On the edge-connectivity of Cartesian product graphs*

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Abstract

A short proof for the closed expression of the edge-connectivity of Cartesian product graphs is given and the structure of minimum edge cuts is described. It is also proved that the connectivity and edge-connectivity of an arbitrary Cartesian power equals its minimum degree.

Key words: Cartesian product of graphs; edge-connectivity; minimum edge cuts;

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1 Introduction

The Cartesian product of graphs is one of the most fundamental graph operations and has been studied since its introduction in the 1950's. Many important classes of graphs such as hypercubes, Hamming graphs, and prisms, are Cartesian products. Even more important is the fact that Cartesian products serve as natural hosts for different embeddings in metric graph theory. A prominent example is the canonical metric representation of an arbitrary graph due to Graham and Winkler [4], see also [3, 5]: every graphs has a unique (irredundant) isometric

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embedding into a Cartesian product of graphs with a largest possible number of factors.

Recently, fundamental results on the connectivity κ and the edge-connectivity κ' of Cartesian product graphs were obtained. In this paper we round these investigations by simplifying some arguments and giving additional structure insights.

We note that these connectivity results present just a fraction of recent extensive developments on the Cartesian product. Let us just mention that Imrich and Peterin designed a linear algorithm for recognizing Cartesian products [7]; that the structure of the automorphism group of Cartesian powers was carefully studied to determine their distinguishing numbers [6]; and that interesting results on colorings and metric properties of Cartesian products were obtained in [1] and [2], respectively.

Back in 1957, Sabidussi [10] observed that for the Cartesian product $G \,\square\, H$ of arbitrary graphs G and H, $\kappa(G \,\square\, H) \geq \kappa(G) + \kappa(H)$. (Interestingly, in [9] it is wrongly claimed that the equality holds here.) About 20 years later, Liouville [8] announced that for any graphs G and H on at least two vertices,

$$\kappa(G \,\square\, H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)\}\,, \tag{1}$$

where δ denoted the minimum degree of a given graph. However, the announced proof never appeared. Finally, after an additional 30 years, the paper [11] gives a proof of (1).

For the edge-connectivity of Cartesian products, Xu and Yang proved in [12] that for any graphs G and H on at least two vertices,

$$\kappa'(G \,\square\, H) = \min\{\kappa'(G)|H|, \kappa'(H)|G|, \delta(G) + \delta(H)\}\,. \tag{2}$$

The basic idea of their proof is to use the edge version of Menger's theorem, that is, to find enough edge-disjoint paths between pairs of vertices of the Cartesian product. This way the arguments are rather technical and lengthy.

In Section 2 we first give a direct, short proof of (2). Then we prove that if S is a minimum edge cut of $G \,\square\, H$, then either S is induced by a minimum edge cut of a factor, or S is a set of edges incident to a vertex of $G \,\square\, H$. In Section 3 we consider powers of graphs (with respect to the Cartesian product) and show that in this case the edge-connectivity as well as the connectivity is always equal to the minimum degree.

In the rest of this section we recall basic properties of the Cartesian product of graphs. Recall that the *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ where vertices (g,h) and (g',h') are adjacent if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. We say that G and H are *factors* of $G \,\square\, H$. Let g be a vertex of G. The subgraph of $G \,\square\, H$ induced by $\{g\} \times V(H)$ is isomorphic to H. It is called an *H-fiber* and denoted ${}^gH$. Similarly one defines the *G-fiber* G^h for a vertex h of H.

The Cartesian product operation is associative, hence products of more than two factors are well-defined. In the special case when all the factors are the same we speak of powers with respect to the Cartesian product. More precisely, for a positive integer n, the n^{th} *Cartesian power* of a graph G is $G^n = \Box_{i=1}^{n} G$.

Note finally that if follows immediately from the definition of the product that $\delta(G \Box H) = \delta(G) + \delta(H)$.

2 A short proof and the structure of edge cuts

In this section we first give a short proof of (2).

Let S' be an edge cut in G and $S = \{(x,h)(y,h) \mid h \in V(H), xy \in S'\}$. Then it is easy to see that S is an edge cut in $G \Box H$; we say that S is *induced* by S'. Therefore,

$$\kappa'(G \Box H) \leq \min\{\kappa'(G)|H|, \kappa'(H)|G|, \delta(G) + \delta(H)\}\,.$$

The nontrivial part of the proof is hence to show the other inequality. Let $S \subseteq E(G \Box H)$ be an edge cut in $G \Box H$, such that $|S| < \min\{\kappa'(G)|H|, \kappa'(H)|G|\}$. We need to prove that $|S| \geq \delta(G) + \delta(H)$.

By our assumption on the size of S there is a G-fiber G^x and an H-fiber ${}^yH$ that are connected in $(G \Box H) - S$. Let $G^x \cup {}^yH$ be contained in the connected component C_1 of $(G \Box H) - S$. Since $(G \Box H) - S$ is not connected there exists a connected component $C_2 \neq C_1$ of $(G \Box H) - S$. Define U and W with

$$U = \{g \in G \mid {}^gH \cap C_2 \neq \emptyset\} \quad \text{and} \quad W = \{h \in H \mid G^h \cap C_2 \neq \emptyset\}\,.$$

Both U and W are nonempty. Let (a,b) be an arbitrary vertex of C_2, and $\overline{U}$ and $\overline{W}$ the complements of U and W, respectively. See Fig. 1.

Then

$$\begin{aligned} \deg_{G \Box H}(a,b) &= \deg_G(a) + \deg_H(b) \\ &= |N(a) \cap U| + |N(a) \cap \overline{U}| + |N(b) \cap W| + |N(b) \cap \overline{W}|\,. \end{aligned}$$

Observe that every edge with one endvertex (a,b) and the other $(a,\bar{w})$ or $(\bar{u},b)$ (here $\bar{u} \in \overline{U}$ and $\bar{w} \in \overline{W}$) is an edge of S. Therefore there are at least $|N(a) \cap \overline{U}| + |N(b) \cap \overline{W}|$ edges of S contained in G^b or ${}^aH$. Moreover, the set S contains at least one edge of every $G^w, w \in W - \{b\}$ and every ${}^uH, u \in U - \{a\}$. Hence there are at least $(|W| - 1) + (|U| - 1)$ edges of S that are not contained in G^b or ${}^aH$. Combining this with $|N(a) \cap U| \leq |U| - 1$ and $|N(b) \cap W| \leq |W| - 1$ we come to the conclusion that $\deg(a,b) \leq |S|$ and the proof is complete.

Remark 2.1 *If in the above proof* $\deg(a,b) = |S|$, *then* $|N(a) \cap U| = |U| - 1$ *and* $|S \cap G^b| = |N(a) \cap \overline{U}|$.

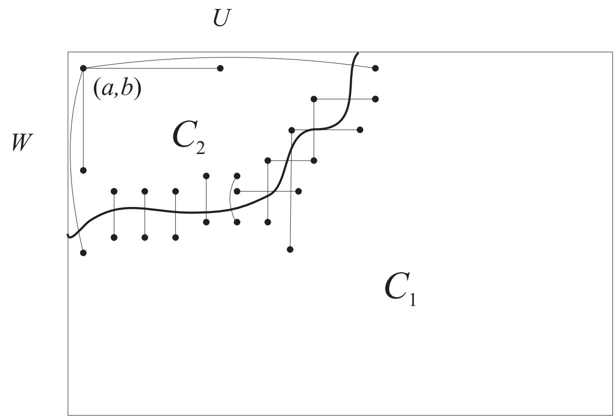

Figure 1: (a, b) is an arbitrary vertex of C_2

We next describe the structure of minimum edge cuts of Cartesian products.

Theorem 2.2 *Let S be a minimum edge cut of $G \,\Box\, H$. Then either S is induced by a minimum edge cut of a factor, or S is the set of edges incident to a vertex of $G \,\Box\, H$.*

Proof. Let S be a minimum edge cut in $G \,\Box\, H$ and C a connected component of $(G \,\Box\, H) - S$. By the minimality of S, $(G \,\Box\, H) - S$ has exactly two connected components. Suppose that every G-fiber has a vertex of C and its complement $\overline{C}$. Then removing the edges of S from any G-fiber disconnects that fiber, and therefore $|S| \geq \kappa'(G)|H|$. Since S is a minimum edge cut $|S| = \kappa'(G)|H|$ and no edges from H-fibers are contained in S. It follows that every H-fiber is either entirely contained in C or entirely in $\overline{C}$, and therefore S is induced by a minimum edge cut of G. If every H-fiber has a vertex of C and $\overline{C}$, analogous arguments prove that S is induced by a minimum edge cut of H.

Suppose now that there is a G-fiber and an H-fiber entirely contained in $\overline{C}$. It follows from the proof of the first part (and from the minimality of S) that then $|S| = \deg(a, b)$ for every $(a, b) \in C$. Therefore, it suffices to prove that if $|C| > 1$, then there are vertices (a, b) and (a', b) in C with $\deg(a, b) \neq \deg(a', b)$. So assume that $|C| > 1$ and let (a, b) be an arbitrary vertex of C. Since $|C| > 1$ there is a vertex $(a', b) \in C$ and at least one neighbor $(a'', b) \in \overline{C}$ of (a, b). We claim that $\deg(a, b) > \deg(a', b)$. Since $\deg(a, b) = \deg_G(a) + \deg_H(b)$ we only have to prove that $\deg_G(a) > \deg_G(a')$. Let

$$U = \{g \in G \mid {}^{g}H \cap C \neq \emptyset\} \quad \text{and} \quad W = \{h \in H \mid G^h \cap C \neq \emptyset\}\,.$$

By remark 2.1 we have $|N(a) \cap U| = |U| - 1$ and $|S \cap G^b| = |N(a) \cap \overline{U}|$. Therefore $|N(a') \cap \overline{U}| = 0$. Since $|N(a) \cap \overline{U}| \geq 1$ and $|N(a') \cap U| \leq |U| - 1$ we conclude that $\deg_G(a) > \deg_G(a')$. We have proved that $|C| = 1$ and hence S is a set of edges incident to a vertex of $G \,\Box\, H$. $\Box$

3 Connectivity of powers of graphs

In this section we prove that in the case when the factors of a Cartesian product are isomorphic, the minimum in (2) is always achieved by the minimum degree. The same holds for the (vertex) connectivity as well.

Theorem 3.1 *Let G be a connected graph on at least two vertices. Then for any $n \geq 2$, $\kappa'(G^n) = n\,\delta(G)$.*

Proof. Assume first that $\kappa'(G) = 1$, that is, G contains a bridge e. Consider a smallest component of $G - e$ to see that $\delta(G) \leq |G|/2$. Using (2) it follows that $2\delta(G) \leq |G| = \kappa'(G)|G|$, thus $\kappa'(G^2) = 2\delta(G)$.

Let $n \geq 3$. Then by induction and associativity of the Cartesian product,

$$\kappa'(G \,\Box\, G^{n-1}) = \min\{|G|^{n-1}, (n-1)\delta(G)|G|, n\delta(G)\}\,. \tag{3}$$

Clearly, $n\delta(G) \leq (n-1)\delta(G)|G|$. Moreover, if $G = K_2$ then $n\delta(G) = n \leq |G^{n-1}| = 2^{n-1}$. And if $|G| \geq 3$ then $n\delta(G) \leq n|G| \leq |G^{n-1}| = |G|^{n-1}$ holds since $n \geq 3$ and $|G| \geq 3$. In any case the minimum in (3) equals $n\delta(G)$ so the results holds for powers of graphs with edge-connectivity 1.

Let $\kappa'(G) \geq 2$, then $2\delta(G) \leq \kappa'(G)|G|$. Hence the assertion is true for $n = 2$. Let $n \geq 3$. Then by the induction hypothesis,

$$\kappa'(G \,\Box\, G^{n-1}) = \min\{\kappa'(G)|G|^{n-1}, (n-1)\delta(G)|G|, n\delta(G)\}\,. \tag{4}$$

Clearly, $n\delta(G) \leq (n-1)\delta(G)|G|$. Moreover,

$$n\delta(G) \leq n(|G| - 1) \leq n|G| \leq 2|G|^{n-1} \leq \kappa'(G)|G|^{n-1}\,,$$

where $n|G| \leq 2|G|^{n-1}$ (that is, $n/2 \leq |G|^{n-2}$) holds since $n \geq 3$ and $|G| \geq 2$. We conclude that the minimum in (4) equals $n\delta(G)$. $\Box$

We conclude the paper by observing that along the same lines as Theorem 3.1 the following result can be proved.

Theorem 3.2 *Let G be a connected graph on at least two vertices. Then for any $n \geq 2$, $\kappa(G^n) = n\,\delta(G)$.*

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