The domination number of exchanged hypercubes

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Abstract

Exchanged hypercubes [Loh et al., IEEE Transactions on Parallel and Distributed Systems 16 (2005) 866–874] are spanning subgraphs of hypercubes with about one half of their edges but still with many desirable properties of hypercubes. Lower and upper bounds on the domination number of exchanged hypercubes are proved which in particular imply that $\gamma(EH(2,t)) = 2^{t+1}$ holds for any $t \geq 2$. Using Hamming codes we also prove that $\gamma(EH(s,2^k-1)) \leq (2^s - 2^k)\gamma(Q_{2^k-1}) + 2^{2^k-1}(\gamma(Q_s^-) + 1)$ holds for $s \geq k \geq 3$.

Key words: interconnection network; hypercube; exchanged hypercube; domination number; Hamming code

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1 Introduction

Hypercubes form a fundamental model for parallel computers and interconnection networks, cf. [22, Chapter 7]. They have many fine properties that are essential for network efficiency, such as recursive decomposition, lots of symmetries, low regularity, and small diameter. Hypercubes also allow straightforward (local) routing and are Hamiltonian. For more information on their fault tolerance with respect to the hamiltonicity see [19, 20] and references therein. Having all this in mind it comes with no big surprise that machines based on hypercubes have actually been implemented, see [22, p. 115] for the list of implementations.

Interconnection networks often require a distribution of limited supply of resources and from this point of view various kinds of dominating sets serve as possible locations for placement of resources. For general aspects of the role of domination in complex networks see the book chapter [1]. Unfortunately, the exact domination number is known only for small dimensional hypercubes and two infinite families: $\gamma(Q_3) = 2$, $\gamma(Q_4) = 4$, $\gamma(Q_5) = 7$, $\gamma(Q_6) = 12$, and $\gamma(Q_n) = 2^{n-k}$ for $n = 2^k - 1$ or $n = 2^k$, see [8]. In general, $\gamma(Q_n) \leq 2^{n-3}$ for $n \geq 7$ [3]. For some variations of domination studied on hypercubes see [3, 7, 17], while for domination of closely related Fibonacci cubes see [4, 18]. Domination was also studied on other types of interconnection networks as for instance on toroidal meshes [21].

Since domination is very difficult on hypercubes, they are not very appropriate when dealing with domination-type problems. In this note we instead study the domination number of exchanged hypercubes $EH(s,t)$. This two-parametric family of graphs was proposed by Loh et al. [13] and constitute a variation of the hypercube networks with numerous appealing properties, see [15] for their bipancyclicity and [10, 14, 16] for their connectivity and super connectivity, important measures for the fault-tolerance of networks. In the special case when $s = t$, the exchanged hypercubes coincide with the so-called dual-cubes, a class of hypercube-like networks studied in [2, 5, 11, 12].

We proceed as follows. In the next section we introduce the exchanged hypercubes, recall some of their properties, and define other concepts used in this note. Then, in Section 3, our results are presented. We prove several bounds on the domination number of exchanged hypercubes and deduce from them that if $t \geq 2$, then $\gamma(EH(2,t)) = 2^{t+1}$. This exact result appears appealing because, as we have noted above, the domination number of the usual hypercubes is an intrinsically difficult problem. Using the fact that Q_{2^k-1} contains a perfect code (which is just a corresponding Hamming code) we also prove that $\gamma(EH(s,2^k-1)) \leq (2^s - 2^k)\gamma(Q_{2^k-1}) + 2^{2^k-1}(\gamma(Q_s^-) + 1)$ holds for $s \geq k \geq 3$.

2 Preliminaries

Graphs considered here are simple, finite, and connected.

If n is a positive integer, then the n-dimensional *hypercube* (or n-*cube*, for short) Q_n is the graph with vertex set $\{0,1\}^n$, two vertices (strings) being adjacent if they differ in exactly one coordinate. Hypercubes are vertex-transitive graphs, hence all vertex-deleted subgraphs $Q_n - v$, $v \in V(Q_n)$, are isomorphic, we denote it with Q_n^-. The distance between vertices $u, v \in V(Q_n)$ is equal to the *Hamming distance* between u and v, denoted $H(u,v)$, that is, the number of coordinates in which u and v differ.

Exchanged hypercubes are spanning subgraphs of hypercubes. Let $u = u_{d-1} \dots u_0 \in \{0,1\}^d$ be a binary string, $d \geq 1$. If $j \geq i$, then we will use the notation $u_{j:i}$ for the substring of u between u_j and u_i, that is, $u_{j:i} = u_j \dots u_i$. For any integers $s \geq 1$ and $t \geq 1$, the *exchanged hypercube* $EH(s,t)$ is the graph with the vertex set $\{0,1\}^{s+t+1}$. Hence, if $u \in V(EH(s,t))$, then its coordinates are $u_{s+t} \dots u_{t+1} u_t \dots u_1 u_0$. Vertices u and v are adjacent if one of the following conditions is satisfied:

(i) $u_{s+t:1} = v_{s+t:1}, u_0 \neq v_0$,

(ii) $u_0 = v_0 = 1, H(u_{t:1}, v_{t:1}) = 1$, and $u_{s+t:t+1} = v_{s+t:t+1}$,

(iii) $u_0 = v_0 = 0, H(u_{s+t:t+1}, v_{s+t:t+1}) = 1$, and $u_{t:1} = v_{t:1}$.

Clearly, $EH(s,t)$ has 2^{s+t+1} vertices. If $u \in V(EH(s,t))$ and $u_0 = 0$, then the degree of u is $s+1$, otherwise the degree of u is $t+1$. It is also straightforward that for any s and t, the exchanged hypercube $EH(s,t)$ is isomorphic to $EH(t,s)$. The ratio of the number of edges in $EH(s,t)$ to that of Q_{s+t+1} is $1/2 + 1/(2(s+t+1))$ [6].

If G is a graph, then $D \subseteq V(G)$ is a *dominating set* if every vertex of $V(G) - D$ is adjacent to some vertex of D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D of G is a *perfect code* if any two vertices from D are at distance at least 3. Hence the closed neighborhoods of the vertices from a perfect code D partition the vertex of G, cf. [9, Theorem 4.1].

A *matching* of a graph G is a set of independent edges and a *perfect matching* is a matching M such that each vertex is an endpoint of an edge from M. Finally, if $X \subseteq V(G)$, then the closed neighborhood $N[X]$ is $\bigcup_{u \in X} N[u]$, where $N[u]$ is the closed neighborhood of u.

3 Results

We begin with the following bounds:

Theorem 3.1 *If* $s, t \geq 1$ *and* $s \leq t$*, then*

$$\max\{2^t\gamma(Q_s), 2^s\gamma(Q_t)\} \leq \gamma(EH(s,t)) \leq (2^s - 1)\gamma(Q_t) + 2^t\gamma(Q_s)\,.$$

Proof. Consider the following edge-subsets of $EH(s,t)$:

$$\begin{aligned}
E_1 &= \{uv: \ u_{s+t:1} = v_{s+t:1}, u_0 \neq v_0\}, \\
E_2 &= \{uv: \ u_{s+t:t+1} = v_{s+t:t+1}, H(u_{t:1}, v_{t:1}) = 1, u_0 = v_0 = 1\}, \\
E_3 &= \{uv: \ u_{t:1} = v_{t:1}, H(u_{s+t:t+1}, v_{s+t:t+1}) = 1, u_0 = v_0 = 0\}.
\end{aligned}$$

Let $EH_1(s,t)$ be the subgraph of $EH(s,t)$ induced by the edges E_2. Then $EH_1(s,t)$ is the disjoint union of 2^s copies of Q_t, we denote these cubes with $Q_t^{(i)}$, $1 \leq i \leq 2^s$. Indeed, fixing the leftmost s bits and fixing the rightmost bit to 1, the induced subgraph is isomorphic to Q_t. Moreover, there are no edges between two such induced subgraphs isomorphic to Q_t. Similarly, the subgraph $EH_0(s,t)$ of $EH(s,t)$ induced by the edges E_3 consists of 2^t subgraphs isomorphic to Q_s denoted with $Q_s^{(j)}$, $1 \leq j \leq 2^t$. Finally, the edges from E_1 form a perfect matching of $EH(s,t)$, it is a matching between $EH_0(s,t)$ and $EH_1(s,t)$. More precisely, for any i, any vertex of $Q_t^{(i)}$ has exactly one neigbor in $EH_0(s,t)$, each of these neighbors belonging to different $Q_s^{(j)}$. See Fig. 1.

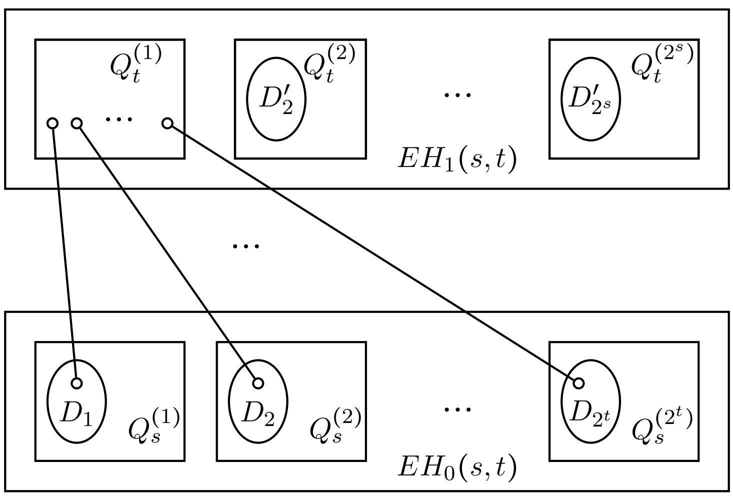

Figure 1: Subgraphs $EH_0(s,t)$ and $EH_1(s,t)$ of $EH(s,t)$

For the upper bound, consider the t-cube $Q_t^{(1)}$. Then each of $Q_s^{(i)}$, $1 \le i \le 2^t$, has a (unique) neighbor in $Q_t^{(1)}$. In each of the cubes $Q_s^{(i)}$ select a minimum dominating set D_i such that if $x \in N[V(Q_t^{(1)})] \bigcap Q_s^{(i)}$ then $x \in D_i$. (Such a dominating set exists since hypercubes are vertex-transitive graphs.) Then $Q_t^{(1)}$ is dominated by $\bigcup_{i=1}^{2^t} D_i$, see Fig. 1 again. For $2 \le i \le 2^s$ let D_i' be a minimum dominating set of $Q_t^{(i)}$. Then $D = \left(\bigcup_{i=1}^{2^t} D_i\right) \bigcup \left(\bigcup_{i=2}^{2^s} D_i'\right)$ is a dominating set of $EH(s,t)$. Clearly, $|D| = 2^t\gamma(Q_s) + (2^s - 1)\gamma(Q_t)$. The upper bound is proved.

Let D be a dominating set of $EH(s,t)$ and let $T_i = D \cap N[V(Q_t^{(i)})]$, $1 \le i \le 2^s$. Then $|T_i| \ge \gamma(Q_t)$, for otherwise $T_i \cap Q_t^{(i)}$ together with the neighbors of the vertices from $T_i - V(Q_t^{(i)})$ that lie in $Q_t^{(i)}$ would form a dominating set of order strictly smaller than $\gamma(Q_t)$. In addition, if $i \ne j$, then $T_i \cap T_j = \emptyset$ because a vertex from $T_i - Q_t^{(i)}$ has exactly one neighbor in $EH_1(s,t)$. It follows that

$$|D| \ge \sum_{i=1}^{2^s} |T_i| \ge 2^s\gamma(Q_t)\,.$$

Applying analogous arguments to $EH_0(s,t)$ we infer that $|D| \ge 2^t\gamma(Q_s)$. This proves the lower bound. □

For another upper bound the following lemma will be useful.

Lemma 3.2 *If $n \ge 3$, then $V(Q_n)$ can be partitioned into 4 (pairwise disjoint) dominating sets.*

Proof. For $n = 3$, the partition $\{\{000,111\},\{100,011\},\{010,101\},\{001,110\}\}$ does the job. We proceed by induction. Let $n \ge 3$ and let $V(Q_n) = \dot\bigcup_{i=1}^4 D_i$, where each D_i ($1 \le i \le 4$) is a dominating set of Q_n, and $\dot\bigcup$ denotes the disjoint union of sets. For $1 \le i \le 4$ set

$$D_i' = \{0u:\ u \in D_i\} \cup \{1u:\ u \in D_i\}\,.$$

Then it is straightforward to verify that each D_i' is a dominating set of Q_{n+1} and that $V(Q_{n+1}) = \dot\bigcup_{i=1}^4 D_i'$. □

Proposition 3.3 *If $2 \le s \le t$ and $t \ge 3$, then*

$$\gamma(EH(s,t)) \le (2^s - 4)\gamma(Q_t) + 2^t(\gamma(Q_s^-) + 1)\,.$$

Proof. Since $t \geq 3$, Lemma 3.2 guarantees the existence of a partition $V(Q_t) = \dot\bigcup_{i=1}^{4} D_i$, where each D_i is a dominating set of Q_t. Since $s \geq 2$, $EH_1(s,t)$ (defined in the proof of Theorem 3.1) contains the four t-cubes $Q_t^{(i)}$, $1 \leq i \leq 4$. For any i with $1 \leq i \leq 4$, let D_i' be the isomorphic copy of D_i in $Q_t^{(i)}$. For $5 \leq i \leq 2^s$, let D_i' be a minimum dominating set of $Q_t^{(i)}$, and for each $j = 1, \ldots, 2^t$, let D_j'' be a minimum dominating set of $\left(Q_s^{(j)}\right)^-$. Note that $|\bigcup_{i=1}^{4} D_i'| = 2^t$ and that each vertex from $\bigcup_{i=1}^{4} D_i'$ is adjacent to exactly one vertex in a private copy of $Q_s^{(j)}$ in $EH_0(s,t)$. It follows that $D = \left(\bigcup_{i=1}^{2^s} D_i'\right) \cup \left(\bigcup_{j=1}^{2^t} D_j''\right)$ is a domination set of $EH(s,t)$. Since

$$|D| = \left|\bigcup_{i=1}^{4} D_i'\right| + \left|\bigcup_{i=5}^{2^s} D_i'\right| + \left|\bigcup_{j=1}^{2^t} D_j''\right| = 2^t + (2^s - 4)\gamma(Q_t) + 2^t\gamma(Q_s^-)\,,$$

the result follows. $\square$

We are now ready for our key insight.

Theorem 3.4 *If $t \geq 2$, then $\gamma(EH(2,t)) = 2^{t+1}$.*

Proof. Let $t = 2$. Then $\gamma(EH(2,2)) \geq 8$ by Theorem 3.1. In Fig. 2 a dominating set of $EH(2,2)$ of size 8 is shown, hence $\gamma(EH(2,2)) = 8$.

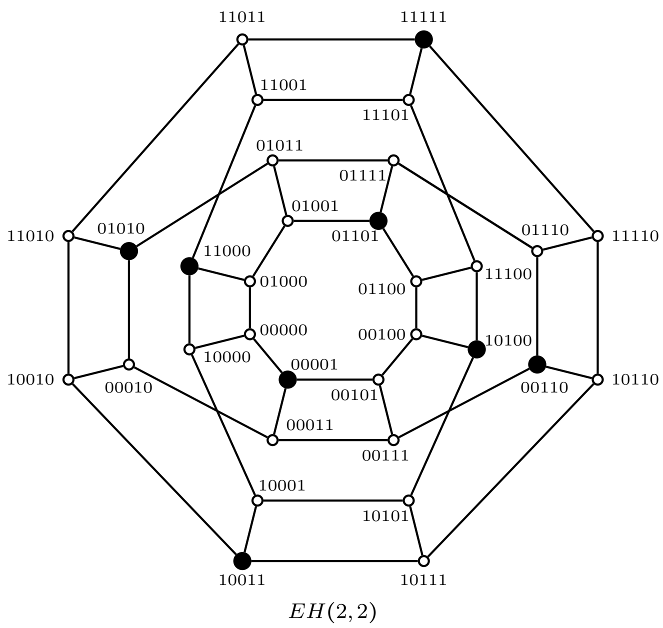

Figure 2: A minimum dominating set of $EH(2,2)$

Let $t \geq 3$. Then the lower bound $\gamma(EH(2,t)) \geq 2^{t+1}$ again follows from Theorem 3.1. On the other hand, $\gamma(EH(2,t)) \leq 2^{t+1}$ follows from Proposition 3.3 having in mind that $s = 2$ and $\gamma(Q_2^-) = 1$. □

In the proof of Proposition 3.3 we have partitioned the vertex set of Q_n into four dominating sets. If $V(Q_n)$ can be partitioned into more disjoint dominating sets, the upper bound can be improved. This is not possible for $n \leq 5$ as we can find out from the exact domination numbers of these cubes. On the other hand, using Hamming codes this can be done in the following special case.

Theorem 3.5 *If $s \geq k \geq 3$, then*

$$\gamma(EH(s, 2^k - 1) \leq (2^s - 2^k)\gamma(Q_{2^k-1}) + 2^{2^k-1}(\gamma(Q_s^-) + 1)\,.$$

Proof. Let $k \geq 3$. Let D_0 be an arbitrary perfect code of Q_{2^k-1}. It is well-known that such a code exists, see [9], in fact, it is just a Hamming code of block length $2^k - 1$. Let $e^{(i)}$, $1 \leq i \leq 2^k - 1$, denote the binary word of length $2^k - 1$ with 1 in the ith coordinate and with 0 in any other coordinate. For each $i = 1, \ldots, 2^k - 1$ we now define

$$D_i = \{u + e^{(i)} :\ u \in D_0\}\,.$$

We claim that $D_i \cap D_j = \emptyset$ for any $i \neq j$, $0 \leq i, j \leq 2^k - 1$. Note first that $D_0 \cap D_i = \emptyset$, because if $u \in D_i$, then there exists an $x \in D_0$ such that $u = x + e^{(i)}$ and hence $H(u, x) = 1$. Since for any other vertex y of D_0 we have $H(x, y) \geq 3$, we conclude that $u \neq y$. Let next $u \in D_i$ and $v \in D_j$, where $i, j \geq 1$ and $i \neq j$. Then $u = x + e^{(i)}$ and $v = y + e^{(j)}$ for some $x, y \in D_0$. Because $H(x, y) \geq 3$ it then follows that $u \neq v$.

The mapping $V(Q_n) \to V(Q_n)$ that changes a fixed coordinate in each of the vertices is an automorphism of Q_n. Since an automorphism maps dominations sets onto dominating set, we infer that D_i, $1 \leq i \leq 2^k - 1$, are dominating sets because D_0 is such. Hence, $V(Q_{2^k-1})$ is partitioned into 2^k dominating sets D_i, $0 \leq i \leq 2^k - 1$.

We now construct a dominating set of $EH(s, 2^k - 1)$ similarly as in the proof of Proposition 3.3. Since $s \geq k$, there exist 2^k cubes $Q_{2^k-1}^{(i)}$, $1 \leq i \leq 2^k$. For any $1 \leq i \leq 2^k - 1$, let D_i' be the isomorphic copy of D_i in $Q_{2^k-1}^{(i)}$, and let D_{2^k}' be the isomorphic copy of D_0 in $Q_{2^k-1}^{(2^k)}$. For $2^k + 1 \leq i \leq 2^s$, let D_i' be a minimum dominating set of $Q_t^{(i)}$, and for $1 \leq j \leq 2^{2^k-1}$, let D_j'' be a minimum dominating set of $\left(Q_s^{(j)}\right)^-$. Then $D =$

$\left(\bigcup_{i=1}^{2^s} D_i'\right) \cup \left(\bigcup_{j=1}^{2^{2^k-1}} D_j''\right)$ is a domination set of $EH(s, 2^k-1)$ of order $(2^s-2^k)\gamma(Q_{2^k-1})+ 2^{2^k-1}(\gamma(Q_s^-)+1)$. □

It is clear that if $\gamma(Q_s^-) < \gamma(Q_s)$, then the upper bounds of Theorem 3.5 and of Proposition 3.3 are better than the upper bound of Theorem 3.1. Unfortunately, it seems difficult to determine whether this indeed holds for some dimensions s. We can show however that $\gamma(Q_s^-) \le \gamma(Q_s)$ holds. To see it, let D be a dominating set of Q_s with $|D| = \gamma(Q_s)$, and let x be an arbitrary vertex of $V(Q_s) - D$. Then D is also a dominating set of $Q_s - x$ which is in turn isomorphic to Q_s^- (because Q_s is vertex-transitive). It follows that $\gamma(Q_s^-) \le \gamma(Q_s)$. An indication that $\gamma(Q_s^-) < \gamma(Q_s)$ might hold for some dimensions s is the fact that there exist vertex-transitive graphs whose vertex-deleted subgraphs have smaller domination number. For instance, $\gamma(C_{3k+1}) = k+1$, while for the vertex-deleted subgraph P_{3k} of C_{3k+1} we have $\gamma(P_{3k}) = k$.

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