The distinguishing chromatic number of Cartesian products of two complete graphs*

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Abstract

A labeling of a graph G is distinguishing if it is only preserved by the trivial automorphism of G. The distinguishing chromatic number of G is the smallest integer k such that G has a distinguishing labeling that is at the same time a proper vertex coloring. The distinguishing chromatic number of the Cartesian product $K_k \,\square\, K_n$ is determined for all k and n. In most of the cases it is equal to the chromatic number, thus answering a question of Choi, Hartke and Kaul whether there are some other graphs for which this equality holds.

Key words: distinguishing chromatic number; graph automorphism; Cartesian product of graphs

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1 Introduction

The distinguishing number of a graph, introduced in 1996 by Albertson and Collins [2], is by today an established and well-studied graph invariant. See [4, 5, 11, 13, 19] for some of the recent results. Ten years later Collins and Trenk [7] followed with a natural variation of the distinguishing number, the distinguishing chromatic number of a graph G, denoted by $\chi_D(G)$. Here not only vertices are distinguished but the corresponding labelings must be proper vertex colorings. Among other results Collins and Trenk determined the distinguishing chromatic number for some basic families of graphs, characterized trees T with $\chi_D(T) = 2$, and obtained an analogue of Brooks theorem proving that $\chi_D(G) \leq 2\Delta(G)$ with a list of the corresponding extremal graphs. (They

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also proved a Brooks-type theorem for the distinguishing number, a result obtained independently in [14].)

Distinguishing numbers of hypercubes were determined in [3]. This result was superceded with a series of papers [1, 15, 11] in which the distinguishing number was determined for all powers of graphs with respect to the Cartesian product. (We note that the paper [11] is the final paper in this series, although it was eventually published before [15].) Moreover, the distinguishing number of Cartesian products of two complete graphs were independently determined in [8, 12].

Choi, Hartke and Kaul [6] studied the distinguishing chromatic number of Cartesian product graphs. They proved that for every graph G there exists a constant d_G such that $\chi_D(G^d) \leq \chi(G) + 1$ for $d \geq d_G$. For hypercubes they proved $\chi_D(Q_3) = 4$, $3 \leq \chi_D(Q_4) \leq 4$, and $\chi_D(Q_n) = 3$, $n \geq 5$. The remaining case Q_4 was settled by Klöckl [16] by showing that $\chi_D(Q_4) = 4$. Choi et al. [6] also showed that the distinguishing chromatic number of the Cartesian product of five or more complete graphs is at most one more than its chromatic number.

Clearly, $\chi_D(G) = \chi(G)$ for any asymmetric graph. This equality holds also for complete graphs and large enough Kneser graphs (due to a personal communication of Füredi to the authors). Choi et al. finish their paper with the following question: Are there some other graphs for which this equality holds? In this paper we prove that this equality holds for almost all graphs $K_k \,\Box\, K_n$. More precisely, we prove the following result.

Theorem 1.1 *Let* $1 \leq k \leq n$. *Then*

$$\chi_D(K_k \,\Box\, K_n) = \begin{cases} n = \chi(K_k \,\Box\, K_n); & k = n = 1, k = n \geq 5, k < n\,, \\ n + 1 = \chi(K_k \,\Box\, K_n) + 1; & k = n = 4\,, \\ n + 2 = \chi(K_k \,\Box\, K_n) + 2; & k = n = 2, k = n = 3\,. \end{cases}$$

In the next section we give concepts needed in this paper and prove the case $k < n$ of the theorem. Then, in Section 3, the distinguishing chromatic number is determined for $K_3 \,\Box\, K_3$ and $K_4 \,\Box\, K_4$, while in the last two sections $K_n \,\Box\, K_n$ is considered for even $n \geq 6$ and odd $n \geq 5$, respectively.

2 Preliminaries and the case $k < n$

Let G be a graph. A labeling (sometimes also called coloring) $\ell : V(G) \to \{0, 1, \ldots, d-1\}$ is *d-distinguishing* if it is invariant only under the trivial automorphism. The *distinguishing number* of a graph G, $D(G)$, is the least integer d such that G has a d-distinguishing labeling. A d-distinguishing labeling $\ell : V(G) \to \{0, 1, \ldots, d-1\}$ is a *proper d-distinguishing labeling* if it is a proper d-coloring of G. The *distinguishing chromatic number* of a graph G, $\chi_D(G)$, is the least integer d such that G has a proper d-distinguishing labeling. Clearly, $\chi_D(G) \geq \max\{\chi(G), D(G)\}$ for any graph G.

The *Cartesian product* $G \,\Box\, H$ of graphs G and H has the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') are adjacent if they are equal in one coordinate and adjacent in the other. The subgraph of $G \Box H$ induced by $\{g\} \times V(H)$ is isomorphic to H and called an *H-fiber*. *G-fibers* are defined analogously.

It is well-known that $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$ for any graphs G and H [18], see [10]. In particular, $\chi(K_k \Box K_n) = \max\{k, n\}$. It is also well-known that the automorphism group of a Cartesian product graph is generated by automorphisms of the factors and transpositions of isomorphic factors [9], see [10].

Let $V(K_k) = \{x_1, x_2, \ldots, x_k\}$ and $V(K_n) = \{y_1, y_2, \ldots, y_n\}$. A d-coloring of the vertices of the graph $K_k \Box K_n$ corresponds to a n by k matrix L with entries from $\{0, 1, \ldots, d-1\}$. The i, j entry of the matrix L is m whenever the vertex (x_i, y_j) in $K_k \Box K_n$ is colored with m. For $k \neq n$ every automorphism of $K_k \Box K_n$ preserves the set of K_k-fibers and the set of K_n-fibers. In this case every automorphism φ of $K_k \Box K_n$ is determined by a permutation $\pi \in S_n$ of K_k-fibers and a permutation $\psi \in S_k$ of K_n-fibers. Let P_π be the permutation matrix representing permutation $\pi \in S_n$ and P_ψ the permutation matrix representing permutation $\psi \in S_k$. Then φ preserves the coloring L if $L = P_\pi L P_\psi$. Moreover, L is a d-distinguishing coloring if $L = P_\pi L P_\psi$ implies $\pi = id$ and $\psi = id$. If $k = n$, every automorphism of $K_k \Box K_n = K_n \Box K_n$ is generated by the automorphisms of the factors and the transpositions of isomorphic factors. In this case φ preserves the coloring L if $L = P_\pi L P_\psi$ or $L = P_\pi L^T P_\psi$, where P_π and P_ψ are defined as before.

In the rest of this section we will determine $\chi_D(K_k \Box K_n)$ for $k < n$. We may assume that $k \geq 2$, since it is already known [7] that $\chi_D(K_1 \Box K_n) = \chi_D(K_n) = n$.

Suppose that $n = \chi(H) \leq \chi(G) = m$, $g : V(G) \to \{0, 1, \ldots, m-1\}$ is a proper m-coloring of G and $h : V(H) \to \{0, 1, \ldots, n-1\}$ is a proper n-coloring of H. Then the coloring $f : V(G \Box H) \to \{0, 1, \ldots, m-1\}$ defined as:

$$f(a, x) = g(a) + h(x) \pmod{m}$$

is a proper m-coloring of $G \Box H$. We call such a coloring the *canonical coloring*. We claim that the canonical n-coloring of $K_k \Box K_n$ is a distinguishing labeling. Since $k < n$, an automorphism ϕ of $K_k \Box K_n$ can only map a K_k-fiber onto a K_k-fiber by a color preserving automorphism ϕ. Moreover (since $k < n$), the labeling of K_k-fibers is pairwise different and hence they must be stabilized by ϕ. But then ϕ stabilizes the K_n-fibers as well.

3 $K_3 \Box K_3$ and $K_4 \Box K_4$

In the rest of the paper we thus need to treat products $K_n \Box K_n$. Since $K_1 \Box K_1 = K_1$, $\chi_D(K_1 \Box K_1) = 1$. Moreover, $K_2 \Box K_2 = C_4$ and hence by [7], $\chi_D(K_2 \Box K_2) = 4$. In this section we determine χ_D for $n = 3$ and $n = 4$.

The matrix

$$L = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

represents the unique 3-coloring (up to color classes) of $K_3 \,\square\, K_3$. But it is not a distinguishing labeling since $P_\pi L^T P_\psi = L$, where

$$P_\pi = P_\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence $\chi_D(K_3 \,\square\, K_3) \geq 4$. Consider next proper 4-colorings of $K_3 \,\square\, K_3$. Since $|V(K_3 \,\square\, K_3)| = 9$, at least one color must appear exactly 3 times. We may assume without loss of generality that the coloring has the form

$$L = \begin{bmatrix} 0 & 1 & 2 \\ - & 0 & - \\ - & - & 0 \end{bmatrix}.$$

The first possibility is that colors 1, 2 and 3 each appears exactly twice. Then there are three possibilities for color 3:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & - \\ - & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ - & 0 & 3 \\ 3 & - & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 2 \\ - & 0 & 3 \\ - & 3 & 0 \end{bmatrix}$$

and hence we get the following possible colorings:

$$L_1 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 3 & 2 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}, \quad L_{3'} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}.$$

The reflection of $L_{3'}$ over the antidiagonal gives the same (up to color classes) coloring as L_2.

The second possibility is that one of the colors 1 and 2 appear three times. We may without loss of generality assume it is color 1. This leads to the following colorings:

$$L_4 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \quad L_5 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \quad L_6 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

But none of the matrices L_i, $1 \leq i \leq 6$, presents a distinguishing labeling of $K_3 \,\square\, K_3$ because L_3 is symmetric, L_1 and L_4 are symmetric with respect to the antidiagonal, while for $i = 2, 5, 6$ we have $P_{\pi_i} L_i^T P_{\psi_i} = L_i$, where

$$P_{\pi_2} = P_{\psi_2} = P_{\pi_6} = P_{\psi_6} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P_{\pi_5} = P_{\psi_5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence $\chi_D(K_3 \,\square\, K_3) \geq 5$. Finally, it is straightforward to verify that the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 0 \end{bmatrix}$$

gives a proper 5-distinguishing labeling.

The product $K_4 \,\square\, K_4$ is considered similarly. Among the different proper 4-colorings

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix},$$

the first three are symmetric over the antidiagonal (the first is also symmetric), while for the last one, denote it with L, we infer that $P_\pi L^T P_\psi = L$, where

$$P_\pi = P_\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore $\chi_D(K_4 \,\square\, K_4) \geq 5$. To conclude that $\chi_D(K_4 \,\square\, K_4) \leq 5$ verify that

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 2 & 4 & 0 & 1 \\ 3 & 2 & 4 & 0 \end{bmatrix}$$

represents a proper 5-distinguishing labeling.

4 Labelings for even $n \geq 6$

In this section we prove that $\chi_D(K_n \,\square\, K_n) = \chi(K_n \,\square\, K_n) = n$ for every even integer $n \geq 6$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and define $\ell_e : V(K_n \,\square\, K_n) \to \{0, 1, \ldots, n-1\}$ as follows:

$$\ell_e(x_i, x_j) = \begin{cases} n-1; & i = j\,, \\ i-2; & j = 1, i \neq 1\,, \\ j-2; & i = 1, j \neq 1\,, \\ (j+i-3) \bmod (n-1); & i < j, i \neq 1, j \neq 2, n\,, \\ (j+i-4) \bmod (n-1); & i > j, j \neq 1, 2\,, \\ (2i-4) \bmod (n-1); & j = 2, i \geq 3\,, \\ (2i-3) \bmod (n-1); & j = n, 2 \leq i \leq n-1\,. \end{cases}$$

In the rest of the section we prove that ℓ_e is a proper n-distinguishing labeling of $K_n \,\square\, K_n$.

Let L_e be the $n \times n$ matrix which corresponds to the coloring ℓ_e. For $i = 1, \ldots, n$ let R_i be the set of labels from the i'th row of L_e and C_j the set of labels from the j'th column of L_e. Denoting $(L_e)_{ij} = a_{i,j}$ we thus have $R_i = \{a_{i,1}, \ldots, a_{i,n}\}$ and $C_j = \{a_{1,j}, \ldots, a_{n,j}\}$.

In the next two lemmas we prove that ℓ_e is a proper vertex coloring.

Lemma 4.1 $R_i = \{0, 1, \dots, n-1\}$ *for every* $1 \le i \le n$ *.*

Proof. Let $i = 1$. Then $a_{1,1} = n-1$ and $a_{1,j} = j-2$, $2 \le j \le n$. For $2 \le j \le n$ we have $0 \le j-2 \le n-2$. It follows that lemma is true for $i = 1$. Consider now the set R_2. Note that $a_{2,1} = 0$, $a_{2,2} = n-1$, $a_{2,n} = 1$ and $a_{2,j} = (j-1) \bmod (n-1)$ for $3 \le j \le n-1$. Last condition implies that $2 \le j-1 \le n-2$ and hence lemma holds also for $i = 2$.

Suppose now that $3 \le i \le n-1$. Then

$$\begin{aligned}
&a_{i,1} = i-2,\\
&a_{i,2} = (2i-4) \bmod (n-1),\\
&a_{i,j} = (j+i-4) \bmod (n-1) \ \text{ for } \ 3 \le j \le i-1,\\
&a_{i,i} = n-1,\\
&a_{i,j} = (j+i-3) \bmod (n-1) \ \text{ for } \ i+1 \le j \le n-1,\\
&a_{i,n} = (2i-3) \bmod (n-1).
\end{aligned}$$

The sequence of numbers $j+i-4$ for $3 \le j \le i-1$ is a sequence of consecutive integers from $i-1$ to $2i-5$. Similarly, the numbers $j+i-3$ for $i+1 \le j \le n-1$ form a sequence of consecutive integers from $2i-2$ to $n+i-4$. Consequently,

$$\begin{aligned}
R_i \setminus \{a_{i,i}\} &= \{a_{i,1}, a_{i,3}, \dots, a_{i,i-1}, a_{i,2}, a_{i,n}, a_{i,i+1}, \dots, a_{i,n-1}\}\\
&= \{i-2, i-1, \dots, 2i-5, 2i-4, 2i-3, 2i-2, \dots, n+i-4\}\\
&= \{0, 1, \dots, n-2\},
\end{aligned}$$

where the elements of the third set are taken modulo $(n-1)$. The last equality holds because $i-2, i-1, \dots, 2i-5, 2i-4, 2i-3, 2i-2, \dots, n+i-4$ is a sequence of $n-1$ consecutive integers. Since $a_{i,i} = n-1$ we conclude that $R_i = \{0, 1, \dots, n-1\}$ for every $3 \le i \le n-1$.

It remains to show that $R_n = \{0, 1, \dots, n-1\}$. In this case we have $a_{n,1} = n-2$, $a_{n,2} = (2n-4) \bmod (n-1)$, $a_{n,j} = (j+n-4) \bmod (n-1)$ for $3 \le j \le n-1$ and $a_{n,n} = n-1$. The numbers $j+n-4$ for $3 \le j \le n-1$ form a sequence of consecutive integers from $n-1$ to $2n-5$. By adding $n-2$ and $2n-4$ we get a sequence of $n-1$ consecutive integers and consequently $R_n \setminus \{a_{n,n}\} = \{0, 1, \dots, n-2\}$ and $R_n = \{0, 1, \dots, n-1\}$. □

Lemma 4.2 $C_j = \{0, 1, \dots, n-1\}$ *for every* $1 \le j \le n$ *.*

Proof. Consider first C_1. By definition of ℓ_e we have $a_{1,1} = n-1$ and $a_{i,1} = i-2$ for $2 \le i \le n$ which implies that $C_1 = \{0, 1, \dots, n-1\}$. C_2 consist of $a_{1,2} = 0$, $a_{2,2} = n-1$ and $a_{i,2} = (2i-4) \bmod (n-1)$ for $3 \le i \le n$. From the last condition we deduce that $\{2i-4\ ;\ 3 \le i \le n\}$ is the set of all even numbers between 2 and $2n-4$ and hence $\{a_{i,2}\ ;\ 3 \le i \le n\} = \{1, 2, \dots, n-2\}$. It follows that $C_2 = \{0, 1, \dots, n-1\}$.

For $3 \le j \le n-1$ we have

$$\begin{aligned}
a_{1,j} &= j-2,\\
a_{i,j} &= (j+i-3) \bmod (n-1) \quad \text{for} \quad 2 \le i \le j-1,\\
a_{j,j} &= n-1,\\
a_{i,j} &= (j+i-4) \bmod (n-1) \quad \text{for} \quad j+1 \le i \le n.
\end{aligned}$$

The sequence of numbers $j+i-3$ for $2 \le i \le j-1$ is a sequence of consecutive integers from $j-1$ to $2j-4$. Similarly, the numbers $j+i-4$ for $j+1 \le i \le n$ form a sequence of consecutive integers from $2j-3$ to $n+j-4$. Putting all of this numbers together and adding $j-2$ gives us a sequence of $n-1$ consecutive integers which are consequently pairwise different by modulo $(n-1)$. Since $a_{j,j} = n-1$ we can conclude that lemma holds for every $3 \le j \le n-1$.

Finally, let $j = n$. Then $a_{1,n} = n-2$, $a_{i,n} = (2i-3) \bmod (n-1)$ for $2 \le i \le n-1$ and $a_{n,n} = n-1$. The set $\{2i-3\ ;\ 2 \le i \le n-1\}$ is the set of all odd numbers between 1 and $2n-5$ and hence $\{a_{i,n}\ ;\ 2 \le i \le n-1\} = \{0, 1, \ldots, n-3\}$. After adding $a_{1,n}$ and $a_{n,n}$ to this set we get $C_n = \{0, 1, \ldots, n-1\}$. $\square$

To complete the proof for even $n \ge 6$, we need to prove that ℓ_e is a distinguishing labeling.

Let φ be an automorphism of $K_n \,\square\, K_n$ that preserves ℓ_e. Suppose first that the factors of the product do not interchange, then φ is determined by a permutation of rows and a permutation of columns of L_e. Let $\pi \in S_n$ be the corresponding permutation of rows. Then the permutation of columns is uniquely determined because the diagonal elements are the only elements labeled with $n-1$ and must hence be mapped onto the diagonal elements. In other words, the permutation of columns is the same as π. The matrix

$$L_e = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

changes to

$$\begin{bmatrix} a_{\pi^{-1}(1),\pi^{-1}(1)} & a_{\pi^{-1}(1),\pi^{-1}(2)} & \cdots & a_{\pi^{-1}(1),\pi^{-1}(n)} \\ a_{\pi^{-1}(2),\pi^{-1}(1)} & a_{\pi^{-1}(2),\pi^{-1}(2)} & \cdots & a_{\pi^{-1}(2),\pi^{-1}(n)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\pi^{-1}(n),\pi^{-1}(1)} & a_{\pi^{-1}(n),\pi^{-1}(2)} & \cdots & a_{\pi^{-1}(n),\pi^{-1}(n)} \end{bmatrix},$$

after the action of π onto rows and onto columns. Since φ is label preserving, the new matrix equals to L_e. Since the first row and the first column of L_e are equal, we have:

$$\begin{aligned}
a_{\pi^{-1}(1),\pi^{-1}(1)} &= a_{\pi^{-1}(1),\pi^{-1}(1)} = a_{1,1} = a_{1,1} = n-1\\
a_{\pi^{-1}(2),\pi^{-1}(1)} &= a_{\pi^{-1}(1),\pi^{-1}(2)} = a_{2,1} = a_{1,2} = 0\\
a_{\pi^{-1}(3),\pi^{-1}(1)} &= a_{\pi^{-1}(1),\pi^{-1}(3)} = a_{3,1} = a_{1,3} = 1\\
&\vdots\\
a_{\pi^{-1}(n),\pi^{-1}(1)} &= a_{\pi^{-1}(1),\pi^{-1}(n)} = a_{n,1} = a_{1,n} = n-2.
\end{aligned}$$

Since $\{\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n)\} = \{1, 2, \ldots, n\}$ and $\pi^{-1}(1) = i$ for some fixed $i \in \{1, \ldots, n\}$, it follows that the i'th column of L_e is equal to the i'th row of L_e. However, we claim that this holds only for $i = 1$. Note first that since $a_{4,2} = 4$ and $a_{2,4} = 3$ we have $i \neq 2$. For $3 \leq i \leq n-2$ we note that $a_{i+1,i} = (2i-3) \bmod (n-1)$ and $a_{i,i+1} = (2i-2) \bmod (n-1)$, therefore $a_{i+1,i} \neq a_{i,i+1}$. Finally, because

$$(2n-6) \bmod (n-1) = a_{n-2,n-1} \neq a_{n-1,n-2} = (2n-7) \bmod (n-1)$$

and

$$(2n-7) \bmod (n-1) = a_{n-2,n} \neq a_{n,n-2} = (2n-6) \bmod (n-1)$$

we also have $i \neq n-1, n$. Therefore,

$$\begin{aligned}
&a_{\pi^{-1}(1),1} = a_{1,\pi^{-1}(1)} = a_{1,1} = a_{1,1} = n-1\\
&a_{\pi^{-1}(2),1} = a_{1,\pi^{-1}(2)} = a_{2,1} = a_{1,2} = 0\\
&a_{\pi^{-1}(3),1} = a_{1,\pi^{-1}(3)} = a_{3,1} = a_{1,3} = 1\\
&\vdots\\
&a_{\pi^{-1}(n),1} = a_{1,\pi^{-1}(n)} = a_{n,1} = a_{1,n} = n-2.
\end{aligned}$$

It follows that $\pi^{-1}(2) = 2$ because $a_{2,1}$ is the only element from the first column of L_e that is 0. Similarly we infer that $\pi^{-1}(i) = i$ for all $3 \leq i \leq n$. But this means that π is the identity and so is φ.

The second case is when φ also exchanges the fibers. This corresponds to the transposition of L_e. The transposition does not preserve ℓ_e because the label of the vertex $(x_{n/2}, x_{(n+2)/2})$ is $n-2$ while the label of $(x_{(n+2)/2}, x_{n/2})$ is $n-3$. Since the transposition does not preserves the labeling, it will also not be preserved by analogous arguments as above additional permutation of rows and columns. Hence φ is the identity automorphism.

5 Labelings for odd $n \geq 5$

The proof of Theorem 1.1 will be complete by proving that $\chi_D(K_n \,\square\, K_n) = n$ for every odd integer $n \geq 5$. Again let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and define $\ell_o : V(K_n \,\square\, K_n) \to \{0, 1, \ldots, n-1\}$ as follows:

$$\ell_o(x_i, x_j) = \begin{cases}
n-1; & i = j,\\
i-2; & j = 1, i \neq 1\,,\\
j-2; & i = 1, j \neq 1\,,\\
(j+i-3) \bmod (n-1); & i < j, i \neq 1, j \neq 2, n\,,\\
(j+i-4) \bmod (n-1); & i > j, j \neq 1, 2\,,\\
(2i-4) \bmod (n-1); & 3 \leq i \leq (n-1)/2, j = 2 \text{ or } i = n, j = 2 \text{ or}\\
& (n+1)/2 \leq i \leq n-1, j = n\,,\\
(2i-3) \bmod (n-1); & (n+1)/2 \leq i \leq n-1, j = 2 \text{ or}\\
& 2 \leq i \leq (n-1)/2, j = n\,.
\end{cases}$$

We proceed similarly as in the previous section. Let L_o be the $n \times n$ matrix which corresponds to the coloring ℓ_o, let R_i be the set of labels from the i'th row of L_o, let C_j be the set of labels from the j'th column of L_o, and let $(L_o)_{ij} = a_{i,j}$. The next two lemmas take care for proper vertex coloring.

Lemma 5.1 $R_i = \{0, 1, \ldots, n-1\}$ *for every* $1 \leq i \leq n$ *.*

Proof. Note first that the cases for $i = 1$, $i = 2$ and $i = n$ are the same as in the proof of Lemma 4.1. Next, according to the definition of ℓ_o, it remains to consider two cases. First one for $3 \leq i \leq \frac{n-1}{2}$ and the second one for $\frac{n+1}{2} \leq i \leq n-1$. The values for $a_{i,j}$ in the first case are exactly the same as the values for $a_{i,j}$ in Lemma 4.1 for $3 \leq i \leq n-1$. Switching the values of $a_{i,2}$ and $a_{i,n}$ in the first case and keeping the rest, gives us the values of $a_{i,j}$ for the second case. Since $R_i = \{0, 1, \ldots, n-1\}$ in the first case (see Lemma 4.1), this holds also in the second case. □

Lemma 5.2 $C_j = \{0, 1, \ldots, n-1\}$ *for every* $1 \leq j \leq n$ *.*

Proof. The proof differs from the one of the Lemma 4.2 only for $j = 2$ and $j = n$.

In the case of $j = 2$ we have

$$\begin{aligned}
&a_{1,2} = 0, \\
&a_{2,2} = n-1, \\
&a_{i,2} = (2i-4) \bmod (n-1) \ \text{ for } \ 3 \leq i \leq \tfrac{n-1}{2} \text{ and } i = n, \\
&a_{i,2} = (2i-3) \bmod (n-1) \ \text{ for } \ \tfrac{n+1}{2} \leq i \leq n-1.
\end{aligned}$$

The set $\{2i-4 \ ; \ 3 \leq i \leq \frac{n-1}{2}\} = \{(2i-4) \bmod (n-1) \ ; \ 3 \leq i \leq \frac{n-1}{2}\}$ is the set of all even numbers from 2 to $n-5$, $(2n-4) \equiv (n-3) \bmod (n-1)$ and $\{2i-3 \ ; \ \frac{n+1}{2} \leq i \leq n-1\}$ is the set of all odd integers from $n-2$ to $2n-5$. It follows that $\{(2i-3) \bmod (n-1) \ ; \ \frac{n+1}{2} \leq i \leq n-1\}$ is the set of all odd numbers from 1 to $n-2$. Hence $C_2 = \{0, 1, \ldots, n-1\}$.

Let $j = n$. Then

$$\begin{aligned}
&a_{1,n} = n-2, \\
&a_{i,n} = (2i-3) \bmod (n-1) \ \text{ for } \ 2 \leq i \leq \tfrac{n-1}{2}, \\
&a_{i,n} = (2i-4) \bmod (n-1) \ \text{ for } \ \tfrac{n+1}{2} \leq i \leq n-1, \\
&a_{n,n} = n-1.
\end{aligned}$$

The set $\{2i-3 \ ; \ 2 \leq i \leq \frac{n-1}{2}\} = \{(2i-3) \bmod (n-1) \ ; \ 2 \leq i \leq \frac{n-1}{2}\}$ is the set of all odd numbers from 1 to $n-4$ and $\{2i-4 \ ; \ \frac{n+1}{2} \leq i \leq n-1\}$ is the set of all even numbers from $n-3$ to $2n-6$. Last observation implies that $\{(2i-4) \bmod (n-1) \ ; \ \frac{n+1}{2} \leq i \leq n-1\}$ is the set of all even numbers from 0 to $n-3$: Hence $C_n = \{0, 1, \ldots, n-1\}$. □

To complete the argument we need to prove that ℓ_o is a distinguishing labeling. Since the proof goes along the same lines as the corresponding proof from Section 4 we only point out the differences and leave the details to the reader. We first show that the i'th column of L_o is equal to the i'th row of L_o if and only if $i = 1$. For

$i = 2$ we have $a_{n,2} = n - 3$ and $a_{2,n} = 1$. In addition, $a_{i+1,i} = (2i - 3) \bmod (n - 1)$ and $a_{i,i+1} = (2i - 2) \bmod (n - 1)$ for $3 \leq i \leq n - 2$, hence $a_{i+1,i} \neq a_{i,i+1}$. Finally, $(2n-6) \bmod (n-1) = a_{n-2,n-1} \neq a_{n-1,n-2} = (2n-7) \bmod (n-1)$ and $(2n-6) \bmod (n-1) = a_{n-1,n} \neq a_{n,n-1} = (2n - 5) \bmod (n - 1)$.

To show that the reflection does not preserve the labeling note that $(x_{(n-1)/2}, x_{(n+1)/2})$ is labeled $n - 3$ while $(x_{(n+1)/2}, x_{(n-1)/2})$ with $n - 4$.

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