Labeling Dot-Cartesian and Dot-Lexicographic Product Graphs with a Condition at Distance Two

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Abstract

If $d(x, y)$ denotes the distance between vertices x and y in a graph G, then an $L(2, 1)$-labeling of a graph G is a function f from vertices of G to nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$, and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$. Griggs and Yeh conjectured that for any graph with maximum degree $\Delta \geq 2$, there is an $L(2, 1)$-labeling with all labels not greater than Δ^2. We prove that the conjecture holds for dot-Cartesian products and dot-lexicographic products of two graphs with possible minor exceptions in some special cases. The bounds obtained are in general much better than the Δ^2-bound.

Key words: frequency assignment; $L(2, 1)$-labeling; graph product; dot-Cartesian product; dot-lexicographic product

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1 Introduction

In the frequency assignment problem, radio transmitters are assigned frequencies with some separation in order to reduce interference. This problem can be formulated as a graph coloring problem [1]. Roberts [2] proposed a new version of the frequency assignment problem with two restrictions: radio transmitters that are "close" must be assigned different frequencies; those that are "very close" must be assigned frequencies at least two apart. To formulate the problem in graph theoretic terms, radio transmitters are represented by vertices of a graph; adjacent vertices are considered "very close" and vertices at distance two are considered "close". Let $d(x, y)$ be the distance between vertices x and y in a graph G. An $L(2, 1)$-labeling of a graph G is a function f from all vertices of G to non-negative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$. For an $L(2, 1)$-labeling, if the maximum label is no greater than k, then it is called

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a k-$L(2,1)$-*labeling*. The $L(2,1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling. The theory of $L(2,1)$-labeling is now already very extensive, see the 2006 survey of Yeh [3] and the 2011 updated survey and annotated bibliography by Calamoneri [4] containing 184 references. From recent results we point to two appealing algorithmic achievements: A linear time algorithm for $L(2,1)$-labeling of trees [5] and a polynomial space algorithm to determine the $L(2,1)$-span in the general case [6].

Griggs and Yeh [7] proved that it is NP-complete to decide whether a given graph G allows an $L(2,1)$-labeling of span at most n. Thus, it is important to obtain good lower and upper bounds for λ. For a diameter two graph G, it is known that $\lambda(G) \leq \Delta^2$, where $\Delta = \Delta(G)$ is the maximum degree of G, and the upper bound can be attained by *Moore graphs*, that is, diameter 2 graphs of order $\Delta^2 + 1$ [7]. Based on the previous research, Griggs and Yeh [7] conjectured that $\lambda(G) \leq \Delta^2$ holds for any graph G with $\Delta \geq 2$. The conjecture is known as the Δ^2-conjecture and considered as the most important open problem in the area. The best general bound $\Delta^2 + \Delta - 2$ so far is due to Gonçalves [8]. Havet, Reed and Sereni [9] proved that the Δ^2-conjecture holds for sufficiently large Δ.

A lot of research regarding $L(2,1)$-labelings (and, more generally of $L(j,k)$-labelings) was done on standard graph products, cf. recent investigations on the Cartesian product [10, 11, 12, 13, 14], the direct product [15, 16], the lexicographic product [17], and the strong product [18]; cf. also references therein. A special emphasize was put on the Δ^2-conjecture. In [19] the conjecture was verified for lexicographic products as well as for Cartesian products with factors of minimum degree at least 2. In [20] the Δ^2-conjecture was confirmed for the strong and the direct product of graphs. The obtained upper bounds on these two products were later improved in [21]. Shiu et al. [22] used an analysis of the adjacency matrices of the graphs to obtain improvements of the previous bounds on all the above four (standard) graph products. Finally, in [23] the Δ^2-conjecture was verified for modular products of two graphs with minor exceptions. Now, modular product is obtained from the strong product by superimposing edges that come from non-edges in both facts. This construction is not really interesting for the direct product. Hence, as there are four standard graph products, there are two natural additional products (w.r.t. the superimposition of the edges that come from non-edges) to consider—the products obtained from the Cartesian product and the lexicographic product, named the dot-Cartesian and the dot-lexicographic (see the next section for formal definitions). In this paper we prove that the Δ^2-conjecture is true also for these products with possible minor exceptions. The bounds obtained are typically much better than the Δ^2-bound.

2 Preliminaries

In this section we first introduce the graph products of our main interest, the dot-Cartesian product and the dot-lexicographic product, and then recall a labeling algorithm of Chang and Kuo that will be a key tool in our proofs. To avoid ambiguities with the definitions of graph products we emphasize that all graphs considered in this paper are without loop.

As we have seen in the introduction (see also [24]), there are many different graph products. In order to simplify their description (and to classify which products are associative and commutative), Imrich and Izbicki [25] (cf. also [24]) introduced the following useful convention. For a graph G, let $\delta : V(G) \times V(G) \to \{\Delta, 1, 0\}$, where Δ is a previously undefined symbol, be a function defined as follows:

$$\delta(g, g') = \begin{cases} \Delta & \text{if } g = g', \\ 1 & \text{if } g \neq g' \text{ and } gg' \in E(G), \\ 0 & \text{if } g \neq g' \text{ and } gg' \notin E(G). \end{cases}$$

So δ encodes the incidence relation of G. An operation $*$ is a *graph product*, if $V(G * H) = V(G) \times V(H)$ and $\delta((g, h), (g', h'))$ is a function of $\delta(g, g')$ and $\delta(h, h')$. Such a function is a binary operation on the set $\{\Delta, 1, 0\}$, and it can be written as $\delta((g, h), (g', h')) = \delta(g, g') * \delta(h, h')$. For example, the multiplication tables for the Cartesian product and the lexicographic product are shown in Tables 1 and 2, respectively.

$\square$	Δ	1	0
Δ	Δ	1	0
1	1	0	0
0	0	0	0

Table 1: Cartesian product

$\circ$	Δ	1	0
Δ	Δ	1	0
1	1	1	1
0	0	0	0

Table 2: lexicographic product

In this way graphs products are defined in a compact way. Indeed, the Cartesian product is usually introduced as follows: The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which the vertex (g, h) is adjacent to the vertex (g', h') if and only if either $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G. The standard (rather clumsy) definition of the lexicographic product $G \circ H$ should now be clear from Table 2. We add here that some authors use the notation $G[H]$ for the lexicographic product. However, we prefer the notation $G \circ H$ because this graph operation is associative. Note also that some authors use the term *composition* for the lexicographic product.

We now introduce the *dot-Cartesian product* $\dot{\square}$ and the *dot-lexicographic product* $\odot$ with the following two tables:

$\boxdot$	Δ	1	0
Δ	Δ	1	0
1	1	0	0
0	0	0	1

Table 3: dot-Cartesian product

$\odot$	Δ	1	0
Δ	Δ	1	0
1	1	1	1
0	0	0	1

Table 4: dot-lexicographic product

Hence the dot-Cartesian product $G \boxdot H$ is obtained from the Cartesian product $G \square H$ by adding the edges $(g,h)(g'h')$, where $gg' \notin E(G)$ and $hh' \notin E(H)$. Analogously, the dot-lexicographic product $G \odot H$ is obtained from the lexicographic product $G \circ H$. (As already mentioned in the introduction, the modular product is obtained in the same manner from the strong product.) As for the notation, note that the strong product $G \boxtimes H$ is obtained from the Cartesian product $G \square H$ by adding the edges of the direct product $G \times H$. So in our case, the central dot means "not an edge in both factors", just like the central cross stands for "an edge in both factors".

Note that K_1 is a unit for both new products, that is, $G \boxdot K_1 = K_1 \boxdot G = G$ and $G \odot K_1 = K_1 \odot G = G$, where by abuse of notation, the equality sigh stands for graph isomorphism. Therefore, we may assume in the rest that all factors have at least two vertices.

We next recall the announced labeling algorithm of Chang and Kuo. For a subset X of $V(G)$, if the distance between any two vertices in X is greater than i, then X is called an *i-stable set* (or *i-independent set*). A 1-stable (independent) set is a usual independent set. A *maximal* 2-stable subset X of a set Y is a 2-stable subset of Y such that X is not a proper subset of any 2-stable subset of Y.

Chang and Kuo [26] introduced the following algorithm to obtain an $L(2,1)$-labeling and the maximum value of that labeling on any given graph. For its statement recall that a vertex subset X of a graph is 2-*stable* (also called a packing) if the distance between any two vertices in X is greater than 2.

Algorithm Label(G)

Input: A graph $G = (V, E)$.

Output: Value k which is the maximum label.

Idea: In each step, find a maximal 2-stable set from all unlabeled vertices which are distance at least two away from the vertices labeled in the previous step. Then label all vertices in this 2-stable set with the same index i. The index i starts from 0 and then increases by 1 in each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.

Iteration:

1. Determine Y_i and X_i.

   - $Y_i = \{x \in V : x \text{ is unlabeled and } d(x, y) \geq 2 \text{ for all } y \in X_{i-1}\}$.

   - X_i a maximal 2-stable subset of Y_i.

   - If $Y_i = \emptyset$ then set $X_i = \emptyset$.

2. Label all vertices in X_i (if there are any) by i.

3. $V \leftarrow V \backslash X_i$.

4. If $V \neq \emptyset$ then $i \leftarrow i + 1$, and go to Step 1.

5. Record the current i as k (which is the maximum label). Stop.

It is clear that the labeling constructed by the above procedure is an $L(2, 1)$-labeling of G (cf. the proof of [26, Theorem 4.1]). It is usually used (as it will be later on in this paper) to obtain (theoretical) upper bounds but it is worth mentioning that the procedure can be implemented in polynomial time. As a preprocessing, distances between vertices of G are computed which can be done in $\mathcal{O}(|V(G)| \cdot |E(G)|)$ time. In the main loop, computing Y_i is a simple task by observing that $x \in Y_i$ if and only if $x \notin N[X_{i-1}]$, that is, $Y_i = V \setminus N[X_{i-1}]$. (Here $N[X]$ denotes the closed neighborhood of X.) Finally, X_i is computed using the greedy approach: start with an empty set, and during the process add to X_i the next vertex from Y_i if it is at distance at least 3 to all already selected vertices. Since distances were precomputed, X_i can be obtained in time $\mathcal{O}(|Y_i)|^2)$ time.

As already mentioned, the value k obtained by the above labeling procedure is an upper bound on $\lambda(G)$. To get a bound in terms of the maximum degree $\Delta(G)$ of G we proceed as follows. Let x be a vertex with the largest label k obtained by Algorithm LABEL. Denote

$I_1 = \{i : 0 \leq i \leq k - 1 \text{ and } d(x, y) = 1 \text{ for some } y \in X_i\}$

$I_2 = \{i : 0 \leq i \leq k - 1 \text{ and } d(x, y) \leq 2 \text{ for some } y \in X_i\}$

$I_3 = \{i : 0 \leq i \leq k - 1 \text{ and } d(x, y) \geq 3 \text{ for all } y \in X_i\}$

It is clear that $|I_2| + |I_3| = k$. For any $i \in I_3$, $x \notin Y_i$; otherwise $X_i \cup \{x\}$ is a 2-stable subset of Y_i, which contradicts the choice of X_i. That is, $d(x, y) = 1$ for some vertex y in X_{i-1}; i.e., $i - 1 \in I_1$. So, $|I_3| \leq |I_1|$. Hence $k \leq |I_2| + |I_3| \leq |I_2| + |I_1|$.

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In order to find k, it suffices to estimate $B = |I_1| + |I_2|$ in terms of $\Delta(G)$. We will investigate the value B for the two classes of graphs introduced in the previous section. The notation introduced in this section will also be used in the remainder of the paper.

3 $L(2,1)$-labelings of dot-Cartesian products

Throughout this section let G_1 and G_2 be graphs of order $n_1 \geq 2$ and $n_2 \geq 2$ and size m_1 and m_2, respectively. We will also simplify the notation $u \in V(G)$ to $u \in G$.

Lemma 3.1 $|E(G_1 \boxdot G_2)| = (n_1 - 1)(n_2 - 1)n_1 n_2/2 - (n_2 - 2)n_2 m_1 - (n_1 - 2)n_1 m_2 + 2m_1 m_2.$

Proof. It is well-known (and easy to see) that $|E(G_1 \square G_2)| = n_1 m_2 + n_2 m_1$ (cf. [24, Exercise 4.2]). The number of the non-Cartesian edges of $G_1 \boxdot G_2$ is $2(\binom{n_1}{2} - m_1)(\binom{n_2}{2} - m_2) = (n_1 - 1)(n_2 - 1)n_1 n_2/2 - (n_2 - 1)n_2 m_1 - (n_1 - 1)n_1 m_2 + 2m_1 m_2$. The total number is then $[(n_1 - 1)(n_2 - 1)n_1 n_2/2 - (n_2 - 1)n_2 m_1 - (n_1 - 1)n_1 m_2 + 2m_1 m_2] + [n_1 m_2 + n_2 m_1] = (n_1 - 1)(n_2 - 1)n_1 n_2 - 2(n_2 - 2)n_2 m_1 - 2(n_1 - 2)n_1 m_2 + 4m_1 m_2.$ $\square$

In the proof of Theorem 3.3 we will make use of the following lower bound on the number of edges in $G_1 \boxdot G_2$ in terms of the orders of the factors.

Corollary 3.2 $|E(G_1 \boxdot G_2)| \geq \min\{n_1 - 1, n_2 - 1\}n_1 n_2/2.$

Proof. By Lemma 3.1, there are $m = (n_1 - 1)(n_2 - 1)n_1 n_2/2 - (n_2 - 2)n_2 m_1 - (n_1 - 2)n_1 m_2 + 2m_1 m_2$ edges in $G_1 \square G_2$. Note that $m = (n_1 - 1)(n_2 - 1)n_1 n_2/2 + 2(m_1 - (n_1 - 2)n_1/2)(m_2 - (n_2 - 2)n_2/2) - (n_1 - 2)n_1(n_2 - 2)n_2/2 = (n_1 + n_2 - 3)n_1 n_2/2 + 2(m_1 - (n_1 - 2)n_1/2)(m_2 - (n_2 - 2)n_2/2)$. Let $q_1 = m_1 - (n_1 - 2)n_1/2$ and $q_2 = m_2 - (n_2 - 2)n_2/2$, so $m = (n_1 + n_2 - 3)n_1 n_2/2 + 2q_1 q_2$. Let us obtain upper and lower bounds for q_1 and q_2. Note that $0 \leq m_1 \leq (n_1 - 1)n_1/2$ and $0 \leq m_2 \leq (n_2 - 1)n_2/2$, thus, $-(n_1 - 2)n_1/2 \leq q_1 = m_1 - (n_1 - 2)n_1/2 \leq (n_1 - 1)n_1/2 - (n_1 - 2)n_1/2) = n_1/2$ and $-(n_2 - 2)n_2/2 \leq q_2 = m_2 - (n_2 - 2)n_2/4 \leq (n_2 - 1)n_2/2 - (n_2 - 2)n_2/2) = n_2/2$. Having these lower and upper bounds for q_1 and q_2, we can obtain a lower bound for $m = (n_1 + n_2 - 3)n_1 n_2/2 + 2q_1 q_2$. Since we assumed $n_1 \geq 2$, the lower bound $-(n_1 - 2)n_1/2$ for q_1 is non-positive and the upper bound $n_1/2$ is positive; the same about the lower bound $-(n_2 - 2)n_2/2$ and the upper bound $n_2/2$ for q_2. Therefore, $q_1 q_2 \geq \min\{-((n_1 - 2)n_1/2) \cdot (n_2/2), \ -((n_2 - 2)n_2/2) \cdot (n_1/2)\} = -(\max\{n_1, n_2\} - 2)n_1 n_2/4$. If $n_1 \geq n_2$, then $m = (n_1 + n_2 - 3)n_1 n_2/2 + 2q_1 q_2 \geq (n_1 + n_2 - 3)n_1 n_2/2 - 2(n_1 - 2)n_1 n_2/4 = (n_2 - 1)n_1 n_2/2$. If $n_1 \leq n_2$, then $m = (n_1 + n_2 - 3)n_1 n_2/2 + 2q_1 q_2 \geq (n_1 + n_2 - 3)n_1 n_2/2 - 2(n_2 - 2)n_1 n_2/4 = (n_1 - 1)n_1 n_2/2$. The statement of the corollary follows immediately. $\square$

Theorem 3.3 *Let* $\Delta = \Delta(G_1 \odot G_2)$, $\Delta_1 = \Delta(G_1)$, *and* $\Delta_2 = \Delta(G_2)$. *Then*

$$\lambda(G_1 \odot G_2) \leq \max_{u \in G_1, v \in G_2} \{\deg(u,v)(\Delta+1) - \deg(u)\deg(v) -$$
$$\min\{n_1 - \deg(u) - 2, n_2 - \deg(v) - 2\}(n_1 - \deg(u) - 1)(n_2 - \deg(v) - 1)\}.$$

Proof. Let $x = (u,v)$ be a vertex of $G_1 \odot G_2$. By the definition of the dot-Cartesian product, $\deg_{G_1 \odot G_2}(x) = \deg_{G_1}(u) + \deg_{G_2}(v) + (n_1 - \deg_{G_1}(u) - 1)(n_2 - \deg_{G_2}(v) - 1)$. To simplify the notations, let $d = \deg_{G_1 \odot G_2}(x)$, $d_1 = \deg_{G_1}(u)$ and $d_2 = \deg_{G_2}(v)$. Hence $d = d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)$ and $\Delta = \Delta(G_1 \odot G_2) = \max_{u \in G_1, v \in G_2}\{d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\}$.

A neighbor (u', v') of (u, v) is called a G_1-*neighbor* if $v = v'$ in G_2 and u is adjacent to u' in G_1. It is called an G_2-*neighbor* if $u = u'$ in G_1 and v is adjacent to v' in G_2. For each G_1-neighbor of x, if $d_2 > 0$, there is an G_2-neighbor of x such that they have a common neighbor other than x in $G_1 \odot G_2$. By the definition of $G_1 \odot G_2$, we have $d_1 d_2$ such "common neighbors".

The number of vertices at distance 1 from x is d. The number of vertices at distance 2 from x is clearly not greater than $d(\Delta - 1)$; let it be $d(\Delta - 1) - r$ for some $r \geq 0$. If two neighbors of x have one common neighbor other than x then this will contribute 1 to r. Hence the number of vertices that are at distance 2 from x cannot be greater than $d(\Delta - 1) - d_1 d_2$.

Now, we will use arguments based on the definition of dot-Cartesian product to further reduce this upper bound. Let ε denote the number of edges of the subgraph F induced by the neighbors of x. The present upper bound on the number of vertices at distance two from x is $d(\Delta - 1) - d_1 d_2$; for each edge in F, this upper bound is decreased by 2. Hence, the number of vertices at distance 2 from x is not greater than $d(\Delta - 1) - d_1 d_2 - 2\varepsilon$. Now we need a good lower bound for ε.

Consider the subgraph Q of F induced by vertices (u^t, v^t), where u^t is not adjacent to u in G_1 and v^t is not adjacent to v in G_2. If $n_1 - d_1 - 1 > 0$ and $n_2 - d_2 - 1 > 0$, then Q is a dot-Cartesian product of two subgraphs of G_1 and G_2, respectively, one with $n_1 - d_1 - 1$ vertices and the other with $n_2 - d_2 - 1$ vertices. By Corollary 3.2, there are at least $\min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)/2$ edges in Q. If $n_1 - d_1 - 1 = 0$ or $n_2 - d_2 - 1 = 0$, then Q does not exist and thus has 0 edges. Combining the above two subcases, we have that there are at least $\min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)/2$ edges in Q. (Note that $n_1 - d_1 - 1 \geq 0$, $n_2 - d_2 - 1 \geq 0$, $n_1 - d_1 - 2$ can be negative only if $n_1 - d_1 - 1 = 0$, and $n_2 - d_2 - 2$ can be negative only if $n_2 - d_2 - 1 = 0$, so $\min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)/2 \geq 0$.) Hence, $\varepsilon \geq \min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)/2$, and the number of vertices at distance 2 from x is not greater than $d(\Delta - 1) - d_1 d_2 - \min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)$.

The number of vertices at distance 1 from x is d. By Algorithm LABEL and by the above,

$$
\begin{aligned}
\lambda(G_1 \square G_2) \quad &= \quad k = |I_2| + |I_3| \leq |I_2| + |I_1| \\
&\leq \quad d + d\Delta - d_1 d_2 - \min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1) \\
&\leq \quad \max_{u \in G_1, v \in G_2} \{d(\Delta + 1) - d_1 d_2 - \\
&\qquad\qquad \min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)\}.
\end{aligned}
$$

and we are done. $\qquad\square$

Corollary 3.4 *Let G_1 and G_2 be graphs without isolated vertices on $n_1 \geq 5$ and $n_2 \geq 5$ vertices, respectively. Let $\Delta = \Delta(G_1 \square G_2)$. If $\Delta(G_1) \leq n_1 - 4$ and $\Delta(G_2) \leq n_2 - 4$, then $\lambda(G_1 \square G_2) \leq \Delta^2$.*

Proof. Since $d = d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)$, $\Delta = \max_{u \in G_1, v \in G_2}\{d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\}$, by Theorem 3.3, we have $\lambda(G_1 \square G_2) \leq \max_{u \in G_1, v \in G_2}\{(d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1))(\max_{u \in G_1, v \in G_2}\{d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\} + 1) - d_1 d_2 - \min\{n_1 - d_1 - 2, n_2 - d_2 - 2\}(n_1 - d_1 - 1)(n_2 - d_2 - 1)\}$.

Define the function $f(s,t) = (s + t + (n_1 - s - 1)(n_2 - t - 1))(\max_{u \in G_1, v \in G_2}\{d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\} + 1) - st - \min\{n_1 - s - 2, n_2 - t - 2\}(n_1 - s - 1)(n_2 - t - 1)$. We consider $f(s,t)$ as defined on the set of pairs (s,t) such that $s = d_1$, $t = d_2$ for some $u \in G_1$ and $v \in G_2$. Note that $(s,t) \in [0, \Delta_{G_1}] \times [0, \Delta_{G_2}]$, and s and t are integer. Suppose that $f(s,t)$ achieves its maximum at some point (p_1, p_2). Setting $X = \Delta^2 - f(p_1, p_2)$ we have:

$$
\begin{aligned}
X \quad &= \quad \Delta^2 - ((p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1))(\Delta + 1) - p_1 p_2 - \\
&\qquad \min\{n_1 - p_1 - 2, n_2 - p_2 - 2\}(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= \quad \Delta^2 - ((p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1))\Delta + (p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) - \\
&\qquad p_1 p_2 - \min\{n_1 - p_1 - 2, n_2 - p_2 - 2\}(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= \quad (\Delta^2 - (p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1))\Delta) - (p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) + \\
&\qquad p_1 p_2 + \min\{n_1 - p_1 - 2, n_2 - p_2 - 2\}(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&\geq \quad (\Delta^2 - \Delta\Delta) - (p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) + p_1 p_2 + \\
&\qquad \min\{n_1 - p_1 - 2, n_2 - p_2 - 2\}(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= \quad -(p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) + p_1 p_2 + \\
&\qquad \min\{n_1 - p_1 - 2, n_2 - p_2 - 2\}(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= \quad p_1 p_2 - p_1 - p_2 + (\min\{n_1 - p_1 - 2, n_2 - p_2 - 2\} - 1)(n_1 - p_1 - 1)(n_2 - p_2 - 1).
\end{aligned}
$$

To show that $\lambda(G_1 \square G_2)$ is bounded by Δ^2, it is sufficient to show that $p_1 p_2 - p_1 - p_2 + (\min\{n_1 - p_1 - 2, n_2 - p_2 - 2\} - 1)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq 0$, since then $\Delta^2 - f(p_1, p_2) \geq 0$ and thus $\lambda(G_1 \square G_2) \leq f(p_1, p_2) \leq \Delta^2$.

Since $\Delta(G_1) \leq n_1 - 4$ and $\Delta(G_2) \leq n_2 - 4$, we infer that $(n_1 - p_1 - 1) \geq (n_1 - \Delta(G_1) - 1) \geq 3$ and $(n_2 - p_2 - 1) \geq (n_2 - \Delta(G_2) - 1) \geq 3$. Since in addition G_1 and G_2 are without isolated vertices, $p_1 \geq 1$ and $p_2 \geq 1$. Then $p_1 p_2 - p_1 - p_2 + (\min\{n_1 - p_1 - 2, n_2 - p_2 - 2\} - 1)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq p_1 p_2 - p_1 - p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1) > 0$. It follows that $\lambda(G_1 \square G_2) \leq \Delta^2$ if $\Delta(G_1) \leq n_1 - 4$ and $\Delta(G_2) \leq n_2 - 4$ and we are done. $\qquad\square$

We conclude this section by noting that it is possible to prove additional cases for which the Δ^2-bound is fulfilled. Here we show two such cases.

Suppose that in the proof of Corollary 3.4 the extreme vertex (u, v) is such that $p_1 = 0$ and $p_2 = 0$, in which case G_1 and G_2 both have isolated vertices. Then $p_1 p_2 - p_1 - p_2 + (\min\{n_1 - p_1 - 2, n_2 - p_2 - 2\} - 1)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq (n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq 9$ and hence the Δ^2-bound holds also in this case.

For another case suppose that $p_1 = 0$ and $p_2 \geq 1$. Then $p_1 p_2 - p_1 - p_2 + (\min\{n_1 - p_1 - 2, n_2 - p_2 - 2\} - 1)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq p_1 p_2 - p_1 - p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1) = (n_1 - 1)(n_2 - p_2 - 1) - p_2 \geq 0$ if $(n_1 - 1)(n_2 - p_2 - 1) \geq p_2$. That is, in this case the bound holds as soon as $(n_1 - 1)(n_2 - p_2 - 1) \geq p_2$.

However, we were not able to cover all the cases and thus these minor exceptions are left as open problems.

4 $L(2,1)$-labelings of dot-lexicographic products

Throughout this section let again G_1 and G_2 be graphs of order $n_1 \geq 2$ and $n_2 \geq 2$ and size m_1 and m_2, respectively. To reduce the number of minor special cases that have to be considered, we also assume that G_1 and G_2 do not have isolated vertices.

Lemma 4.1 $|E(G_1 \odot G_2)| = (n_1 - 1)(n_2 - 1)n_1 n_2/2 + n_2 m_1 - (n_1 - 2)n_1 m_2 + 2m_1 m_2$.

Proof. Recall that $|E(G[H])| = n_1 m_2 + n_2^2 m_1$, cf. [24, Exercise 4.2]. Hence, similarly as in the proof of Lemma 3.1, $|E(G_1 \odot G_2)| = [(n_1 - 1)(n_2 - 1)n_1 n_2/2 - (n_2 - 1)n_2 m_1 - (n_1 - 1)n_1 m_2 + 2m_1 m_2] + [n_1 m_2 + n_2^2 m_1] = (n_1 - 1)(n_2 - 1)n_1 n_2/2 + n_2 m_1 - (n_1 - 2)n_1 m_2 + 2m_1 m_2$. $\qquad\square$

For the proof of the main result of this section, Theorem 4.3, we need the following lower bound.

Corollary 4.2 $|E(G_1 \odot G_2)| \geq (n_2 - 1)n_1 n_2/2$.

Proof. By Lemma 4.1, there are $m = (n_1-1)(n_2-1)n_1n_2/2+n_2m_1-(n_1-2)n_1m_2+2m_1m_2$ edges in $G_1 \odot G_2$. This can be written as $m = ((n_1-1)n_2-1)n_1n_2/2+2(m_1-(n_1-2)n_1/2)(m_2+n_2/2)$. Let $q_1 = m_1 - (n_1-2)n_1/2$ and $q_2 = m_2+n_2/2$, then $m = ((n_1-1)n_2-1)n_1n_2/2+2q_1q_2$. Let us obtain upper and lower bounds for q_1 and q_2. Note that $0 \leq m_1 \leq (n_1-1)n_1/2$ and $0 \leq m_2 \leq (n_2-1)n_2/2$, thus, $-(n_1-2)n_1/2 \leq q_1 = m_1 - (n_1-2)n_1/2 \leq (n_1-1)n_1/2 - (n_1-2)n_1/2 = n_1/2$ and $n_2/2 \leq q_2 = m_2+n_2/2 \leq (n_2-1)n_2/2+n_2/2 = n_2^2/2$. Since we assumed $n_1 \geq 2$ and $n_2 \geq 2$, the lower bound $-(n_1-2)n_1/2$ for q_1 is non-positive and the bounds $n_1/2$, $n_2/2$, $\nu_2^2/2$ are positive; therefore, $q_1q_2 \geq (-(n_1-2)n_1/2)(n_2^2/2)$ and $m \geq ((n_1-1)n_2-1)n_1n_2/2 - (n_1-2)n_1n_2^2/2 = ((n_1-1)n_2-1-(n_1-2)n_2)n_1n_2/2 = (n_2-1)n_1n_2/2$. $\qquad\square$

Theorem 4.3 *Let* $\Delta = \Delta(G_1 \odot G_2)$, $\Delta_1 = \Delta(G_1)$, *and* $\Delta_2 = \Delta(G_2)$. *Then*

$$
\begin{aligned}
\lambda(G_1 \odot G_2) \quad \leq \quad & \max_{u \in G_1, v \in G_2} \{\deg(u,v)(\Delta+1) - \deg(v)(\Delta_2-1)n_2\deg(u) - \\
& \deg(u)(\Delta_1-1)(n_2-1) - 2\deg(u)\Delta_2 - 2n_2\deg(u)\deg(v) - \\
& (n_2 - \deg(v) - 2)(n_1 - \deg(u) - 1)(n_2 - \deg(v) - 1)\}.
\end{aligned}
$$

Proof. Let $x = (u,v)$ be a vertex of $G_1 \odot G_2$. By the definition of the dot-lexicographic product, $\deg_{G_1 \odot G_2}(x) = n_2\deg_{G_1}(u) + \deg_{G_2}(v) + (n_1 - \deg_{G_1}(u) - 1)(n_2 - \deg_{G_2}(v) - 1)$. To simplify the notations, let $d = \deg_{G_1 \odot G_2}(x)$, $d_1 = \deg_{G_1}(u)$ and $d_2 = \deg_{G_2}(v)$. Hence $d = n_2d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)$ and $\Delta = \Delta(G_1 \odot G_2) = \max_{u \in G_1, v \in G_2}\{n_2d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\}$. The number of vertices at distance 2 from (u,v) in $G_1 \odot G_2$ is not greater than $d(\Delta-1)$. Now we will strengthen this straightforward bound by analyzing the structure of $G_1 \odot G_2$.

For this paragraph, see Fig. 1, where d_u denotes the degree of u in G_1. For any vertex v' in G_2 at distance 2 from v, there must be a path $v'v''v$ of length two between v' and v in G_2; since the degree of u in G_1 is d_1, i.e., u has d_1 adjacent vertices in G_1, by the definition of a dot-lexicographic product $G_1 \odot G_2$, for each vertex v_k, $1 \leq k \leq n_2$, of G_2, there must be d_1 internally-disjoint paths $(u,v')-(u_i,v_k)-(u,v)$ of length two between (u,v') and (u,v), and these do not include the path of length two between (u,v') and (u,v) via (u,v''). Hence for any vertex v' in G_2 with distance 2 from v, there must be at least $n_2d_1 + 1$ internally-disjoint paths of length 2 from $x = (u,v)$ to (u,v') in $G_1 \odot G_2$. Speaking informally, at least $n_2d_1 + 1$ potential vertices at distance 2 from (u,v) coincide in $G_1 \odot G_2$, for any vertex v' in G_2 at distance 2 from v. On the contrary, whenever such a vertex in G_2 with distance 2 from v in G_2 is missing, there will not exist the corresponding $n_2d_1 + 1$ potential vertices with distance 2 from $x = (u,v)$ in $G_1 \odot G_2$. In the former case, since such $n_2d_1 + 1$ vertices with distance 2 from $x = (u,v)$ coincide in $G_1 \odot G_2$ and hence can only be counted once, we have

to deduct $n_2 d_1 + 1 - 1$ from the upper bound $d(\Delta - 1)$ on the number of distance 2 vertices from x in $G_1 \odot G_2$; in the latter case, since such $n_2 d_1 + 1$ potential vertices with distance 2 from $x = (u, v)$ in $G_1 \odot G_2$ do not exist at all, we have to deduct $n_2 d_1 + 1$ from the upper bound $d(\Delta - 1)$. Let the number of vertices in G_2 with distance 2 from v be t, then $t \in [0, d_2(\Delta_2 - 1)]$. The minimum number we have to deduct from the upper bound $d(\Delta - 1)$ occurs when $t = d_2(\Delta_2 - 1)$; then, the upper bound is reduced by $d_2(\Delta_2 - 1)(n_2 d_1 + 1 - 1) = d_2(\Delta_2 - 1)n_2 d_1$ from the value $d(\Delta - 1)$, and $d(\Delta - 1) - d_2(\Delta_2 - 1)n_2 d_1$ is now the improved upper bound for the number of vertices with distance 2 from $x = (u, v)$ in $G_1 \odot G_2$.
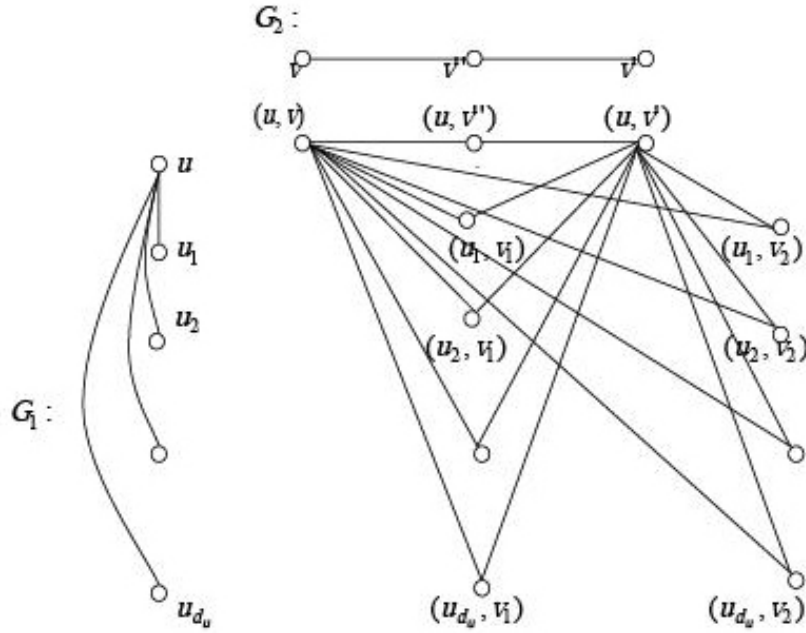

Figure 1: Local structure of $G_1 \odot G_2$

For this paragraph see Fig. 2. For any vertex u' in G_1 with distance 2 from u, there must be a path $u'u''u$ of length two between u' and u in G_1. Since the number of vertices of G_2 is n_2, by the definition of a dot-lexicographic product $G_1 \odot G_2$, there must exist n_2 internally-disjoint paths of length two between (u', v) and (u, v) in $G_1 \odot G_2$. Hence for any vertex in G_1 with distance 2 from u, there must be the corresponding n_2 potential vertices with distance 2 from $x = (u, v)$ which coincide in $G_1 \odot G_2$. On the contrary, whenever such a vertex in G_1 with distance 2 from u is missing, there will not exist the corresponding n_2 potential vertices with distance 2 from $x = (u, v)$.

In the former case, since such n_2 vertices with distance 2 from $x = (u, v)$ coincide in $G_1 \odot G_2$ and hence can only be counted once, we have to deduct $n_2 - 1$ from the upper bound on the number of vertices at distance 2 from x in $G_1 \odot G_2$; in the latter case, since such n_2 potential vertices with distance 2 from $x = (u, v)$ do not exist at all, we have to deduct n_2 from the upper bound. Let the number of vertices in G_1 with distance 2 from u be t, then $t \in [0, d_1(\Delta_1 - 1)]$. The minimum number we have to deduct from the upper bound occurs when $t = d_1(\Delta_1 - 1)$, so the upper bound on the number of vertices with distance 2 from $x = (u, v)$ in $G_1 \odot G_2$ will decrease at least by $d_1(\Delta_1 - 1)(n_2 - 1)$. Therefore, the number $d(\Delta - 1) - d_2(\Delta_2 - 1)n_2d_1 - d_1(\Delta_1 - 1)(n_2 - 1)$ is now the improved upper bound on the number of vertices with distance 2 from $x = (u, v)$ in $G_1 \odot G_2$.
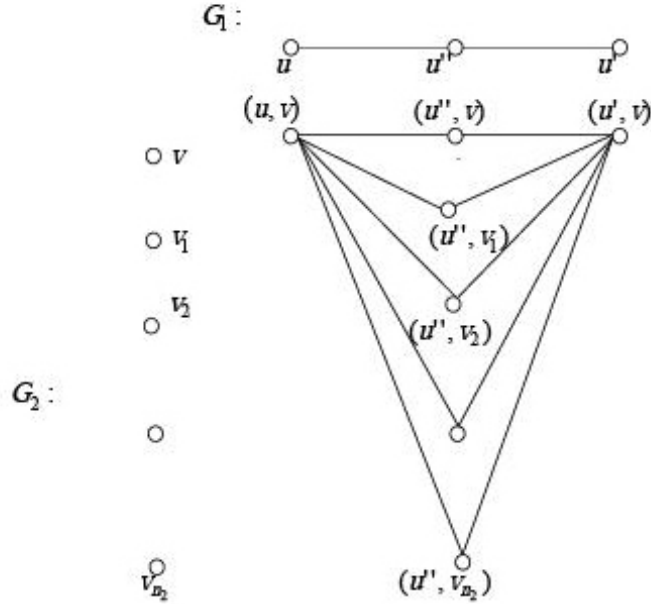

Figure 2: Local structure of $G_1 \odot G_2$

Moreover, we can further analyze as follows. Let ε denote the number of edges of the subgraph F induced by the neighbors of x. For the edges of the subgraph F, we consider the following cases.

Case 1. (See Fig. 3 for this paragraph.) If u' is adjacent to u in G_1, then (u, v) must be adjacent to (u', v_1) and (u', v_2) for any two vertices v_1 and v_2 of G_2, hence all (u', v) where $v \in V(G_2)$ are neighbors of x and vertices of F. Because $\Delta(G_2) = \Delta_2$ and there are totally d_1 neighbors u' of u, there should be at least $d_1\Delta_2$ edges in F of the type $((u', v_1), (u', v_2))$.

Case 2. (See Fig 4 for this paragraph.) For each neighbor (u, v') of $x = (u, v)$ where v' is adjacent to v in G_2, and any vertex (u', v_t) where u' is adjacent to u in G_1 and v_t is any vertex of G_2, there
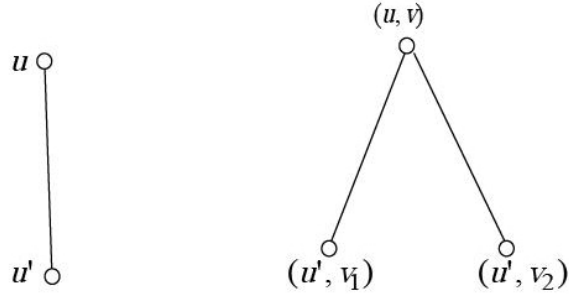
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Figure 3: Situation from Case 1

must be an edge between (u', v_t) and (u, v'). But there are in total $n_2 d_1$ neighbors (u', v_t) (where u' is adjacent to u in G_1) of $x = (u, v)$ and d_2 neighbors (u, v') (where v' is adjacent to v in G_2) of $x = (u, v)$, hence the number of edges of the type $((u', v_t), (u, v'))$ of the subgraph F should be at least $n_2 d_1 d_2$.
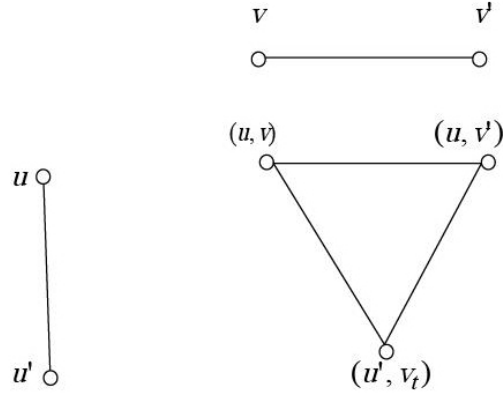

Figure 4: Situation from Case 2

Case 3. Let us estimate the number of edges in the subgraph of F induced by vertices (u^t, v^t), where u^t is not adjacent to u in G_1 and v^t is not adjacent to v in G_2. If $n_1 - d_1 - 1 > 0$ and $n_2 - d_2 - 1 > 0$, then this subgraph is a dot-lexicographic product of two subgraphs of G_1 and G_2, one with vertex number $n_1 - d_1 - 1$ and the other with vertex number $n_2 - d_2 - 1$, respectively. By Corollary 4.2, there are at least $(n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1)/2$ edges in this subgraph. If $n_1 - d_1 - 1 = 0$ or $n_2 - d_2 - 1 = 0$, then this subgraph does not exist and has 0 edges. Thus, there are at least $(n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1)/2$ edges in this subgraph. Note that $n_1 - d_1 - 1 \geq 0$, $n_2 - d_2 - 1 \geq 0$, and $n_2 - d_2 - 2$ can be negative only if $n_2 - d_2 - 1 = 0$, thus $(n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1) \geq 0$.

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The current upper bound on the number of vertices at distance two from x is $d(\Delta - 1) - d_2(\Delta_2 - 1)n_2d_1 - d_1(\Delta_1 - 1)(n_2 - 1)$. For each edge in F, this upper bound is decreased by 2. Hence, by the analysis of the above cases, the upper bound on the number of vertices with distance 2 from $x = (u, v)$ in $G_1 \odot G_2$ can be decreased by at least $2d_1\Delta_2 + 2n_2d_1d_2 + (n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1)$ and becomes $d(\Delta - 1) - d_2(\Delta_2 - 1)n_2d_1 - d_1(\Delta_1 - 1)(n_2 - 1) - 2d_1\Delta_2 - 2n_2d_1d_2 - (n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1)$.

The number of vertices with distance 1 from x is not greater than d. Then by Algorithm LABEL,

$$
\begin{aligned}
\lambda(G_1 \odot G_2) \;\leq\; & |I_2| + |I_3| \leq |I_2| + |I_1| \\
\leq\; & d + d\Delta - d_2(\Delta_2 - 1)n_2d_1 - d_1(\Delta_1 - 1)(n_2 - 1) - 2d_1\Delta_2 - 2n_2d_1d_2 - \\
& (n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1) \\
\leq\; & \max_{u \in G_1, v \in G_2} \{ d(\Delta + 1) - d_2(\Delta_2 - 1)n_2d_1 - d_1(\Delta_1 - 1)(n_2 - 1) - 2d_1\Delta_2 - \\
& 2n_2d_1d_2 - (n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1) \},
\end{aligned}
$$

where $d = \deg_{G_1 \odot G_2}(u, v)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 4.4 *Let G_1 and G_2 be graphs with $n_1 \geq 4$ and $n_2 \geq 4$ vertices, respectively. If $\Delta(G_1) \leq n_1 - 3$ and $\Delta(G_2) \leq n_2 - 3$, then $\lambda(G_1 \odot G_2) \leq \Delta^2$.*

Proof. By Theorem 4.3 and using the expressions for d and Δ from the beginning of the proof of Theorem 4.3, $\lambda(G_1 \odot G_2) \leq \max_{u \in G_1, v \in G_2} \{(n_2d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1))(\max_{u \in G_1, v \in G_2}\{n_2d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\} + 1) - d_2(\Delta_2 - 1)n_2d_1 - d_1(\Delta_1 - 1)(n_2 - 1) - 2d_1\Delta_2 - 2n_2d_1d_2 - (n_2 - d_2 - 2)(n_1 - d_1 - 1)(n_2 - d_2 - 1)\}$.

Define the function $f(s, t) = (n_2s + t + (n_1 - s - 1)(n_2 - t - 1))(\max_{u \in G_1, v \in G_2}\{n_2d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\} + 1) - t(\Delta_2 - 1)n_2s - s(\Delta_1 - 1)(n_2 - 1) - 2s\Delta_2 - 2n_2st - (n_2 - t - 2)(n_1 - s - 1)(n_2 - t - 1)$. We consider $f(s, t)$ as defined on the set of pairs (s, t) such that $s = d_1$, $t = d_2$ for some $u \in G_1$ and $v \in G_2$. Note that $(s, t) \in [1, \Delta_{G_1}] \times [1, \Delta_{G_2}]$, and s and t are integer. Suppose that $f(s, t)$ achieves its maximum at some point (p_1, p_2). Then, $\lambda(G_1 \odot G_2) \leq f(p_1, p_2)$. Note that $p_1 \geq 1$,

$p_2 \geq 1$. Setting $X = \Delta^2 - f(p_1, p_2)$, we now have

$$
\begin{aligned}
X &= \Delta^2 - ((n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1))(\Delta + 1) - p_2(\Delta_2 - 1)n_2 p_1 - \\
&\quad p_1(\Delta_1 - 1)(n_2 - 1) - 2p_1 \Delta_2 - 2n_2 p_1 p_2 - (n_2 - p_2 - 2)(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= \Delta^2 - ((n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1))\Delta + \\
&\quad (n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) - p_2(\Delta_2 - 1)n_2 p_1 - p_1(\Delta_1 - 1)(n_2 - 1) - \\
&\quad 2p_1 \Delta_2 - 2n_2 p_1 p_2 - (n_2 - p_2 - 2)(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= (\Delta^2 - (n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1))\Delta) - \\
&\quad (n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) + p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + \\
&\quad 2p_1 \Delta_2 + 2n_2 p_1 p_2 + (n_2 - p_2 - 2)(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&\geq (\Delta^2 - \Delta\Delta) - (n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) + p_2(\Delta_2 - 1)n_2 p_1 + \\
&\quad p_1(\Delta_1 - 1)(n_2 - 1) + 2p_1 \Delta_2 + 2n_2 p_1 p_2 + (n_2 - p_2 - 2)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \\
&= -(n_2 p_1 + p_2 + (n_1 - p_1 - 1)(n_2 - p_2 - 1)) + p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + \\
&\quad 2p_1 \Delta_2 + 2n_2 p_1 p_2 + (n_2 - p_2 - 2)(n_1 - p_1 - 1)(n_2 - p_2 - 1)) \\
&= p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) - n_2 p_1 - p_2 + 2p_1 \Delta_2 + 2n_2 p_1 p_2 + \\
&\quad (n_2 - p_2 - 3)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \\
&= p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - p_2) + n_2 p_1(2p_2 - 1) + \\
&\quad (n_2 - p_2 - 3)(n_1 - p_1 - 1)(n_2 - p_2 - 1).
\end{aligned}
$$

To show that $\lambda(G_1 \odot G_2)$ is bounded by Δ^2, we only need to show that $p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - p_2) + n_2 p_1(2p_2 - 1) + (n_2 - p_2 - 3)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq 0$ since the left side of this inequality is $\Delta^2 - f(p_1, p_2)$ and $\lambda(G_1 \odot G_2) \leq f(p_1, p_2)$. We have supposed that $\Delta(G_1) \leq n_1 - 3$ and $\Delta(G_2) \leq n_2 - 3$, then $(n_1 - p_1 - 1) \geq (n_1 - \Delta(G_1) - 1) \geq 2$ and $(n_2 - p_2 - 1) \geq (n_2 - \Delta(G_2) - 1) \geq 2$. We consider the following three cases:

Case 1. $(n_1 - p_1 - 1) = 2$ and $(n_2 - p_2 - 1) \geq 2$.

Then $p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - p_2) + n_2 p_1(2p_2 - 1) + (n_2 - p_2 - 3)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - p_2) + n_2 p_1(2p_2 - 1)$.

Subcase 1. $p_1 = 1$ and $p_2 \geq 1$. Then $p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - p_2) + n_2 p_1(2p_2 - 1) = (\Delta_2 - 1)n_2 p_2 + (\Delta_1 - 1)(n_2 - 1) + (2\Delta_2 - p_2) + n_2(2p_2 - 1) > 0$.

Subcase 2. $p_2 = 1$ and $p_1 \geq 1$. Then $p_2(\Delta_2 - 1)n_2 p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - p_2) + n_2 p_1(2p_2 - 1) = p_1(\Delta_2 - 1)n_2 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1 \Delta_2 - 1) + n_2 p_1 > 0$.

Subcase 3. $p_1 \geq 2$ and $p_2 \geq 2$. Hence $\Delta_1 \geq 2$ and $\Delta_2 \geq 2$. Then $p_2(\Delta_2 - 1)n_2p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1\Delta_2 - p_2) + n_2p_1(2p_2 - 1) = p_2(\Delta_2 - 1)n_2p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + ((2p_1 - 1)\Delta_2 + \Delta_2 - p_2) + n_2p_1(2p_2 - 1) \geq p_1n_1p_2 + p_1(n_2 - 1) + (2p_1 - 1)\Delta_2 + n_2p_1(2p_2 - 1) > 0$.

Case 2. $(n_2 - p_2 - 1) = 2$ and $(n_1 - p_1 - 1) \geq 2$.

Now $p_2(\Delta_2 - 1)n_2p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1\Delta_2 - p_2) + n_2p_1(2p_2 - 1) + (n_2 - p_2 - 3)(n_1 - p_1 - 1)(n_2 - p_2 - 1) = p_2(\Delta_2 - 1)n_2p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1\Delta_2 - p_2) + n_2p_1(2p_2 - 1)$. We can consider the same three subcases as in Case 1, with the same analysis and conclusions.

Case 3. $(n_1 - p_1 - 1) \geq 3$ and $(n_2 - p_2 - 1) \geq 3$.

Then $p_2(\Delta_2 - 1)n_2p_1 + p_1(\Delta_1 - 1)(n_2 - 1) + (2p_1\Delta_2 - p_2) + n_2p_1(2p_2 - 1) + (n_2 - p_2 - 3)(n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq 2\Delta_2 - p_2 + n_2p_1 + (n_1 - p_1 - 1)(n_2 - p_2 - 1) \geq n_2p_1 + 9$.

By above three cases, we have proved that $\lambda(G_1 \odot G_2) \leq \Delta^2$ if $\Delta(G_1) \leq n_1 - 3$ and $\Delta(G_2) \leq n_2 - 3$. □

Note that Case 3 is the most general case and the bound from the proof of Corollary 4.4 is much better than the Δ^2-bound.

We conclude the paper by proving that the Δ^2-bound also holds for dot-lexicographic products in which G_1 has large maximum degree. More precisely, the following result holds.

Proposition 4.5 *Let G_1 and G_2 be graphs with $n_1 \geq 3$ and $n_2 \geq 2$ vertices, respectively. If $\Delta(G_1) > n_1 - 3$, then $\lambda(G_1 \odot G_2) \leq \Delta^2$.*

Proof. Recall from [7] that if G is an arbitrary graph on n vertices with chromatic number χ, then $\lambda(G) \leq n + \chi - 2$ holds. If $\Delta = 2$, then G is a path or a cycle and we have $\lambda(G) \leq \Delta^2$. Thus, we only need to consider graphs with $\Delta \geq 3$. Suppose that $\Delta \geq (n-1)/2$, then $\lambda(G) \leq n + \chi - 2 \leq 2\Delta + 1 + \Delta - 2 = 3\Delta - 1 < \Delta^2$ (note that $\Delta \geq 3$). We will use this result to prove the proposition. Since $\Delta(G_1) > n_1 - 3$, we need to consider the following two cases. (Recall that $|V(G_1 \odot G_2)| = n_1n_2$).

Case 1. $\Delta(G_1) = n_1 - 1$. Then

$\Delta - (n_1n_2 - 1)/2 = \Delta(G_1 \odot G_2) - (n_1n_2 - 1)/2 = \max_{u \in G_1, v \in G_2}\{n_2d_1 + d_2 + (n_1 - d_1 - 1)(n_2 - d_2 - 1)\} - (n_1n_2 - 1)/2 \geq n_2\Delta(G_1) + \Delta(G_2) + (n_1 - \Delta(G_1) - 1)(n_2 - \Delta(G_2) - 1) - (n_1n_2 - 1)/2 = n_2(n_1 - 1) + \Delta(G_2) - (n_1n_2 - 1)/2 \geq n_2(n_1 - 1) - (n_1n_2 - 1)/2 = ((n_1 - 2)n_2 + 1)/2 > 0$.

Case 2. $\Delta(G_1) = n_1 - 2$. Then

$\Delta - (n_1n_2 - 1)/2 \geq n_2\Delta(G_1) + \Delta(G_2) + (n_1 - \Delta(G_1) - 1)(n_2 - \Delta(G_2) - 1) - (n_1n_2 - 1)/2 = n_2(n_1 - 2) +$

$$\Delta(G_2) + n_2 - \Delta(G_2) - 1 - (n_1 n_2 - 1)/2 = n_2(n_1 - 2) + n_2 - 1 - (n_1 n_2 - 1)/2 = ((n_1 - 2)n_2 - 1)/2 > 0.$$
$\square$

Acknowledgments

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