Average distance, surface area, and other structural properties of exchanged hypercubes

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Abstract

Exchanged hypercubes [Loh et al., IEEE Transactions on Parallel and Distributed Systems 16 (2005) 866–874] are spanning subgraphs of hypercubes with about one half of their edges but still with many desirable properties of hypercubes. In this paper it is shown that distance properties of exchanged hypercubes are also comparable to the corresponding properties of hypercubes. The average distance and the surface area of exchanged hypercubes are computed and it is shown that exchanged hypercubes have asymptotically the same average distance as hypercubes. Several additional metric and other properties are also deduced and it is proved that exchanged hypercubes are prime with respect to the Cartesian product of graphs.

Key words: interconnection network; exchanged hypercube; Wiener index; average distance; surface area; Cartesian product of graphs

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1 Introduction

Hypercubes form one of the fundamental models for interconnection networks. They are universal in the sense that the binary strings are naturally encoded into their topology. Consequently, they possess numerous fine network properties such as small diameter, small density, and high connectivity. At the same time, straightforward local routing is possible. Nevertheless, due to several reasons different variations of hypercubes were proposed. We mention here only some of these such as Möbious cubes [6], folded hyper-star networks [20], Fibonacci cubes [21], folded cubes [24], k-ary n-cubes [36], and twisted-cubes [37]. An additional variation of hypercubes—exchanged hypercubes—is of our prime interest here.

The exchanged hypercubes $EH(s,t)$, proposed by Loh et al. [28], are graphs obtained by systematically removing edges from hypercubes. The number of vertices in $EH(s,t)$ is equal to that of the $(s+t+1)$-dimensional hypercube, but the ratio of the number of edges in $EH(s,t)$ to that of the $(s+t+1)$-dimensional hypercube is $1/2+1/(2(s+t+1))$ [5]. Different properties of exchanged hypercubes were investigated by now. The paths and cycles embedding was investigated in [29, 35]. To measure the fault-tolerance of them, the connectivity and the super connectivity were determined in [25, 30, 31]. Upper and lower bounds for the domination number of them were given in [22]. These results, in addition to those obtained in the seminal paper [28], indicate that the exchanged hypercubes keep numerous desirable properties of the hypercubes. In this paper we take a closer look to metric properties of exchanged hypercubes and some other related properties.

Among the metric properties we study the surface area, the average distance, the (average) eccentricity and some related invariants. All these invariants are of lasting interest when it comes to networks and wider, let us list here just a selection of related, recent papers. Very recently the concept of the surface area for an interconnection network was generalized to the one of the edge-centered surface area [4]. The relation of the average distance with the independence number and spanning trees was investigated in [32]. For a couple of recent studies of average eccentricity see [12, 17].

Graphs considered here are simple, finite, and connected. The *distance* $d_G(u,v)$ between vertices u and v of a (connected) graph G is the usual shortest-path distance. If G will be clear from the context, we will simply write $d(u,v)$. The *Wiener index*

$W(G)$ of a graph G is the sum of the distances between all unordered pairs of vertices of G. For instance, $W(K_n) = \binom{n}{2}$, that is, the number of its edges. This graph invariant is one of the fundamental properties of (interconnection) networks, cf. [2, 5, 7, 23, 33]. On the other hand, the Wiener index of a graph is a central and one of the most studied invariants in mathematical chemistry, see for instance surveys [10, 11].

A concept closely related to the Wiener index is the surface area. If v is a vertex of a graph G and r a positive integer, then the *r-surface area* $B_{G,v}(r)$ of G centered at v is the number of vertices at distance exactly r from the fixed base vertex v. That is, the surface area is the size of the r-sphere around v. If G is vertex-transitive, then the surface area is independent of a selected vertex. The surface area of a network is interesting because many network properties and algorithms are directly related to it. Consequently, it has been studied for a variety of networks, cf. [7, 8, 18].

The paper is organized as follows. In the next section we introduce the exchanged hypercubes in two equivalent ways and recall or deduce some of their basic properties. Then, in Section 3, we first determine the Wiener index of exchanged hypercubes. As a consequence it is demonstrated that asymptotically, exchanged hypercubes and hypercubes have the same average distance. Then we determine the surface area and indicate that this result yields an alternative way to determine the Wiener index. In Section 4 we first consider several other distance-based invariants and then prove that the exchanged hypercubes are prime with respect to the Cartesian product operation. In Section 5 we apply our previous result to obtain some results on a related class of interconnection networks, the dual-cubes. We also compare the max degree, the diameter, the eccentricity, the average distance, and the surface area of the hypercube, the exchanged hypercube and the dual-cube.

2 Exchanged hypercubes: two definitions and some properties

In this section we first introduce exchanged hypercubes and list some of their properties. Then we equivalently describe these cubes as the graphs obtained by adding a perfect matching between two collections of hypercubes and show how this point of view can be used to infer additional properties.

Exchanged hypercubes are spanning subgraphs of hypercubes. Recall that if d is a positive integer, then the d-dimensional *hypercube* (or *d-cube*, for short) Q_d is the graph with vertex set $\{0,1\}^d$, two vertices (strings) being adjacent if they differ in exactly one coordinate. The *Hamming distance* $H(b,c)$ between binary strings b and c of the same length is the number of positions in which b and c differ. It is well known that $d_{Q_d}(u,v) = H(u,v)$ holds for any two vertices (alias strings) of Q_d.

Let $u = u_{d-1}\dots u_0 \in \{0,1\}^d$ be a binary string, $d \geq 1$. If $j \geq i$, then we will use the notation $u_{j:i}$ for the substring of u between u_j and u_i, that is, $u_{j:i} = u_j \dots u_i$. For any integers $s \geq 1$ and $t \geq 1$, the *exchanged hypercube* $EH(s,t)$ is the graph with the vertex set $\{0,1\}^{s+t+1}$. Hence, if $u \in V(EH(s,t))$, then its coordinates are $u_{s+t}\dots u_{t+1}u_t\dots u_1u_0$. Vertices u and v are adjacent if one of the following conditions is satisfied:

(i) $u_{s+t:1} = v_{s+t:1}, u_0 \neq v_0$,

(ii) $u_0 = v_0 = 1, H(u_{t:1}, v_{t:1}) = 1$, and $u_{s+t:t+1} = v_{s+t:t+1}$,

(iii) $u_0 = v_0 = 0, H(u_{s+t:t+1}, v_{s+t:t+1}) = 1$, and $u_{t:1} = v_{t:1}$.

For instance, $EH(1,2)$ and $EH(2,2)$ are shown in Fig. 1.

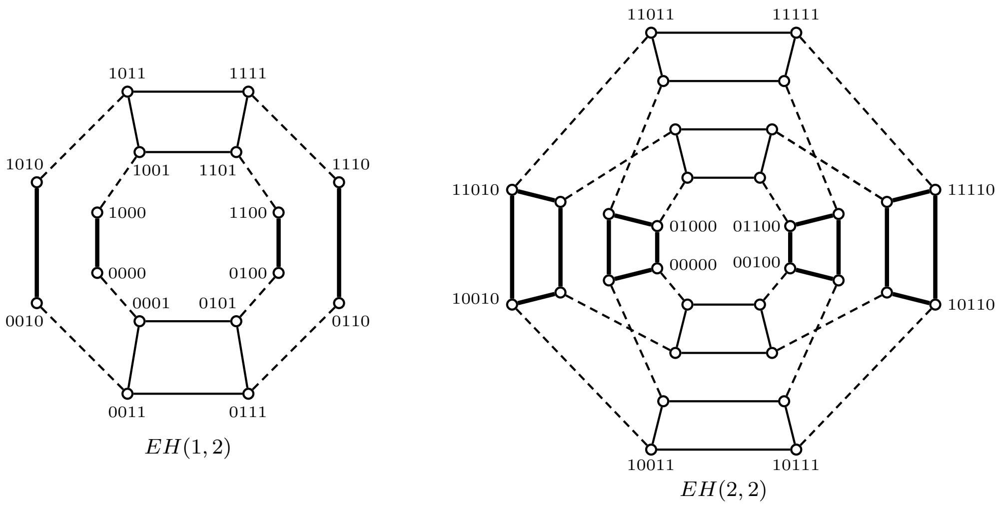

Figure 1: Exchanged hypercubes $EH(1,2)$ and $EH(2,2)$

Clearly, $EH(s,t)$ has 2^{s+t+1} vertices. Note also that if $u \in V(EH(s,t))$ and $u_0 = 0$, then the degree of u is $s+1$, otherwise the degree of u is $t+1$. It is also straightforward that for any s and t, the exchanged hypercube $EH(s,t)$ is isomorphic to $EH(t,s)$.

For practical purposes, another description of the exchanged hypercubes is useful. Note that the edge set of $EH(s,t)$ is the disjoint union of sets E_1, E_2, E_3, where

$$E_1 = \{uv : \ u_{s+t:1} = v_{s+t:1}, u_0 \neq v_0\},$$

$E_2 = \{uv:\ u_{s+t:t+1} = v_{s+t:t+1}, H(u_{t:1}, v_{t:1}) = 1, u_0 = v_0 = 1\}$,
$E_3 = \{uv:\ u_{t:1} = v_{t:1}, H(u_{s+t:t+1}, v_{s+t:t+1}) = 1, u_0 = v_0 = 0\}$.

In Fig. 1 the edges from E_1, E_2 and E_3 are represented by dashed lines, thin lines, and thick lines, respectively.

Let $EH_1(s,t)$ be the subgraph of $EH(s,t)$ induced by the edges E_2. Then $EH_1(s,t)$ is the disjoint union of 2^s copies of Q_t. Indeed, fixing the leftmost s bits and fixing the rightmost bit to 1, the induced subgraph is isomorphic to Q_t. Moreover, there are no edges between two such induced subgraphs isomorphic to Q_t. Similarly, the subgraph $EH_0(s,t)$ of $EH(s,t)$ induced by the edges E_3 consists of 2^t subgraphs isomorphic to Q_s. Finally, the edges from E_1 form a perfect matching of $EH(s,t)$, it is a matching between $EH_0(s,t)$ and $EH_1(s,t)$. More precisely, let Q be an arbitrary copy of Q_t from $EH_1(s,t)$. Then each vertex of Q has exactly one neighbor in $EH_0(s,t)$, each of these neighbors belongs to a different copy of Q_s from $EH_0(s,t)$. For instance, in the graph $EH(2,2)$ in Fig. 1, the subgraph $EH_1(2,2)$ consists of 4 copies of Q_2 drawn with thin lines, the subgraph $EH_0(2,2)$ consists of 4 copies of Q_2 drawn with thick lines, while the matching edges (that is, the edges from E_1) are drawn with dashed lines.

Using the above representation of the exchanged hypercubes it is immediate that $|E(EH(s,t))| = 2^t(s2^{s-1}) + 2^s(t2^{t-1}) + 2^t2^s = 2^{s+t-1}(s+t+2)$, cf. [5]. Since Q_{s+t+1} has $(s+t+1)2^{s+t}$ edges, it thus follows that $EH(s,t)$ has about one half of the edges of the hypercube of the same dimension.

To emphasize that the newly proposed representation of exchanged hypercubes can be efficiently used to obtain their structural properties, we present the following result and its short proof. Exchanged hypercubes are bipartite (as subgraphs of hypercubes) and hence in particular triangle-free. Therefore, the next proposition gives the number of shortest cycles of exchanged hypercubes, another important property of interconnection networks.

Proposition 2.1 *If $s,t \geq 1$, then the number of 4-cycles of $EH(s,t)$ is*

$$2^{s+t-2}\left(\binom{s}{2} + \binom{t}{2}\right).$$

Proof. From the above representation we infer that a 4-cycle of $EH(s,t)$ contains no edge between $EH_0(s,t)$ and $EH_1(s,t)$. It follows that any 4-cycle lies either in $EH_0(s,t)$ or in $EH_1(s,t)$. As these two subgraphs are disjoint unions of hypercubes Q_s

and Q_t, respectively, and since Q_n contains exactly $2^{n-2}\binom{n}{2}$ 4-cycles, cf. [19, Exercise 2.4], the assertion follows immediately. □

We have noted that the edges from E_1 form a perfect matching of $EH(s,t)$. But exchanged hypercubes contain many additional perfect matchings. Indeed, hypercubes contain a huge number of perfect matchings, cf. [14], hence any combination of separate perfect matchings in copies of Q_s and Q_t gives a perfect matching in $EH(s,t)$. Moreover, Fink [13] proved that every perfect matching of Q_d, $d \geq 2$, is contained in a Hamiltonian cycle. This result together with the second description of the exchanged hypercubes can be used to construct many Hamiltonian cycles in exchanged hypercubes $EH(s,s)$, see [3].

3 Average distance and surface area

The Wiener index of a graph is the most famous and one of the most studied distance-based graph invariants with many applications, in particular in network theory and in mathematical chemistry. Equivalent to the study of the Wiener index of a graphs is the study of the average distance of a graph. The investigation of the Wiener index is now going on for several decades and is still a hot topic. The reader may first look to the recent paper [15] and references therein. A closely related invariant, the surface area, is an important tool for network performance evaluation. Consequently it has been earlier studied for numerous graphs and networks as for instance the star graph, the k-ary n-cube, the (n,k)-star graph, and the arrangement graph, see [4, 18, 34]. In this section we determine both, the Wiener index and the surface area, for the exchanged hypercubes.

Using the second description of exchanged hypercubes from the previous section we prove the following key lemma. It is implicitly given in [28, Table 2], but since it is the key lemma, we give a formal proof of it.

Lemma 3.1 *If $s,t \geq 1$ and $u,v \in V(EH(s,t))$, then*

$$d(u,v) = \begin{cases} H(u,v)+2, & u_0 = v_0 = 0, u_{t:1} \neq v_{t:1}, \text{ or} \\ & u_0 = v_0 = 1, u_{s+t:t+1} \neq v_{s+t:t+1}; \\ H(u,v), & \text{otherwise.} \end{cases}$$

Proof. Note first that $d(u,v) \geq H(u,v)$ because $EH(s,t)$ is a spanning subgraph of Q_{s+t+1} and $d_{Q_{s+t+1}}(u,v) = H(u,v)$.

Suppose first that $u_0 = v_0 = 0$ and $u_{t:1} = v_{t:1}$. Then u and v belong to the same subgraph Q_s of $EH_0(s,t)$ and hence $d(u,v) \leq H(u,v)$. Thus $d(u,v) = H(u,v)$. Analogously we get the same conclusion when $u_0 = v_0 = 1$ and $u_{s+t:t+1} = v_{s+t:t+1}$. Let next $u_0 \neq v_0$ and assume without loss of generality that $u_0 = 0$ and $v_0 = 1$. Then a u,v-path of length $H(u,v)$ can be constructed as follows. First change one by one the bits of u between u_{s+t} and u_{t+1} in which u differs from v. Then change the rightmost bit to 1, and finally change the remaining bits in which u differs from v. Hence in all these cases we have $d(u,v) = H(u,v)$.

Assume now that $u_0 = v_0 = 0$ and $u_{t:1} \neq v_{t:1}$. Then u and v belong to different copies of Q_s from $EH_0(s,t)$. Hence any u,v-path necessarily contains at least two matching edges between $EH_0(s,t)$ and $EH_1(s,t)$. Then it follows that $d(u,v) \geq H(u,v)+2$. We can find a u,v-path of length $H(u,v)+2$ as follows: change one by one the bits of u between u_{s+t} and u_{t+1} in which u differs from v, then change the rightmost bit to 1, change the remaining bits in which u differs from v, and finally change the rightmost bit to 0. We conclude that $d(u,v) = H(u,v) + 2$. The case $u_0 = v_0 = 1$ and $u_{s+t:t+1} \neq v_{s+t:t+1}$ is treated analogously. □

Theorem 3.2 *If $s,t \geq 1$, then*

$$W(EH(s,t)) = (s+t+3)2^{2(s+t)} - 2^{2s+t} - 2^{s+2t}\,.$$

Proof. By Lemma 3.1, the contribution of a pair $\{u,v\} \in \binom{V(EH(s,t))}{2}$ to $W(EH(s,t))$ is either $H(u,v)$ or $H(u,v)+2$. Thus $W(EH(s,t))$ is the sum of $W(Q_{s+t+1})$ and twice the number of pairs of vertices $\{u,v\}$ with $d(u,v) = H(u,v) + 2$.

Consider first pairs with $u_0 = v_0 = 0$ and $u_{t:1} \neq v_{t:1}$. In the subcase when $u_{s+t:t+1} = v_{s+t:t+1}$, there are 2^s possible substrings for the first s coordinates and hence the number of (unordered) pairs of vertices with $u_0 = v_0 = 0$, $u_{t:1} \neq v_{t:1}$, and $u_{s+t:t+1} = v_{s+t:t+1}$, is

$$\binom{2^t}{2} 2^s\,.$$

If on the other hand $u_{s+t:t+1} \neq v_{s+t:t+1}$, we have $\binom{2^s}{2}$ pairs with respect to the first s coordinates and $\binom{2^t}{2}$ for the consecutive t coordinates. As such pairs can be combined in two ways, the number of such (unordered) pairs of vertices is

$$2\binom{2^s}{2}\binom{2^t}{2}.$$

Similarly, the number of pairs $\{u, v\}$ with $u_0 = v_0 = 1$ and $u_{s+t:t+1} \neq v_{s+t:t+1}$ is equal to

$$\binom{2^s}{2}2^t + 2\binom{2^s}{2}\binom{2^t}{2}.$$

It is well-known (cf. [16, Exercise 19.3]) that $W(Q_{s+t+1}) = (s+t+1)2^{2(s+t)}$. Putting all this together we obtain that

$$W(EH(s,t)) = (s+t+1)2^{2(s+t)} + 2\left(\binom{2^t}{2}2^s + \binom{2^s}{2}2^t + 4\binom{2^s}{2}\binom{2^t}{2}\right),$$

which, after a routine computation, reduces to the claimed expression. □

The *average distance* $\mu(G)$ of a graph G is defined with

$$\mu(G) = \frac{1}{\binom{|V(G)|}{2}}W(G).$$

We note that some authors prefer to define the average distance as $2W(G)/|V(G)|^2$. The definitions are almost equivalent and are in fact equivalent in the asymptotic sense, which we consider next.

Corollary 3.3

$$\lim_{s,t\to\infty} \frac{\mu(EH(s,t))}{s+t+1} = \lim_{d\to\infty} \frac{\mu(Q_d)}{d} = \frac{1}{2}.$$

Proof. Using Theorem 3.2 we find that

$$\frac{W(EH(s,t))}{W(Q_{s+t+1})} = \frac{1}{s+t+1}\left(s+t+3-2^{-s}-2^{-t}\right),$$

hence $\lim_{s,t\to\infty} \frac{\mu(EH(s,t))}{s+t+1} = \lim_{s,t\to\infty} \frac{\mu(Q_{s+t+1})}{s+t+1}$. For the latter limit we h ave

$$\lim_{d\to\infty} \frac{\mu(Q_d)}{d} = \lim_{d\to\infty} \frac{d\,2^{2d-2}}{\binom{2^d}{2}d} = \frac{1}{2},$$

and we have the desired result. □

We next determine the surface area of exchanged hypercubes:

Theorem 3.4 *If $s, t \geq 1$, then*

$$B_{EH(s,t),v}(r) = \begin{cases} \binom{s+t+1}{r-1} + \binom{s}{r} - \binom{s}{r-2}, & v_0 = 0; \\ \binom{s+t+1}{r-1} + \binom{t}{r} - \binom{t}{r-2}, & v_0 = 1. \end{cases}$$

Proof. Assume $v_0 = 0$ and $d(u, v) = r$ $(0 \leq r \leq s + t + 2)$. We count the number of u according to the three cases.

Case 1: $u_0 = 0$, $u_{t:1} = v_{t:1}$.
By Lemma 3.1, there are r different bits between $u_{s+t:t+1}$ and $v_{s+t:t+1}$. The number of u is $\binom{s}{r}$. Note that $r \leq s$ in this case.

Case 2: $u_0 = 0$, $u_{t:1} \neq v_{t:1}$.
By Lemma 3.1, there are $r-2$ different bits between $u_{s+t:1}$ and $v_{s+t:1}$. Since $u_{t:1} \neq v_{t:1}$, the number of different bits between $u_{s+t:t+1}$ and $v_{s+t:t+1}$ is less than $r - 2$. That is, not all different bits lie in the leftmost s bits. Hence, the number of u is $\binom{s+t}{r-2} - \binom{s}{r-2}$.

Case 3: $u_0 = 1$.
By Lemma 3.1, there are $r - 1$ different bits between $u_{s+t:1}$ and $v_{s+t:1}$. The number of u is $\binom{s+t}{r-1}$.

Combining the above three cases we conclude that

$$B_{EH(s,t),v}(r) = \binom{s}{r} + \binom{s+t}{r-2} - \binom{s}{r-2} + \binom{s+t}{r-1} = \binom{s+t+1}{r-1} + \binom{s}{r} - \binom{s}{r-2}.$$

We proceed analogously when $v_0 = 1$. □

We conclude the section by noting that Theorem 3.4 yields another proof of Theorem 3.2. The computations are rather technical, so we state only the key steps. If $v_0 = 0$, the total distance of v is

$$\begin{aligned} \sum_{u \in V(EH(s,t))} d(u, v) &= \sum_{r=1}^{s+t+2} r \cdot B_{EH(s,t),v}(r) \\ &= \sum_{r=1}^{s+t+2} r \left[\binom{s+t+1}{r-1} + \binom{s}{r} - \binom{s}{r-2} \right] \\ &= (s + t + 3)2^{s+t} - 2^{s+1}. \end{aligned}$$

Analogously, if $v_0 = 1$, then the total distance of v is $(s+t+3)2^{s+t} - 2^{t+1}$. The number of vertices with $v_0 = 0$ (or $v_0 = 1$) is 2^{s+t}. Hence, the Wiener index of $EH(s, t)$ is $2^{s+t}[(s + t + 3)2^{s+t} - 2^{s+1} + (s + t + 3)2^{s+t} - 2^{t+1}]/2 = (s+t+3)2^{2(s+t)} - 2^{2s+t} - 2^{s+2t}$.

4 Additional properties

In this section we first consider the eccentricity of vertices of exchanged hypercubes. This is an essential invariant for networks due to the worst case behaviour of different algorithms, for instance in broadcasting procedures when a message from a given vertex must be distributed to all the other vertices of the network. The exchanged hypercubes are very stable in this respect since the eccentricity of every vertex is the same, cf. Proposition 4.1. This also yields the average eccentricity, which is in turn important for the average time complexity of algorithms.

As we have seen in Section 2, $EH(s,t)$ consists of the disjoint union of 2^s copies of Q_t, the disjoint union of 2^t subgraphs isomorphic to Q_s, and a matching between these two unions. One way of describing hypercubes is that they are Cartesian products of complete graphs on two vertices. Hence it is natural to ask whether exchanged hypercubes also posses a product structure. In Proposition 4.3 we answer this question in negative.

Recall that the *eccentricity* $\mathrm{ecc}(u)$ of a vertex $u \in V(G)$ is the maximum distance between u and the vertices of G. The *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ of G are the minimum and the maximum eccentricity in G, respectively.

From Lemma 3.1 we can also determine the eccentricity of vertices of $EH(s,t)$. Let u be an arbitrary vertex, then its binary complement $\overline{u}$ is the unique vertex with respect to u with the property $H(u,\overline{u}) = s+t+1$. By Lemma 3.1 we infer that $d(u,\overline{u}) = s+t+1$. On the other hand, the lemma implies that $d(u,\overline{u}') = s+t+2$, where $\overline{u}'$ is the vertex obtained from $\overline{u}$ by complementing $\overline{u}_0$. Hence, $\mathrm{ecc}(u) \geq s+t+2$. Moreover, using Lemma 3.1 again, we also observe that $d(u,v) < s+t+2$ for any vertex v other than $\overline{u}'$. Hence, $\mathrm{ecc}(u) = s+t+2$, and the vertex $\overline{u}'$ is the unique vertex to u such that $\mathrm{ecc}(u) = d(u,\overline{u}')$. We collect these facts into the following result, where the average eccentricity (cf. [17]) is defined in the natural way. (We note that the diameter of $EH(s,t)$ was earlier determined in [28, Theorem 6].)

Proposition 4.1 *If $u \in V(EH(s,t))$, then $\mathrm{ecc}(u) = s+t+2$ and hence the average eccentricity of $EH(s,t)$ is $s+t+2$. Moreover, every vertex has a unique antipodal vertex.*

Corollary 4.2 *If $s,t \geq 1$, then $\mathrm{diam}(EH(s,t)) = \mathrm{rad}(EH(s,t)) = s+t+2$.*

This should be compared with the fact that $\mathrm{diam}(Q_{s+t+1}) = \mathrm{rad}(Q_{s+t+1}) = s+t+1$. Hence again the exchanged hypercubes keep these fine properties of hypercubes.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g,h) and (g',h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. For more information on this graph operation see [16]. For our purpose it is essential to recall that Q_d can be represented as the d-fold Cartesian product of K_2. Hence hypercubes are in a way the simplest possible Cartesian product graphs. A graph that cannot be represented as the Cartesian product of graphs on at least two vertices, is called *prime (with respect to the Cartesian product).* We conclude this note with the following results that contrasts exchanged hypercubes from hypercubes.

Proposition 4.3 *If $s,t \geq 1$, then $EH(s,t)$ is a prime graph.*

Proof. Consider vertices $u = 00\ldots00$ and $v = 00\ldots01$. Clearly, $uv \in E(EH(s,t))$. The open neighborhood $N(u)$ of u is $\{v, u^{(t+1)}, \ldots, u^{(s+t)}\}$, where $u_i^{(i)} = 1$ for $t+1 \leq i \leq s+t$, and $u_j^{(i)} = 0$ for $t+1 \leq i \leq s+t$, $0 \leq j \leq s+t$, $j \neq i$. Similarly, $N(v) = \{u, v^{(1)}, \ldots, v^{(t)}\}$, where $v_i^{(i)} = v_0^{(i)} = 1$ for $1 \leq i \leq t$, and $v_j^{(i)} = 0$ for $1 \leq i \leq t$, $1 \leq j \leq s+t$, $j \neq i$. It follows that if $x \in N(u) - \{v\}$ and $y \in N(v) - \{u\}$, then $d(x,y) = 3$. Consequently, the edge uv is contained in no 4-cycle of $EH(s,t)$. Since in the Cartesian product of two nontrivial graphs every edge is contained in at least one 4-cycle, the result is proved. □

5 Applications to dual cubes

Another closely related class of cubes studied in the literature is formed by dual-cubes D_n, $n \geq 1$. These cubes were by now well studied, see for instance [1, 3, 26, 27]. Here we follow the notation used in [1]. As it turns out, D_n is isomorphic to $EH(n-1, n-1)$. Hence all our results can be directly applied to the dual-cubes. For instance:

Corollary 5.1 *If $n \geq 1$, then $W(D_{n+1}) = (2n+3)2^{4n} - 2^{3n+1}$.*

Dual-cubes are vertex-transitive graphs. For such graphs G the surface area is independent of a selected vertex u, hence the notation $B_{G,u}(r)$ can be simplified to $B_G(r)$. Then we have:

Corollary 5.2 *If* $n \geq 1$, *then* $B_{D_{n+1}}(r) = \binom{2n+1}{r-1} + \binom{n+1}{r} - \binom{n+1}{r-1}$.

Proof. Apply Theorem 3.4 together with the facts that D_{n+1} is isomorphic to $EH(n,n)$ and that $\binom{n}{r} - \binom{n}{r-2} = \binom{n+1}{r} - \binom{n+1}{r-1}$. □

In Table 1 we summarize the max degree, diameter, eccentricity, average distance, and surface area of the hypercube, the exchanged hypercube and the dual-cube, assuming that all networks have the same number of vertices 2^{s+t+1}, where $s \leq t$. Note that $s = t$ in the dual-cube. (For a general approach to the degree/diameter problem in networks see [9].)

Table 1 Topological Properties of Hypercube, Dual-cube and Exchanged hypercube

Networks	Max degree	Diameter	Eccentricity	Wiener index	Surface area
Hypercube	$s+t+1$	$s+t+1$	$s+t+1$	$(s+t+1)2^{2(s+t)}$	$\binom{s+t+1}{r}$
Dual-cube	$t+1$	$2t+2$	$2t+2$	$(2t+3)2^{4t} - 2^{3t+1}$	$\binom{2t+1}{r-1} + \binom{t+1}{r} - \binom{t+1}{r-1}$
Exchanged hypercube	$t+1$	$s+t+2$	$s+t+2$	$(s+t+3)2^{2(s+t)}$ $-2^{2s+t} - 2^{s+2t}$	$\binom{s+t+1}{r-1} + \binom{s}{r} - \binom{s}{r-2}, v_0 = 0$ $\binom{s+t+1}{r-1} + \binom{t}{r} - \binom{t}{r-2}, v_0 = 1$

6 Concluding remarks

In this paper we have obtained several properties of exchanged hypercubes with respect to the distance function. We determined Wiener index and surface area of exchanged hypercubes, and showed that the exchanged hypercubes and hypercubes have, asymptotically, the same average distance. The exchanged hypercube is a prime graph with respect to the Cartesian product. We also deduced some related conclusions about the dual-cubes since the dual-cube is a special kind of the exchanged hypercube.

Very recently, dual-cubes D_n were generalized to *dual-cube-like networks* DC_n in [1]. DC_n consists of 2^n disjoint copies of Q_{n-1}. In addition, it contains a perfect matching, where the endpoints belong to different copies of Q_{n-1}, and between any two copies of Q_{n-1} there is at most one edge of the perfect matching. This generalization of dual-cubes to dual-cube-like networks can be naturally extended to generalize extended hypercubes to extended hypercube-like networks. We think it would be worth studying the extended hypercube-like networks.

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