ_n is given in [3]. It depends on the recursive structure of Γ_n and the value of the integer that represents the given vertex (= binary number). This approach was further developed in [14] where the degrees are used to determine the domination number of the Fibonacci cubes. In the main result on the degrees ([14, Theorem 2.6]) vertices of degrees n, $n-1$, $n-2$, and $n-3$ are explicitly described. However, the approach in general does not give the number of vertices of Γ_n of a given degree, a fundamental property of a given family of graphs.

For $n, k \geq 0$ let $f_{n,k}$ denote the number of vertices of Γ_n having degree k. Then our first main result is:

Theorem 1.1 *For all $n \geq k \geq 0$,*

$$f_{n,k} = \sum_{i=0}^{k} \binom{n-2i}{k-i}\binom{i+1}{n-k-i+1}. \tag{1}$$

Note that only the terms with i between $\lceil (n-k)/2 \rceil$ and $\min(k, n-k)$ are nonzero which could be useful when evaluating these numbers. An analogous remark holds for the subsequent summation formulas as well.

In the next section we prove Theorem 1.1 by deriving and solving a corresponding system of linear recurrences. Then, in Section 3, several consequences of Theorem 1.1 are presented. A special emphasis is given on vertices of small and large degrees. For

instance, Corollary 3.4 in particular covers the degrees of the above-mentioned [14, Theorem 2.6]. In Section 4 we give a direct approach to Theorem 1.1 by considering degrees via the partition of $V(\Gamma_n)$ into strings of a given weight. In this way not only Theorem 1.1 is reproved, but (i) the vertices of a given degree and weight are enumerated thus giving an additional information on the Fibonacci semilattice [4] (and the Lucas semilattice [16]) and (ii) the way to our second main theorem is paved.

Denoting by $\ell_{n,k}$, $n, k \geq 0$, the number of vertices of Λ_n having degree k, we prove in Section 5:

Theorem 1.2 *For all $n \geq k \geq 0$ with $n \geq 2$,*

$$\ell_{n,k} = \sum_{i=0}^{k} \left[2 \binom{i}{2i+k-n} \binom{n-2i-1}{k-i} + \binom{i-1}{2i+k-n} \binom{n-2i}{k-i} \right]. \qquad (2)$$

Finally, in Section 6, we reprove Theorem 1.2 by the method of generating functions. This approach is somewhat more involved than the one taken in Section 5, however it can be further used to obtain several additional properties of the sequence of degrees of the Fibonacci and Lucas cubes.

Throughout the paper, we follow the definition of binomial coefficients given in [5]. In particular, $\binom{m}{0} = 1$ and $\binom{m}{k} = 0$ for all $m, k \in \mathbb{Z}$ with $k < 0$. We find this remark important since not all currently used computer algebra systems follow this convention.

2   Proof of Theorem 1.1

The vertex set of Γ_n naturally decomposes into the sets A_n and B_n consisting of those strings that start with a 1, and those strings that do not start with a 1, respectively. Hence $A_0 = \emptyset$, $B_0 = \{\lambda\}$ (where λ is the empty string), and for $n \geq 1$,

$$A_n = \{1\alpha \mid \alpha \in B_{n-1}\} \quad \text{and} \quad B_n = \{0\alpha \mid \alpha \in A_{n-1} \cup B_{n-1}\}.$$

Since every vertex in A_n, $n \geq 2$, necessarily starts with 10, A_n induces Γ_{n-2} in Γ_n. On the other hand, B_n induces Γ_{n-1} in Γ_n. Moreover, each vertex 1α of A_n has exactly one neighbor in B_n, namely 0α.

We now give the key definition that will enable us to compute the degree sequence of Γ_n. For any $n \geq 1$ and any $0 \leq k \leq n$, let $a_{n,k}$, respectively $b_{n,k}$, be the number of vertices of A_n, respectively B_n, of degree k. Consider a vertex $x \in A_n$ of degree k. Then it is of degree $k-1$ in the subgraph Γ_{n-2} of Γ_n induced by A_n. Since x lies in exactly one of the corresponding sets A_{n-2} and B_{n-2}, we get

$$a_{n,k} = a_{n-2,k-1} + b_{n-2,k-1}.$$

Similarly, a vertex $y \in B_n$ either has a neighbor in A_n (if it starts with 00) or has no neighbor in A_n. In the first case, it is a vertex of the corresponding set B_{n-1}, in

the second case, a vertex of A_{n-1}. Therefore,

$$b_{n,k} = b_{n-1,k-1} + a_{n-1,k}.$$

Hence the degree sequences in the subgraphs induced by A_n and B_n satisfy the system of linear recurrences and initial conditions

$$
\begin{aligned}
a_{n,k} &= a_{n-2,k-1} + b_{n-2,k-1} \quad (n \geq 2,\ k \geq 1), & (3) \\
b_{n,k} &= b_{n-1,k-1} + a_{n-1,k} \quad (n \geq 1,\ k \geq 1), & (4) \\
a_{0,k} &= a_{n,0} = 0 \quad (n \geq 0,\ k \geq 0), \quad a_{1,1} = 1, \quad a_{1,k} = 0 \quad (k \geq 2), \\
b_{0,0} &= 1, \quad b_{0,k} = b_{n,0} = 0 \quad (n \geq 1,\ k \geq 1).
\end{aligned}
$$

Their generating functions $a(x,y) = \sum_{n,k \geq 0} a_{n,k} x^n y^k$ and $b(x,y) = \sum_{n,k \geq 0} b_{n,k} x^n y^k$ therefore satisfy the system of linear algebraic equations

$$
\begin{aligned}
a(x,y) - xy &= x^2 y\, a(x,y) + x^2 y\, b(x,y), \\
b(x,y) - 1 &= xy\, b(x,y) + x\, a(x,y)
\end{aligned}
$$

whose solution is

$$
\begin{aligned}
a(x,y) &= \frac{xy(1 + x - xy)}{(1 - xy)(1 - x^2 y) - x^3 y}, & (5) \\
b(x,y) &= \frac{1}{(1 - xy)(1 - x^2 y) - x^3 y}.
\end{aligned}
$$

Write $u = 1 - xy$, $v = 1 - x^2 y$. Then

$$
\begin{aligned}
b(x,y) &= \frac{1}{uv - x^3 y} = \frac{(uv)^{-1}}{1 - x^3 y (uv)^{-1}} = \sum_{h \geq 0} x^{3h} y^h (uv)^{-h-1} \\
&= \sum_{h \geq 0} \frac{(xy)^h}{(1 - xy)^{h+1}} \frac{(x^2 y)^h}{(1 - x^2 y)^{h+1}} y^{-h} \\
&= \sum_{h \geq 0} \sum_{i,j \geq h} \binom{i}{h}\binom{j}{h} x^{i+2j} y^{i+j-h}.
\end{aligned}
$$

In the last step we used the well-known expansion

$$\frac{x^k}{(1 - x)^{k+1}} = \sum_{i \geq k} \binom{i}{k} x^i.$$

Now replace summation variables h and i by $n = i + 2j$ and $k = i + j - h$. Then $i = n - 2j$ and $h = n - k - j$, so

$$b(x,y) = \sum_{n,k,j \geq 0} \binom{n - 2j}{n - k - j}\binom{j}{n - k - j} x^n y^k,$$

4

hence

$$b_{n,k} = \sum_{j=0}^{k} \binom{n-2j}{k-j}\binom{j}{n-k-j}. \tag{6}$$

From (3) and (4) (or, alternatively, from (5)) we obtain

$$a_{n,k} = b_{n-1,k-1} - b_{nk-2,k-2} + b_{n-2,k-1} \quad (n \geq 2, \ k \geq 2). \tag{7}$$

Denote

$$t_{n,k,j} = \binom{n-2j}{k-j}\binom{j}{n-k-j}.$$

Then

$$t_{n-1,k-1,j} - t_{n-2,k-2,j} + t_{n-2,k-1,j} =$$
$$= \left[\binom{n-2j-1}{k-j-1} - \binom{n-2j-2}{k-j-2}\right]\binom{j}{n-k-j} + t_{n-2,k-1,j}$$
$$= \binom{n-2j-2}{k-j-1}\binom{j}{n-k-j} + \binom{n-2j-2}{k-j-1}\binom{j}{n-k-j-1}$$
$$= \binom{n-2j-2}{k-j-1}\binom{j+1}{n-k-j}$$

by using Pascal's identity twice, hence it follows from (6) and (7) that

$$a_{n,k} = \sum_{j \geq 0} \binom{n-2j-2}{k-j-1}\binom{j+1}{n-k-j}$$
$$= \sum_{j=0}^{k} \binom{n-2j}{k-j}\binom{j}{n-k-j+1} \quad (n \geq 2, k \geq 2). \tag{8}$$

Here we replaced j by $j-1$ and noted that $\binom{n}{k}\binom{0}{n-k+1} = 0$ for $n \neq -1$. It is easy to check that (8) holds when $k \in \{0,1\}$ or $n \in \{0,1\}$ as well. Finally, from (6) and (8) we obtain

$$f_{n,k} = a_{n,k} + b_{n,k} = \sum_{j=0}^{k} \binom{n-2j}{k-j}\binom{j+1}{n-k-j+1}$$

by using Pascal's identity once more.

3   Consequences of Theorem 1.1

Let F_n be the nth Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Since $|V(\Gamma_n)| = F_{n+2}$, Theorem 1.1 immediately implies:

Corollary 3.1 *For any $n \geq 0$,*

$$F_{n+2} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{i+1}{n-k-i+1} \binom{n-2i}{k-i}. \tag{9}$$

We next give an alternative proof (avoiding Fibonacci cubes) of Corollary 3.1. Set $F(n) = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{i+1}{n-k-i+1} \binom{n-2i}{k-i}$. If $k > n$ and $i \geq 1$ then $n-k-i+1 < 0$, thus $\binom{i+1}{n-k-i+1} = 0$. If $k > n$ and $i = 0$, or if $i > k$, then $\binom{n-2i}{k-i} = 0$. Thus, after interchanging the order of summation and using Vandermonde's convolution,

$$F(n) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \binom{i+1}{n-k-i+1} \binom{n-2i}{k-i} = \sum_{i=0}^{\infty} \binom{n-i+1}{n-2i+1}.$$

Since for $i > (n+1)/2$ we have $\binom{n-i+1}{n-2i+1} = 0$, we can restrict our summation range to, say, $0 \leq i \leq n+1$, and obtain

$$F(n) = \sum_{i=0}^{n+1} \binom{n-i+1}{n-2i+1} = \sum_{i=0}^{n+1} \binom{n-i+1}{i},$$

where the last equality holds because $n-i+1 \geq 0$. Using the well-known identity $\sum_{i=0}^{m} \binom{m-i}{i} = F_{m+1}$, see [5, p. 289, equation 6.130], we conclude that $F(n) = F_{n+2}$.

An interesting problem, useful for applications such as domination or coloring, is to determine for some fixed integer m the number of vertices of degree $\Delta(\Gamma_n) - m$ and $\delta(\Gamma_n) + m$. (As usually, Δ and δ denote the maximum and the minimum degree.) When m is small, the following two corollaries show that in both cases almost all the terms in the sum of Theorem 1.1 vanish.

Corollary 3.2 *For $0 \leq m \leq n$,*

$$f_{n,n-m} = \sum_{i=\lceil m/2 \rceil}^{m+1} \binom{n-2i}{n-m-i} \binom{i+1}{m-i+1}. \tag{10}$$

Proof. By Theorem 1.1,

$$f_{n,n-m} = \sum_{i=0}^{n-m} \binom{n-2i}{n-m-i} \binom{i+1}{m-i+1}.$$

If $m - i + 1 < 0$ we have $\binom{i+1}{m-i+1} = 0$, thus we can assume that $i \leq m+1$. If $2i < m$ we have $m - i + 1 > i + 1$ and again $\binom{i+1}{m-i+1} = 0$. $\qquad \square$

Corollary 3.3 *Let $\delta = \delta(\Gamma_n) = \lfloor \frac{n+2}{3} \rfloor$. For $n > 0$ and $m \leq n - \delta$,*

$$f_{n,\delta+m} = \sum_{i=\delta-\lfloor \frac{m}{2} \rfloor - 1}^{\delta+m} \binom{n-2i}{\delta+m-i} \binom{i+1}{\delta+m+2i-n}.$$

Proof. We have $i + 1 > 0$, thus $\binom{i+1}{n-k-i+1} = \binom{i+1}{k+2i-n}$. Rewrite the sum in Theorem 1.1 for $k = \delta + m$ and observe that if $i \leq \delta - \lfloor \frac{m}{2} \rfloor - 2$ then $\delta + m + 2i - n \leq 3\delta + m - 2\lfloor \frac{m}{2} \rfloor - n - 4 \leq 3\delta - n - 3 \leq -1$. Hence in this case $\binom{i+1}{\delta+m+2i-n} = 0$. $\square$

Our last two results in particular give the asymptotic behavior of the number of vertices of degrees $\Delta(\Gamma_n) - m$ and $\delta(\Gamma_n) + m$ when $n \to \infty$.

Corollary 3.4 *Let $m \geq 0$ and let $n \geq 2m + 2$. Then*

$$
f_{n,n-m} = \begin{cases}
1; & m = 0, \\
2; & m = 1, \\
n + 1; & m = 2, \\
3n - 8; & m = 3, \\
n^2/2 + 3n/2 - 21; & m = 4, \\
2n^2 - 16n + 10; & m = 5.
\end{cases}
$$

More generally, $f_{n,n-m}$ is a polynomial in n of degree $\lfloor m/2 \rfloor$. Its leading coefficient is $\frac{1}{(m/2)!}$ when m is even, and $\frac{\lceil m/2 \rceil + 1}{\lfloor m/2 \rfloor!}$ when m is odd.

Proof. When $i \leq m+1$ and $n \geq 2m+2$ we have $n - 2i \geq 0$, thus $\binom{n-2i}{n-m-i} = \binom{n-2i}{m-i}$. Hence, having in mind Corollary 3.2, the first values are thus obtained directly from

$$
\sum_{i=\lceil m/2 \rceil}^{m} \binom{n - 2i}{m - i} \binom{i + 1}{m - i + 1}.
$$

Consider now this sum for some fixed m. For all i, $\binom{n-2i}{m-i}$ is a polynomial in n with leading term $\frac{n^{m-i}}{(m-i)!}$, and $\binom{i+1}{m-i+1}$ is independent of n. Thus $f_{n,n-m}$ is a polynomial in n. Its leading monomial is obtained from the term corresponding to the minimal i such that $\binom{i+1}{m-i+1} \neq 0$, which is equivalent to $2i \geq m$ and further to $i \geq \lceil m/2 \rceil$. Hence the minimal such i is $\lceil m/2 \rceil$, and $\deg f_{n,n-m} = m - \lceil m/2 \rceil = \lfloor m/2 \rfloor$.

If m is even, then $\binom{i+1}{m-i+1} = 1$ when $i = \lceil m/2 \rceil = m/2$, thus the leading term is $\frac{1}{(m/2)!} n^{m/2}$. If m is odd, then $\binom{i+1}{m-i+1} = \lceil m/2 \rceil + 1$ when $i = \lceil m/2 \rceil$, thus in this case the leading term is $\frac{\lceil m/2 \rceil + 1}{\lfloor m/2 \rfloor!} n^{\lfloor m/2 \rfloor}$. $\square$

Corollary 3.5 *Let $\delta = \delta(\Gamma_n) = \lfloor \frac{n+2}{3} \rfloor$. Then*

$$
f_{n,\delta+m} = \begin{cases}
1; & m = 0, \ n = 3p, \\
\frac{1}{2}(p+1)(p+4); & m = 0, \ n = 3p+1, \\
p + 2; & m = 0, \ n = 3p+2, \\
\frac{1}{6}p(p+1)(p+8); & m = 1, \ n = 3p, \\
\frac{1}{120}p(p+1)\left(p^3 + 24p^2 + 81p + 14\right); & m = 1, \ n = 3p+1, \\
\frac{1}{24}(p+1)(p+2)\left(p^2 + 15p + 12\right); & m = 1, \ n = 3p+2.
\end{cases}
$$

More generally, for all $m \geq 0$, $f_{n,\delta+m}$ is :

- *a polynomial in p of degree $3m$ and leading coefficient $\frac{1}{(3m)!}$ for $n = 3p$;*

- *a polynomial in p of degree $3m+2$ and leading coefficient $\frac{1}{(3m+2)!}$ for $n = 3p+1$;*

- *a polynomial in p of degree $3m+1$ and leading coefficient $\frac{1}{(3m+1)!}$ for $n = 3p+2$.*

Proof. The first values are obtained by direct use of Corollary 3.3.

Let m be fixed and consider the general case when $n = 3p$ for some fixed m. Then $\delta = p$, and by introducing a new summation variable $j = i - p$ we can rewrite the sum of Corollary 3.3 as

$$f_{3p,p+m} \;=\; \sum_{j=-\lfloor \frac{m}{2} \rfloor -1}^{m} \binom{p-2j}{m-j}\binom{p+j+1}{2j+m}.$$

Notice that $\binom{p-2j}{m-j}$ is a polynomial in p of degree $m-j$, and $\binom{p+j+1}{2j+m}$ is a polynomial in p of degree $2j+m$, therefore their product is of degree $2m+j$. The maximum degree will be obtained when j is maximum, i.e., $j = m$. Then $\binom{p-2j}{m-j} = 1$ and $\binom{p+j+1}{2j+m} = \binom{p+m+1}{3m}$, thus the leading term is $\frac{p^{3m}}{(3m)!}$.

The cases $n = 3p+1$ and $n = 3p+2$ are treated similarly. The maximum degree is obtained when j is maximum, which in these two cases is $j = m+1$. When $n = 3p+1$ we have $\binom{p-2j+1}{m-j+1} = 1$ and $\binom{p+j+1}{2j+m} = \binom{p+m+2}{3m+2}$, thus the leading term is $\frac{p^{3m+2}}{(3m+2)!}$. When $n = 3p+2$ then $\binom{p-2j+2}{m-j+1} = 1$ and $\binom{p+j+1}{2j+m-1} = \binom{p+m+2}{3m+1}$, thus the leading term is $\frac{p^{3m+1}}{(3m+1)!}$. $\qquad\square$

4 Enumeration of vertices in Γ_n by weight

The purpose of this section is to determine the number of vertices in Γ_n with a given weight and degree, where the *weight* of a binary string is the number of 1's in it. This could be done by means of generating functions as in Section 2, nevertheless we use a direct approach which along the way gives some additional information about Fibonacci strings. As a consequence, we are able to give an alternative proof of Theorem 1.1 as well as a combinatorial interpretation of the summation expression. From this approach we can also describe easily the set of vertices of a given weight and degree, and deduce quickly the degree sequence of Lucas cubes. We leave the latter task for the next section and continue here with the study of the structure of Fibonacci strings.

For $n \geq 0$ denote

$$
\begin{aligned}
\mathcal{F}_n &= \quad \text{the set of all Fibonacci strings of length } n, \\
\mathcal{L}_n &= \quad \text{the set of all Lucas strings of length } n, \\
S_n^{i,j} &= \quad \{\alpha \in \mathcal{F}_n;\ \alpha \text{ starts with } i \text{ and ends with } j\},\ i,j \in \{0,1\},\,\text{where } S_n^{0,0} \text{ also} \\
&\qquad \text{includes the empty string } \lambda.
\end{aligned}
$$

Note that in the notation of Section 2, $S_n^{1,1} \cup S_n^{1,0} = A_n$ and $S_n^{0,1} \cup S_n^{0,0} = B_n$. In addition, for any integer $m \geq 0$ we introduce the following Fibonacci strings:

- $\alpha_m = (01)^m 0$

- $\beta_m = (10)^m$

- $\gamma_m = (01)^m$

- $\delta_m = (10)^m 1$

We call the strings δ_m *degenerate Fibonacci strings.*

Lemma 4.1 *Every nondegenerate Fibonacci string can be uniquely decomposed as*

$$\beta_{m_0} 0^{l_0} \alpha_{m_1} 0^{l_1} \alpha_{m_2} 0^{l_2} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}} \, ,$$

where $p \geq 0$, $l_0, \ldots, l_p \geq 0$, $m_1, \ldots, m_p \geq 1$, and $m_0, m_{p+1} \geq 0$. Moreover, m_0 and m_{p+1} determine to which of the sets $S_n^{1,1}$, $S_n^{1,0}$, $S_n^{0,1}$, or $S_n^{0,0}$ the string belongs.

Proof. The proof of the existence of such a decomposition is by induction on the length of the string. This is clearly true for strings of length $n \leq 2$.

Consider now a string s of length $n > 2$. Suppose first that $s = 0s'$, where $s' \in \mathcal{F}_{n-1}$. By induction, we have the following possibilities for s':

- $s' = \beta_{m_0} 0^{l_0} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$, hence $s = \alpha_{m_0} 0^{l_0} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$;

- $s' = 0^{l_0} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$, hence $s = 0^{l_0+1} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$;

- $s' = \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$, hence $s = 0 \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$;

- $s' = \gamma_{m_1}$, thus now $s = 0\gamma_{m_1}$;

- $s' = \delta_{m_1}$, and hence $s = \gamma_{m_1+1}$.

Similarly, if $s = 1s'$, $s' \in \mathcal{F}_{n-1}$, we have the following cases:

- $s' = 0^{l_0} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$, hence $s = \beta_1 0^{l_0-1} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$;

- $s' = \alpha_{m_1} 0^{l_1} \alpha_{m_2} 0^{l_2} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$, hence $s = \beta_{m_1+1} 0^{l_1} \alpha_{m_2} 0^{l_2} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$;

- $s' = \gamma_{m_1}$, but then $s = \delta_{m_1}$ is degenerate.

Hence in each of the cases we have obtained a decomposition of s in the expected form.

It is immediate to verify that strings from $S_n^{1,1}$ satisfy $m_0 > 0$ and $m_{p+1} > 0$; strings from $S_n^{1,0}$ satisfy $m_0 > 0$ and $m_{p+1} = 0$; strings from $S_n^{0,1}$ satisfy $m_0 = 0$ and $m_{p+1} > 0$; and strings from $S_n^{0,0}$ satisfy $m_0 = 0$ and $m_{p+1} = 0$.

9

To prove uniqueness, consider first a string $\beta_{m_0} 0^{l_0} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$ from $S_n^{1,1}$, thus with $m_0 > 0$ and $m_{p+1} > 0$. In the three possible cases ($l_0 > 0, l_0 = 0$ and $m_1 > 0, p = 0$) such a string contains at least two consecutive 0's, so the string is not degenerate. On the other hand, it is clear that a nondegenerate string cannot be decomposed in two ways as $\beta_{m_0} 0^{l_0} \alpha_{m_1} 0^{l_1} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$. $\qquad\square$

Note also that the degenerate Fibonacci string δ_m is of length $n = 2m+1$, weight $w = m+1$, and the corresponding vertex of Γ_n is of degree $k = m+1$. For all the other strings we have:

Proposition 4.2 *A Fibonacci string $\beta_{m_0} 0^{l_0} \alpha_{m_1} 0^{l_1} \alpha_{m_2} 0^{l_2} \ldots \alpha_{m_p} 0^{l_p} \gamma_{m_{p+1}}$ is of length $n = \sum_{i=0}^{p} l_i + 2 \sum_{i=0}^{p+1} m_i + p$ and weight $w = \sum_{i=0}^{p+1} m_i$, and the corresponding vertex of Γ_n is of degree $k = \sum_{i=0}^{p} l_i + \sum_{i=0}^{p+1} m_i$.*

Proof. The assertion for the length and the weight follows immediately from definitions. As for the degree, use the fact that changing a 1 to 0 in a vertex from $\mathcal{F}_n$ gives a vertex in $\mathcal{F}_n$, while a 0 can be changed to 1 only if it is not adjacent to 1, and thus not inside a block of the form α_m, β_m, and γ_m. $\qquad\square$

We will use the following classical results about composition of integers.

Lemma 4.3 *Let $a, b \geq 0$. Then the number of solutions of $x_1 + x_2 + \cdots + x_a = b$, with $x_1, x_2, \ldots, x_a$ nonnegative integers, is $\binom{b+a-1}{b}$.*

Lemma 4.4 *Let $a, b \geq 0$. Then the number of solutions of $x_1 + x_2 + \cdots + x_a = b$, with $x_1, x_2, \ldots, x_a$ positive integers, is $\binom{b-1}{b-a}$.*

In the rest we will use some more notation. Let

$$s_{n,k}^{1,1}, \ s_{n,k}^{1,0}, \ s_{n,k}^{0,1}, \text{ and } s_{n,k}^{0,0}$$

be the number of vertices of degree k in

$$S_n^{1,1}, \ S_n^{1,0}, \ S_n^{0,1}, \text{ and } S_n^{0,0},$$

respectively. Let in addition

$$S_{n,w}^{1,1}, \ S_{n,w}^{1,0}, \ S_{n,w}^{0,1}, \text{ and } S_{n,w}^{0,0},$$

be the corresponding sets where each vertex is of weight w, and let

$$s_{n,k,w}^{1,1}, \ s_{n,k,w}^{1,0}, \ s_{n,k,w}^{0,1}, \text{ and } s_{n,k,w}^{0,0}$$

be the number of vertices of degree k in these sets, respectively.

Lemma 4.5 *For all integers k, n, w*

$$s_{n,k,w}^{0,0} = \binom{w-1}{2w+k-n}\binom{n-2w}{k-w},$$

$$s_{n,k,w}^{0,1} = s_{n,k,w}^{1,0} = \binom{w-1}{2w+k-n-1}\binom{n-2w}{k-w},$$

$$s_{n,k,w}^{1,1} = \binom{w-1}{2w+k-n-2}\binom{n-2w}{k-w}.$$

Proof. Assume first that $w \leq k \leq n$.

A string from $S_{n,w}^{0,0}$ is decomposable as $0^{l_0}\alpha_{m_1}0^{l_1}\ldots\alpha_{m_p}0^{l_p}$ where $p \geq 0$, $l_0, l_1 \ldots, l_p \geq 0$, and $m_1 \ldots, m_p > 0$. By Proposition 4.2 there is a 1-1 mapping between $S_{n,w}^{0,0}$ and the solutions of

$$
\begin{cases}
p = n - k - w \geq 0, \\
l_0 + \cdots + l_p = k - w \text{ with } l_0, \ldots, l_p \geq 0, \\
m_1 + \cdots + m_p = w \text{ with } m_1 \ldots, m_p \geq 1.
\end{cases}
\tag{11}
$$

A string from $S_{n,w}^{1,0}$ is decomposable as $\beta_{m_0}0^{l_0}\alpha_{m_1}0^{l_1}\alpha_{m_2}0^{l_2}\ldots\alpha_{m_p}0^{l_p}$ with $p \geq 0$, $l_0, l_1 \ldots, l_p \geq 0$, $m_0, m_1 \ldots, m_p > 0$. Thus there is a 1-1 mapping between $S_{n,w}^{1,0}$ and the solutions of

$$
\begin{cases}
p = n - k - w \geq 0, \\
l_0 + \cdots + l_p = k - w \text{ with } l_0, \ldots, l_p \geq 0, \\
m_0 + \cdots + m_p = w \text{ with } m_0, \ldots, m_p \geq 1.
\end{cases}
\tag{12}
$$

A nondegenerate string from $S_{n,w}^{1,1}$ is decomposable as $\beta_{m_0}0^{l_0}\alpha_{m_1}0^{l_1}\alpha_{m_2}0^{l_2}\ldots\alpha_{m_p}0^{l_p}\gamma_{m_{p+1}}$ where $p \geq 0$, $l_0, \ldots, l_p \geq 0$, $m_0, \ldots, m_p > 0$. Thus there is a 1-1 mapping between these strings and the solutions of

$$
\begin{cases}
p = n - k - w \geq 0, \\
l_0 + \cdots + l_p = k - w \text{ with } l_0, \ldots, l_p \geq 0, \\
m_0 + \cdots + m_{p+1} = w \text{ with } m_0, \ldots, m_{p+1} \geq 1.
\end{cases}
\tag{13}
$$

Assume that $p = n - k - w \geq 0$, then by Lemmas 4.3 and 4.4 the number of solutions of (11), (12) and (13) are $\binom{w-1}{2w+k-n}\binom{n-2w}{k-w}$, $\binom{w-1}{2w+k-n-1}\binom{n-2w}{k-w}$, and $\binom{w-1}{2w+k-n-2}\binom{n-2w}{k-w}$, respectively.

Assume now that $n - k - w < 0$. Then there are no solutions of (11), (12) and (13), thus there are no nondegenerate strings of degree k in $S_{n,w}^{0,0}$, $S_{n,w}^{1,0}$ and $S_{n,w}^{1,1}$. Notice that we have $w \geq 1$ because $w = 0$ implies $n - k < 0$, a contradiction.

Suppose $n - k - w \leq -2$. Then we can write

$$2w+k-n > 2w+k-n-1 > 2w+k-n-2 = w+(w+k-n-2) \geq w > w-1 \geq 0,$$

thus $\binom{w-1}{2w+k-n} = \binom{w-1}{2w+k-n-1} = \binom{w-1}{2w+k-n-2} = 0$.

Assume that $n - k - w = -1$. Then

$$2w + k - n > 2w + k - n - 1 = w > w - 1 \geq 0,$$

therefore $\binom{w-1}{2w+k-n} = \binom{w-1}{2w+k-n-1} = 0$. Consider now $\binom{w-1}{2w+k-n-2}\binom{n-2w}{k-w} = \binom{w-1}{w-1}\binom{n-2w}{k-w} = \binom{n-2w}{k-w} = \binom{k-w-1}{k-w}$. This number is zero if $k > w$. Otherwise (if $k = w$ and $n = 2k - 1$) it is 1, which corresponds to the degenerate string δ_{k-1}.

By symmetry we have $s^{0,1}_{n,k,w} = s^{1,0}_{n,k,w}$.

A vertex of weight w has degree at least k, thus there are no vertices of degree k in the sets $S^{1,1}_{n,w}$, $S^{1,0}_{n,w}$, $S^{0,1}_{n,w}$, $S^{0,0}_{n,w}$ if $w \leq k \leq n$ is not satisfied. It is immediate to verify that the four formulas hold also. $\square$

Let $f_{n,k,w}$ be the number of vertices of Γ_n having degree k and weight w. Then we have:

Theorem 4.6 *For all integers k, n, w with $k, w \leq n$,*

$$f_{n,k,w} = \binom{w+1}{n-w-k+1}\binom{n-2w}{k-w}.$$

Proof. Clearly, $f_{n,k,w} = s^{1,1}_{n,k,w} + s^{1,0}_{n,k,w} + s^{0,1}_{n,k,w} + s^{0,0}_{n,k,w}$. Applying Lemma 4.5 and (three times) the identity $\binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}$, we arrive at

$$f_{n,k,w} = \binom{w+1}{2w+k-n}\binom{n-2w}{k-w}.$$

Because $w + 1 > 0$, we have $\binom{w+1}{2w+k-n} = \binom{w+1}{n-w-k+1}$. $\square$

Note that by the convention we are using for the binomial coefficients, $f_{n,k,w} = 0$ when $w > (n+1)/2$.

Theorem 1.1 immediately follows from Theorem 4.6.

5 Proof of Theorem 1.2

Let $\ell_{n,k,w}$ be the number of vertices of Λ_n of degree k and weight w, and let $\ell^{p,q}_{n,k,w}$, for $p, q \in \{0, 1\}$, be the number of such strings in the set $S^{p,q}_n$.

Lemma 5.1 *For all n, k, w such that $n \geq 2$, $1 \leq k \leq n$ and $0 \leq w \leq n$,*

$$
\begin{aligned}
\ell^{0,0}_{n,k,w} &= s^{0,0}_{n-1,k-1,w} + s^{1,0}_{n-1,k,w}, \\
\ell^{0,1}_{n,k,w} &= \ell^{1,0}_{n,k,w} = s^{0,1}_{n-1,k,w} + s^{1,1}_{n-1,k,w}, \\
\ell^{1,1}_{n,k,w} &= 0.
\end{aligned}
$$

12

Proof. A Lucas string that starts and ends with 0 can be written as $0s$, where either $s \in S_{n-1,w}^{0,0}$ is of degree $k-1$, or $s \in S_{n-1,w}^{1,0}$ is of degree k. This gives the first equality. Similarly we obtain the second equality, while the last one is obvious. $\square$

Theorem 5.2 *For all n, k, w such that $n \geq k, w \geq 0$ and $n \geq 2$,*

$$\ell_{n,k,w} = \binom{w-1}{2w+k-n}\binom{n-2w}{k-w} + 2\binom{w}{2w+k-n}\binom{n-2w-1}{k-w}. \qquad (14)$$

Proof. Assume first that $k \geq 1$. Since $\ell_{n,k,w} = \ell_{n,k,w}^{0,0} + 2\ell_{n,k,w}^{0,1}$, Lemmas 4.5 and 5.1 imply that

$$
\begin{aligned}
\ell_{n,k,w} = {} & \binom{w-1}{2w+k-n}\binom{n-2w-1}{k-w} + \binom{w-1}{2w+k-n}\binom{n-2w-1}{k-w-1} + \\
& 2\binom{w-1}{2w+k-n-1}\binom{n-2w-1}{k-w} + 2\binom{w-1}{2w+k-n}\binom{n-2w-1}{k-w}.
\end{aligned}
$$

Using Pascal's identity we can group the first term with one half of the third term, the second term with one half of the fourth term, and the remaining half of the third term with the remaining half of the fourth term to obtain (14).

The only Lucas strings of degree $k = 0$ are λ and 0, hence $\ell_{n,0,w} = 0$ when $n \geq 2$. But in this case the right-hand side of (14) evaluates to 0 as well. $\square$

Note again that by the convention we are using for the binomial coefficients, $\ell_{n,k,w} = 0$ when $w > n/2$.

Theorem 1.2 now follows immediately from Theorem 5.2.

Corollary 5.3 *Let $n \geq 1$. The number of vertices of weight $w \leq n$ in L_n is*

$$\sum_{k=0}^{n} \ell_{n,k,w} = \binom{n-w}{w} + \binom{n-w-1}{n-2w}.$$

Proof. Note first that the result is true when $w \leq 0$ or $n = 1$. Assume now that $w \geq 1$ and $n \geq 2$. Then $\binom{w-1}{2w+k-n} = \binom{w-1}{n-k-w-1}$ and $\binom{w}{2w+k-n} = \binom{w}{n-k-w}$. Hence we obtain from Theorem 5.2 by Vandermonde's convolution

$$
\begin{aligned}
\sum_{k=0}^{n} \ell_{n,k,w} &= \sum_{k=0}^{n}\left[\binom{w-1}{n-k-w-1}\binom{n-2w}{k-w} + 2\binom{w}{n-k-w}\binom{n-2w-1}{k-w} \right] \\
&= \binom{n-w-1}{n-2w-1} + 2\binom{n-w-1}{n-2w}.
\end{aligned}
$$

Using Pascal's identity and $\binom{n-w}{n-2w} = \binom{n-w}{w}$ we have the final expression. $\square$

Similarly as Theorem 1.1 yields special cases for specific degrees in Fibonacci cubes, one can apply Theorem 1.2 to obtain the number of vertices of certain degrees

13

in Lucas cubes. For instance, $\ell_{n,n} = 1$ $(n \geq 2)$, $\ell_{n,n-1} = 0$ $(n \geq 3)$, and $\ell_{n,n-2} = n$ $(n \geq 5)$. For the minimal degree, if $n \geq 2$, then

$$\ell_{n,\lfloor (n+2)/3 \rfloor} = \begin{cases} 3; & n \equiv 0 \pmod 3, \\ n(n+5)/6; & n \equiv 1 \pmod 3, \\ n; & n \equiv 2 \pmod 3. \end{cases}$$

6 The method of generating functions

In this section we approach Theorem 1.2 using generating functions. It is relatively more complicated than the approach from the previous two sections. On the other hand, it enables us to obtain many additional results as demonstrated at the end of the section by several examples.

Clearly,

$$\begin{align} \mathcal{F}_n &= S_n^{1,1} \cup S_n^{1,0} \cup S_n^{0,1} \cup S_n^{0,0} \quad \text{(for } n \geq 0\text{)}, \tag{15} \\ \mathcal{L}_n &= S_n^{1,0} \cup S_n^{0,1} \cup S_n^{0,0} \quad \text{(for } n \geq 0\text{)}, \tag{16} \\ S_n^{1,1} &= 10\mathcal{F}_{n-4}01 \quad \text{(for } n \geq 4\text{)}, \tag{17} \\ S_n^{1,0} &= 10\mathcal{F}_{n-3}0 \quad \text{(for } n \geq 3\text{)}, \tag{18} \\ S_n^{0,1} &= 0\mathcal{F}_{n-3}01 \quad \text{(for } n \geq 3\text{)}, \tag{19} \\ S_n^{0,0} &= 0\mathcal{F}_{n-2}0 \quad \text{(for } n \geq 2\text{)}. \tag{20} \end{align}$$

Equation (15) shows that $V(\Gamma_n) = \mathcal{F}_n$ can be partitioned into four blocks which, by (17) – (20), induce in Γ_n with $n \geq 4$ a Γ_{n-4}, a Γ_{n-3}, a Γ_{n-3}, and a Γ_{n-2}, respectively. By (15) again, each of these blocks can be further partitioned into four subblocks

$$\begin{align} S_n^{1,1} &= 10S_{n-4}^{1,1}01 \ \cup \ 10S_{n-4}^{1,0}01 \ \cup \ 10S_{n-4}^{0,1}01 \ \cup \ 10S_{n-4}^{0,0}01, \tag{21} \\ S_n^{1,0} &= 10S_{n-3}^{1,1}0 \ \cup \ 10S_{n-3}^{1,0}0 \ \cup \ 10S_{n-3}^{0,1}0 \ \cup \ 10S_{n-3}^{0,0}0, \tag{22} \\ S_n^{0,1} &= 0S_{n-3}^{1,1}01 \ \cup \ 0S_{n-3}^{1,0}01 \ \cup \ 0S_{n-3}^{0,1}01 \ \cup \ 0S_{n-3}^{0,0}01, \tag{23} \\ S_n^{0,0} &= 0S_{n-2}^{1,1}0 \ \cup \ 0S_{n-2}^{1,0}0 \ \cup \ 0S_{n-2}^{0,1}0 \ \cup \ 0S_{n-2}^{0,0}0. \tag{24} \end{align}$$

Proposition 6.1 *The set of those edges of Γ_n not contained within one of the four blocks in (15) equals $\bigcup_{i=1}^{8} M_i$ where each M_i is a perfect matching between a subblock and the union of a pair of subblocks of different blocks, as follows (see Fig. 1):*

1. *M_1 is a perfect matching between $0S_{n-2}^{0,0}0$ and $0S_{n-3}^{0,0}01 \cup 0S_{n-3}^{0,1}01$,*

2. *M_2 is a perfect matching between $0S_{n-2}^{1,0}0$ and $0S_{n-3}^{1,0}01 \cup 0S_{n-3}^{1,1}01$,*

3. *M_3 is a perfect matching between $0S_{n-3}^{0,0}01$ and $10S_{n-4}^{0,0}01 \cup 10S_{n-4}^{1,0}01$,*

4. *M_4 is a perfect matching between $0S_{n-3}^{0,1}01$ and $10S_{n-4}^{0,1}01 \cup 10S_{n-4}^{1,1}01$,*

5. *M_5 is a perfect matching between $0S_{n-2}^{0,0}0$ and $10S_{n-3}^{0,0}0 \cup 10S_{n-3}^{1,0}0$,*

14

6. M_6 is a perfect matching between $0S_{n-2}^{0,1}0$ and $10S_{n-3}^{0,1}0 \cup 10S_{n-3}^{1,1}0$,

7. M_7 is a perfect matching between $10S_{n-3}^{0,0}0$ and $10S_{n-4}^{0,0}01 \cup 10S_{n-4}^{0,1}01$,

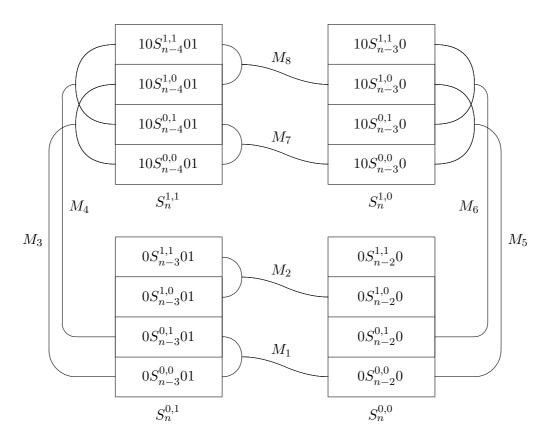8. M_8 is a perfect matching between $10S_{n-3}^{1,0}0$ and $10S_{n-4}^{1,0}01 \cup 10S_{n-4}^{1,1}01$.

Figure 1: Perfect matchings between subblocks and unions of subblocks of Γ_n.

Proof. We need to analyze the external connections of each of the 16 subblocks of Γ_n. By way of example we do this for the subblock $10S_{n-3}^{1,0}0$, in all the other cases the analysis is similar. Each string $\sigma \in 10S_{n-3}^{1,0}0$ is of the form $\sigma = 101\tau00$ where $\tau \in \mathcal{F}_{n-5}$. So σ is adjacent to

- precisely one vertex $101\tau01 \in S_n^{1,1}$ (if τ ends with 1 then $101\tau01 \in 10S_{n-4}^{1,1}01$, otherwise $101\tau01 \in 10S_{n-4}^{1,0}01$);

- no vertices in $S_n^{0,1}$, since each vertex of $S_n^{1,0}$ is at distance 2 or more from each vertex of $S_n^{0,1}$;

15

- precisely one vertex in $S_n^{0,0}$, namely $001\tau00 \in 0S_{n-2}^{0,0}0$.

When analyzing other subblocks, we find out in a similar way that

- each vertex in $10S_{n-4}^{1,1}01 \cup 10S_{n-4}^{1,0}01$ is adjacent to precisely one vertex in $10S_{n-3}^{1,0}0$;

- each vertex in $10S_{n-3}^{0,0}0$ is adjacent to precisely one vertex in $0S_{n-2}^{0,0}0$;

- each vertex in $0S_{n-2}^{0,0}0$ is adjacent to precisely one vertex in $10S_{n-3}^{1,0}0 \cup 10S_{n-3}^{0,0}0$.

Taken together, these facts imply that the external connections of the subblock $10S_{n-3}^{1,0}0$ are precisely the edges of $M_5 \cup M_8$ with one endpoint in $10S_{n-3}^{1,0}0$. $\qquad\square$

It follows from (21) – (24) and from Proposition 6.1 that

$$
\begin{aligned}
s_{n,k}^{1,1} &= s_{n-4,k-2}^{1,1} + s_{n-4,k-2}^{1,0} + s_{n-4,k-2}^{0,1} + s_{n-4,k-2}^{0,0} \quad (n \geq 4, k \geq 2),\\
s_{n,k}^{1,0} &= s_{n-3,k-1}^{1,1} + s_{n-3,k-2}^{1,0} + s_{n-3,k-1}^{0,1} + s_{n-3,k-2}^{0,0} \quad (n \geq 3, k \geq 2),\\
s_{n,k}^{0,1} &= s_{n-3,k-1}^{1,1} + s_{n-3,k-1}^{1,0} + s_{n-3,k-2}^{0,1} + s_{n-3,k-2}^{0,0} \quad (n \geq 3, k \geq 2),\\
s_{n,k}^{0,0} &= s_{n-2,k}^{1,1} + s_{n-2,k-1}^{1,0} + s_{n-2,k-1}^{0,1} + s_{n-2-2,k-2}^{0,0} \quad (n \geq 2, k \geq 2).
\end{aligned}
$$

Together with the corresponding initial conditions, this system of recurrences implies that the generating functions,

$$
\begin{aligned}
s^{1,1}(x,y) &= \sum_{n,k\geq0} s_{n,k}^{1,1} x^n y^k,\\
s^{1,0}(x,y) &= \sum_{n,k\geq0} s_{n,k}^{1,0} x^n y^k,\\
s^{0,1}(x,y) &= \sum_{n,k\geq0} s_{n,k}^{0,1} x^n y^k,\\
s^{0,0}(x,y) &= \sum_{n,k\geq0} s_{n,k}^{0,0} x^n y^k
\end{aligned}
$$

satisfy the system of linear algebraic equations

$$
\begin{aligned}
s^{1,1}(x,y) &= xy + x^3 y^2 + x^4 y^2 \left(s^{1,1}(x,y) + s^{1,0}(x,y) + s^{0,1}(x,y) + s^{0,0}(x,y) \right),\\
s^{1,0}(x,y) &= x^2 y + x^3 y \left(s^{1,1}(x,y) + s^{0,1}(x,y) \right) + x^3 y^2 \left(s^{1,0}(x,y) + s^{0,0}(x,y) \right),\\
s^{0,1}(x,y) &= x^2 y + x^3 y \left(s^{1,1}(x,y) + s^{1,0}(x,y) \right) + x^3 y^2 \left(s^{0,1}(x,y) + s^{0,0}(x,y) \right),\\
s^{0,0}(x,y) &= 1 + xy + x^2 s^{1,1}(x,y) + x^2 y \left(s^{1,0}(x,y) + s^{0,1}(x,y) \right) + x^2 y^2 s^{0,0}(x,y)
\end{aligned}
$$

whose solution is

$$
s^{1,1}(x,y) = \frac{xy(1-xy)}{(1-xy)(1-x^2 y)-x^3 y}, \tag{25}
$$

$$
s^{1,0}(x,y) = s^{0,1}(x,y) = \frac{x^2 y}{(1-xy)(1-x^2 y)-x^3 y}, \tag{26}
$$

$$
s^{0,0}(x,y) = \frac{1-x^2 y}{(1-xy)(1-x^2 y)-x^3 y}. \tag{27}
$$

16

Expanding these rational functions into power series we obtain

$$s_{n,k}^{1,1} = \sum_{w=0}^{k} \binom{w-1}{2w+k-n-2}\binom{n-2w}{k-w},$$

$$s_{n,k}^{1,0} = s_{n,k}^{0,1} = \sum_{w=0}^{k} \binom{w-1}{2w+k-n-1}\binom{n-2w}{k-w},$$

$$s_{n,k}^{0,0} = \sum_{w=0}^{k} \binom{w-1}{2w+k-n}\binom{n-2w}{k-w}. \tag{28}$$

By noting that $f_{n,k} = s_{n,k}^{1,1}+s_{n,k}^{1,0}+s_{n,k}^{0,1}+s_{n,k}^{0,0}$ and by using Pascal's identity repeatedly, we obtain (1) again.

To recompute $\ell_{n,k}$, note that for $n \geq 3$,

$$\mathcal{L}_n = 10\mathcal{F}_{n-3}0 \cup 0\mathcal{F}_{n-1}$$
$$= 10\mathcal{F}_{n-3}0 \cup (0S_{n-1}^{1,1} \cup 0S_{n-1}^{1,0} \cup 0S_{n-1}^{0,1} \cup 0S_{n-1}^{0,0}).$$

Each $\sigma \in 10\mathcal{F}_{n-3}0$ is of the form $\sigma = 10\tau0$ with $\tau \in \mathcal{F}_{n-3}$. Hence σ is adjacent to precisely one vertex in $0\mathcal{F}_{n-1}$, namely $00\tau0 \in 0S_{n-1}^{0,0}$. Conversely, each vertex $00\tau0 \in 0S_{n-1}^{0,0}$ is adjacent to $10\tau0 \in 10\mathcal{F}_{n-3}0$. So for $n \geq 3, k \geq 1$,

$$\ell_{n,k} = f_{n-3,k-1} + s_{n-1,k}^{1,1} + s_{n-1,k}^{1,0} + s_{n-1,k}^{0,1} + s_{n-1,k-1}^{0,0}$$
$$= f_{n-3,k-1} + f_{n-1,k} + s_{n-1,k-1}^{0,0} - s_{n-1,k}^{0,0}. \tag{29}$$

Using (1) and (28), this formula can be shown equivalent to (2).

From (25) – (29) and the values $\ell_{0,0} = \ell_{1,0} = 1$, $\ell_{1,1} = 0$ it is straightforward to compute the generating functions

$$f(x,y) = \sum_{n,k\geq 0} f_{n,k}x^n y^k = \frac{1+xy+(1-y)x^2y}{(1-xy)(1-x^2y)-x^3y},$$

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k}x^n y^k = \frac{1+(1-y)x+x^2y^2+(1-y)x^3y-(1-y)^2x^4y}{(1-xy)(1-x^2y)-x^3y}$$

from which additional interesting information concerning the degree sequences $\langle f_{n,k}\rangle_{k=0}^{n}$ and $\langle \ell_{n,k}\rangle_{k=0}^{n}$ can be obtained easily. For instance:

1. Since the generating functions $f(x,y)$, $\ell(x,y)$, $s^{1,1}(x,y)$, $s^{1,0}(x,y)$, $s^{0,1}(x,y)$, $s^{0,0}(x,y)$ all have $(1-xy)(1-x^2y)-x^3y = 1-xy-x^2y^2-x^3y+x^3y^2$ as their denominator, each of the sequences $s_{n,k} \in \{f_{n,k}, \ell_{n,k}, s_{n,k}^{1,1}, s_{n,k}^{1,0}, s_{n,k}^{0,1}, s_{n,k}^{0,0}\}$ satisfies the same recurrence

$$s_{n,k} = s_{n-1,k-1} + s_{n-2,k-1} + s_{n-3,k-1} - s_{n-3,k-2}$$

for all large enough n and k.

17

2. From

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} f_{n,k} \;=\; f(x,y)|_{y=1} \;=\; \frac{1+x}{1-x-x^2} \;=\; \sum_{n\geq 0} F_{n+2} x^n$$

it follows that $|V(\Gamma_n)| = F_{n+2}$, and from

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} \ell_{n,k} \;=\; \ell(x,y)|_{y=1} \;=\; \frac{1+x^2}{1-x-x^2} \;=\; \sum_{n\geq 0} L_n x^n - 1$$

it follows that $|V(\Lambda_0)| = L_0 - 1 = 1$, $|V(\Lambda_n)| = L_n$ for $n \geq 1$.

3. From

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} k f_{n,k} \;=\; \frac{\partial}{\partial y} f(x,y)|_{y=1}$$

$$=\; \frac{2x}{(1-x-x^2)^2} \;=\; 2\sum_{n\geq 0} \frac{nF_{n+1} + 2(n+1)F_n}{5} x^n$$

it follows that $|E(\Gamma_n)| = (nF_{n+1} + 2(n+1)F_n)/5$, and from

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} k \ell_{n,k} \;=\; \frac{\partial}{\partial y} \ell(x,y)|_{y=1} \;=\; \frac{2(2-x)x^2}{(1-x-x^2)^2} \;=\; \sum_{n\geq 0} 2n F_{n-1} x^n$$

it follows that $|E(\Lambda_n)| = n F_{n-1}$.

4. More generally, for each $p \geq 0$ one can easily compute the generating functions of the sequences of the p-th moments $\langle \sum_{k=0}^{n} k^p f_{n,k} \rangle_{n=0}^{\infty}$ resp. $\langle \sum_{k=0}^{n} k^p \ell_{n,k} \rangle_{n=0}^{\infty}$ of the degree sequences $\langle f_{n,k} \rangle_{k=0}^{n}$ resp. $\langle \ell_{n,k} \rangle_{k=0}^{n}$ from the higher derivatives of $f(x,y)$ resp. $\ell(x,y)$. Since

$$\frac{\partial^p}{\partial y^p} f(x,y)|_{y=1} \;=\; \sum_{n\geq 0} x^n \sum_{k=0}^{n} k^{\underline{p}} f_{n,k}$$

where $k^{\underline{p}} = \prod_{j=0}^{p-1}(k-j)$ is the p-th falling power of k, we have

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} k^p f_{n,k} \;=\; \sum_{n\geq 0} x^n \sum_{k=0}^{n} \sum_{j=0}^{p} S_{p,j} k^{\underline{j}} f_{n,k}$$

$$=\; \sum_{j=0}^{p} S_{p,j} \sum_{n\geq 0} x^n \sum_{k=0}^{n} k^{\underline{j}} f_{n,k} \;=\; \sum_{j=0}^{p} S_{p,j} \frac{\partial^j}{\partial y^j} f(x,y)|_{y=1}$$

where $S_{p,j}$ denotes Stirling numbers of the second kind. Similarly,

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} k^p \ell_{n,k} \;=\; \sum_{j=0}^{p} S_{p,j} \frac{\partial^j}{\partial y^j} \ell(x,y)|_{y=1}.$$

18

Acknowledgement

References

[1] S. Cabello, D. Eppstein, and S. Klavžar. The Fibonacci dimension of a graph. submitted. arXiv:0903.2507v1 [math.CO].

[2] E. Dedó, D. Torri, and N. Zagaglia Salvi. The observability of the Fibonacci and the Lucas cubes. *Discrete Math.*, 255(1-3):55–63, 2002.

[3] J. A. Ellis-Monaghan, D. A. Pike, and Y. Zou. Decycling of Fibonacci cubes. *Australas. J. Combin.*, 35:31–40, 2006.

[4] S. Fontanesi, E. Munarini, and N. Zagaglia Salvi. On the Fibonacci semilattices. In *Algebras and combinatorics (Hong Kong, 1997)*, pages 237–245. Springer, Singapore, 1999.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1989.

[6] P. Gregor. Recursive fault-tolerance of Fibonacci cube in hypercubes. *Discrete Math.*, 306(13):1327–1341, 2006.

[7] W.-J. Hsu. Fibonacci cubes—a new interconnection technology. *IEEE Trans. Parallel Distrib. Syst.*, 4(1):3–12, 1993.

[8] S. Klavžar. On median nature and enumerative properties of Fibonacci-like cubes. *Discrete Math.*, 299(1-3):145–153, 2005.

[9] S. Klavžar and I. Peterin. Edge-counting vectors, Fibonacci cubes, and Fibonacci triangle. *Publ. Math. Debrecen*, 71(3-4):267–278, 2007.

[10] S. Klavžar and P. Žigert. Fibonacci cubes are the resonance graphs of Fibonaccenes. *Fibonacci Quart.*, 43(3):269–276, 2005.

[11] E. Munarini, C. Perelli Cippo, and N. Zagaglia Salvi. On the Lucas cubes. *Fibonacci Quart.*, 39(1):12–21, 2001.

[12] E. Munarini and N. Salvi Zagaglia. Structural and enumerative properties of the Fibonacci cubes. *Discrete Math.*, 255(1-3):317–324, 2002.

[13] D. Offner. Some Turán type results on the hypercube. *Discrete Math.*, 309(9):2905–2912, 2009.

[14] D. A. Pike and Y. Zou. The domination number of Fibonacci cubes. Manuscript, 2009.

[15] A. Taranenko and A. Vesel. Fast recognition of Fibonacci cubes. *Algorithmica*, 49(2):81–93, 2007.

[16] N. Zagaglia Salvi. The automorphism group of the Lucas semilattice. *Bull. Inst. Combin. Appl.*, 34:11–15, 2002.