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An Euler-type formula for median graphs

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Abstract

Let G be a median graph on n vertices and m edges and let k be the number of equivalence classes of the Djoković's relation Θ defined on the edge-set of G. Then $2n-m-k \leqslant 2$. Moreover, $2n-m-k=2$ if and only if G is cube-free.

A *median graph* is a connected graph such that, for every triple of vertices u, v, w, there is a unique vertex x lying on a geodesic (i.e. shortest path) between each pair of u, v, w. By now, the class of median graphs is well studied and a rich structure theory is available, see e.g. [5]. In this note, we present an Euler-type formula for median graphs, which involves the number of vertices n, the number of edges m, and the number of Θ-classes k (or, equivalenty, the number of cutsets in the cutset coloring, cf. [6,7]). The formula is an inequality, where equality is attained if and only if the median graph is cube-free.

For $u, v \in V(G)$ let $d_G(u, v)$ denote the length of a shortest path in G from u to v. A subgraph H of G is *convex*, if for any $u, v \in V(H)$, all shortest paths between u and v belong to H. Clearly, a convex subgraph is connected.

The Djoković's relation Θ introduced in [1] is defined on the edge-set of a graph in the following way. Two edges $e = xy$ and $f = uv$ of a graph G are in relation Θ if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u).$$

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Clearly, Θ is reflexive and symmetric. If G is bipartite, relation Θ can be rewritten as follows: $e = xy$ and $f = uv$ are in relation Θ if

$$d(x,u) = d(y,v) \quad \text{and} \quad d(x,v) = d(y,u).$$

Among bipartite graphs, Θ is transitive precisely for partial cubes (i.e. isometric subgraphs of hypercubes), as proved by Winkler in [9]. Since median graphs form a subclass of partial cubes, Θ is in particular transitive for median graphs. Thus Θ is a congruence on median graphs. For more information on Θ we refer to [2,3].

Let $G = (V,E)$ be a connected graph. For two subgraphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ of G, the *intersection* $G_1 \cap G_2$ is the subgraph of G with vertex-set $V_1 \cap V_2$ and edge-set $E_1 \cap E_2$, and the *union* $G_1 \cup G_2$ is the subgraph of G with vertex-set $V_1 \cup V_2$ and edge-set $E_1 \cup E_2$. A *convex cover* G_1, G_2 of G consists of two convex subgraphs G_1 and G_2 of G such that $G_0 = G_1 \cap G_2$ is non-empty and $G = G_1 \cup G_2$. Note that there are no edges between $G_1 - G_2$ and $G_2 - G_1$, and that G_0 is convex.

Let G' be a connected graph, and let G_1', G_2' be a convex cover of G' with $G_0' = G_1' \cap G_2'$. The *expansion* of G' with respect to G_1', G_2' is the graph G constructed as follows. Let G_i be an isomorphic copy of G_i', for $i = 1,2$, and, for any vertex u' in G_0', let u_i be the corresponding vertex in G_i, for $i = 1,2$. Then, G is obtained from the disjoint union $G_1 \cup G_2$, where u_1 and u_2 are joined by an edge for each u' in G_0'. For example, if $G' = G_1' = G_2' = Q_n$, then $G = Q_{n+1}$, and if G' is a tree and $G_1' = G'$ and $G_2' = \{u'\}$, then G is obtained from G' by adding a new vertex pending at u'. We call G_1, G_2 a *split* in G. Note that the above notion of expansion is called *convex Cartesian expansion* in the much more general setting of [8]. We will say that G is obtained from a graph H by an *expansion procedure* if we obtain G from H by a sequence of expansions.

An important tool in the study of median graphs is the following theorem from [6,7].

Theorem 1. *A graph is a median graph if and only if it can be obtained from the one vertex graph by an expansion procedure.*

Here we need two consequences of this theorem. For the first one see, for instance [3,6,7].

Corollary 2. *Let G be a median graph and let F be a Θ-equivalence class. Then F consists of the edges between the parts of a split in G.*

A graph is called *cube-free* if it does not contain the 3-cube as an induced subgraph. Suppose G is a median graph which is an expansion with respect to G_1', G_2' and assume that $G_0' = G_1' \cap G_2'$ contains a cycle. Let C_n, $n \geqslant 4$, be a shortest cycle in G_0'. Clearly, C_n is an even isometric cycle. Let u, v and w be three consecutive vertices of C and let x be the antipodal vertex of v on C. Then the median y of u, w and x together with u, v and w form a C_4 in G_0' and by Theorem 1 we find a 3-cube in G. In conclusion, we have the following:

Corollary 3. *A graph is a cube-free median graph if and only if it can be obtained from the one vertex graph by an expansion procedure, in which every expansion step is done with respect to a convex cover with a convex tree as intersection.*

We are now ready to state our principal result.

Theorem 4. *Let G be a median graph with n vertices, m edges and k equivalence classes of the relation Θ. Then*

$$2n - m - k \leqslant 2.$$

In addition,

$$2n - m - k = 2$$

if and only if G is cube-free.

Proof. We prove the inequality by induction on the number of vertices using Theorem 1. The inequality reduces to $2 \leqslant 2$ if $G = K_1$. So assume that G is the expansion of the median graph G' with respect to its convex subgraphs G_1', G_2' with $G_0' = G_1' \cap G_2'$. By induction we have $2n' - m' - k' \leqslant 2$ for G', where k', n', m' are the corresponding parameters of G'. Let t be the number of vertices in G_0', so that G_0', being connected, has at least $t-1$ edges. Then we have $n = n' + t$ and $m \geqslant m' + 2t - 1$. Moreover, by Corollary 2, we also have $k = k' + 1$. So

$$\begin{aligned} 2n - m - k &\leqslant 2(n' + t) - (m' + 2t - 1) - (k' + 1) \\ &= 2n' - m' - k' \\ &\leqslant 2. \end{aligned}$$

Clearly, we have equality $2n - m - k = 2$ if and only if, in all of the expansions to obtain G from K_1, the expansion was with respect to two isometric subgraphs having a tree as intersection. By Corollary 3, this is equivalent with G being a cube-free median graph. □

For the simplest example of how to use Theorem 4, consider the 6-cycle C_6. Then $n = m = 6$ and $k = 3$, hence $2n - m - k = 3$ and we can conclude that C_6 is not a median graph. For a less trivial example consider the graph G from Fig. 1. We have $n = 14$, $m = 21$ and $k = 3$. Thus $2n - m - k = 4$ and so also G is not a median graph.

Consider the Cartesian product of the claw-graph $K_{1,3}$ with the path on three vertices P_3 to see that cube-free median graphs are not planar in general. However, if the graph in question is a planar, cube-free median graph, then combining Euler's formula $n - m + f = 2$ with equality $2n - m - k = 2$ of Theorem 4 we immediately get the following corollary due to Janaqi [4]:

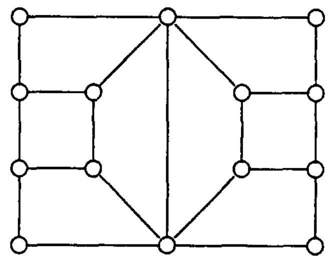

Fig. 1. A graph which is not median.

Corollary 5. *Let G be a planar, cube-free median graph with n vertices and k equivalence classes of the relation Θ. Then the number of faces in its planar embedding is equal to n-k.*

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