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Cartesian powers of graphs can be distinguished by two labels

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Abstract

The distinguishing number $D(G)$ of a graph G is the least integer d such that there is a d-labeling of the vertices of G which is not preserved by any nontrivial automorphism. For a graph G let G^r be the rth power of G with respect to the Cartesian product. It is proved that $D(G^r) = 2$ for any connected graph G with at least 3 vertices and for any $r \geq 3$. This confirms and strengthens a conjecture of Albertson. Other graph products are also considered and a refinement of the Russell and Sundaram motion lemma is proved.

MSC: 05C25

1. Introduction

A labeling $\ell : V(G) \to \{1, \ldots, d\}$ of a graph G is d-*distinguishing* if no nontrivial automorphism of G preserves the labeling. Such a labeling thus uniquely identifies vertices of G, that is, vertices are "distinguished" among themselves. The *distinguishing number*, $D(G)$, of a graph G, is the minimum d such that G has a d-distinguishing labeling.

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Since its introduction in [2], the distinguishing number of a graph, and more generally of a group action on a set, became an active research area within graphs and groups. The distinguishing numbers of Cartesian products of graphs have been investigated in [1,3]. Let us, as is nowadays more or less standard, denote the Cartesian product of graphs G and H by $G \square H$. For a graph G and an integer r, let G^r be defined as $G^1 = G$ and $G^r = G^{r-1} \square G$. Then the r-cube Q_r, $r \geq 1$, is defined as K_2^r.

Bogstad and Cowen [3] proved that $D(Q_2) = D(Q_3) = 3$ and $D(Q_d) = 2$ for $d \geq 4$. Their result has been widely generalized by Albertson [1] as follows. We say that a graph is *prime* (with respect to the Cartesian product) if it cannot be written as the Cartesian product of two nontrivial graphs.

Theorem 1.1 ([1]). *If G is a connected prime graph, then $D(G^r) = 2$ for $r \geq 4$. If, in addition, $|V(G)| \geq 5$, then $D(G^3) = 2$.*

Then Albertson conjectured that the condition that G be a prime graph might not be necessary.

Conjecture 1.2 ([1]). *For any graph G (not necessarily prime), there is an integer $R = R(G)$ such that $D(G^r) = 2$ for any $r \geq R(G)$.*

We confirm this conjecture by proving that $D(G^r) = 2$ for any connected graph G with at least 3 vertices and for any $r \geq 3$. Along the way, other graph products are considered and a refinement of the motion lemma from [8] is proved. We add that very recently, it has been proved in [7] that $D(G^2) = 2$ for any connected graph $G \neq K_2, K_3$.

2. An upper bound on $D(G * H)$

In this section we give an upper bound on the distinguishing number of an arbitrary graph product in which the automorphisms preserve the layer structure. This result (in the case of the Cartesian product) will then be used in the next section to obtain our main result.

Let G and H be graphs. Then by a *graph product* $G * H$ in the sense of [5] we mean any operation for which $V(G * H) = V(G) \times V(H)$ and the adjacency of two vertices in $G * H$ depends only on the adjacencies of the corresponding vertices in the factors. In particular, the *Cartesian product* $G \square H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined on the vertex set $V(G \square H) = V \times W$ while $E(G \square H) = \{\{(a, x), (b, y)\} \mid ab \in E \text{ and } x = y, \text{ or } xy \in F \text{ and } a = b\}$. Observe that the Cartesian product is commutative and associative.

Let G and H be graphs and $a \in V(G)$. Then the subgraph of $G * H$ induced by the vertex set $\{(a, x) : x \in V(H)\}$, is called an *H-layer* of $G * H$ and denoted by H_a. Analogously one defines G-layers. Note that the layers of the Cartesian product are isomorphic to the factor graphs. Among the four standard graph products [6] the strong product and the lexicographic product also have this property. However, the automorphism groups of lexicographic products generally do not satisfy the conditions of the following theorems.

Theorem 2.1. *Let G and H be connected graphs with $2 \leq D(G) \leq D(H)$. Let $G * H$ be a graph product for which the layers are isomorphic to the corresponding factors. If every* *automorphism* φ *of* G *is of the form* $\varphi(a, x) = (\varphi_G(a), \varphi_H(x))$, *where* $\varphi_G \in \mathrm{Aut}(G)$ *and* $\varphi_H \in \mathrm{Aut}(H)$, *then*

$$D(G * H) \leq \max\{D(G), D(H) - (2^{D(G)} - D(G) - 1)\}.$$

Proof. Let $n = D(G)$ and $k = D(H)$. Let ℓ_G be an n-distinguishing labeling of G and ℓ_H a k-distinguishing labeling of H. Define a labeling ℓ of $G * H$ in the following way. First set

$$\ell(a, x) = \begin{cases} \ell_H(x), & 1 \leq \ell_H(x) \leq k - (2^n - n - 1); \\ \ell_G(a), & \ell_H(x) = k. \end{cases}$$

We still need to define ℓ for the vertices (a, x) with $k - 2^n + n + 2 \leq \ell_H(x) \leq k - 1$. Let $A_1, \ldots, A_t$ be the subsets of $\{1, 2, \ldots, n\}$ with $2 \leq |A_i| \leq n - 1$. To each integer in the interval $[k - 2^n + n + 2, k - 1]$ assign a unique set A_i. Complete the definition of ℓ such that the vertices of the layer G_x, where $k - 2^n + n + 2 \leq \ell_H(x) \leq k - 1$, are (arbitrarily) labeled using all the labels from the set A_i which is assigned to $\ell_H(x)$.

We claim that ℓ is a distinguishing labeling of $G * H$. Let φ be an automorphism of $(G * H, \ell)$. Then by the theorem assumption, $\varphi = (\varphi_G, \varphi_H)$. We need to show that $\varphi = \mathrm{id}$, that is, the identity map.

Claim 1. For any $x \in V(H)$: $\ell_H(\varphi_H(x)) = \ell_H(x)$.
If $\ell_H(x) = k$, then G_x receives n labels. If $\ell(H_x) < k$, then G_x receives at most $n - 1$ labels. As $\varphi(H_x) = H_{\varphi_H(x)}$ and φ preserves the labels, it follows that if $\ell_H(x) = k$, then $\ell_H(\varphi_H(x)) = k$. Similarly, if $\ell_H(x), \ell_H(y) < k$, then the construction of ℓ implies that if $\ell_H(x) \neq \ell_H(y)$, then the sets of labels of layers G_x and G_y are different. Because φ preserves labels and maps G-layers onto G-layers, we conclude that $\ell_H(\varphi_H(x)) = \ell_H(x)$.

Claim 2. $\varphi_G = \mathrm{id}$.
Let x be a vertex of H with $\ell_H(x) = k$ and let $\varphi_H(x) = y$. (It is possible that $x = y$.) By Claim 1, $\ell_H(y) = k$. Hence φ_G induces a label preserving isomorphism between the layers $G_x \cong G$ and $G_y \cong G$. As ℓ_G is an n-distinguishing labeling of G, we conclude that $\varphi_G = \mathrm{id}$.

Claim 3. $\varphi_H = \mathrm{id}$.
Let u be a vertex of G and consider the layer $H_u \cong H$. By Claim 2, φ maps H_u onto H_u. But then φ_H induces an isomorphism $H_u \to H_u$. Moreover, by Claim 1, this isomorphism gives us a label preserving automorphism of (H, ℓ_H). Thus as ℓ_H is a k-distinguishing labeling of H, the claim follows. □

Theorem 2.1 can, for instance, be applied to the strong product of connected, prime, and so-called thin graphs; cf. [6, Theorem 5.22]. For our purposes the most important special case in which the conditions of the theorem are fulfilled is given in the next corollary.

Sabidussi [9] and Vizing [10] proved that every connected graph has a unique prime factor decomposition with respect to the Cartesian product. Hence it makes sense to define graphs G and H to be *relatively prime* (with respect to the Cartesian product) if there is no nontrivial graph that is a factor in the prime factor decomposition of G and in the decomposition of H. Clearly, two prime graphs are relatively prime. We refer the reader to [6, Corollary 4.17] for the fact that the automorphisms of the Cartesian product of connected, relatively prime graphs preserve the layer structure.

Corollary 2.2. *Let G and H be connected graphs with* $2 \le D(G) \le D(H)$. *If G and H are relatively prime then*

$$D(G \,\Box\, H) \le \max\{D(G), D(H) - (2^{D(G)} - D(G) - 1)\}.$$

Consider the products $K_2 \,\Box\, C_3$ and $K_2 \,\Box\, C_5$. Since $D(K_2) = 2$ and $D(C_3) = D(C_5) = 3$, Corollary 2.2 gives $D(K_2 \,\Box\, C_3) = D(K_2 \,\Box\, C_5) = 2$. On the other hand, $D(K_2 \,\Box\, C_4) = 3$ since $K_2 \,\Box\, C_4 \cong Q_3$. This example shows that in general we cannot drop the assumption that the factors are relatively prime.

When the distinguishing numbers of G and H are both small, the bound of Corollary 2.2 is often exact. For instance:

Corollary 2.3. *Let G and H be connected, relatively prime graphs with* $D(G) = 2$ *and* $2 \le D(H) \le 3$. *Then* $D(G \,\Box\, H) = 2$.

Proof. Since $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ are nontrivial, so is $\mathrm{Aut}(G \,\Box\, H)$; hence $D(G \,\Box\, H) \ge 2$. Corollary 2.2 completes the argument. □

On the other hand, $D(G \,\Box\, H)$ could be much smaller than $\max\{D(G), D(H)\}$. Let G be a graph on k vertices with $D(G) = 1$, that is, an asymmetric graph. Then it is easy to see that

$$D(G \,\Box\, K_n) = \left\lceil n^{\frac{1}{k}} \right\rceil,$$

while $\max\{D(G), D(K_n)\} = n$.

3. A refinement of the motion lemma and the main result

Albertson's proof of Theorem 1.1 uses a result of Russell and Sundaram [8] that is known as the motion lemma. In this section we first prove a refinement of the motion lemma that might be of independent interest. We then simplify this result for the case of vertex transitive graphs and complete the section with a proof of our main theorem.

For $\phi \in \mathrm{Aut}(G)$, let $m(\phi) = |\{x \in V(G) : \phi(x) \ne x\}|$, and $m(G) = \min\{m(\phi) : \phi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}_V\}\}$. Then the *motion lemma* asserts that if $d^{\frac{m(G)}{2}} > |\mathrm{Aut}(G)|$, then $D(G) \le d$. Before we present a refinement of this result some preparation is needed.

Suppose $\phi \in \mathrm{Aut}(G)$ is decomposed into a product of disjoint cycles:

$$\phi = (v_{11} v_{12} \cdots v_{1\ell_1})(v_{21} v_{22} \cdots v_{2\ell_2}) \cdots (v_{t1} v_{t2} \cdots v_{t\ell_t});$$

then the *cycle norm* of ϕ is defined as

$$c(\phi) = \sum_{i=1}^{t} (\ell_i - 1).$$

A d-labeling ℓ of G is preserved by ϕ if and only if the vertices in each cycle of ϕ are labeled by the same label. So the probability that a random d-labeling ℓ is preserved by

ϕ is equal to $d^{-c(\phi)}$. It is obvious that $c(\phi) \geq m(\phi)/2$. So the probability that a random d-labeling ℓ is preserved by ϕ is at most $d^{-m(\phi)/2}$.

Lemma 3.1. *Let G be a graph with $|V(G)| = n$. Suppose* Aut(G) *acting on $V(G)$ has k orbits and $d \geq 2$ is an integer. If $n - m(G) \geq 3$, and*

$$\left(|\mathrm{Aut}(G)| - \frac{k|\mathrm{Aut}(G)| - n}{n - m(G)} - 1\right) d^{-n/2} + \frac{k|\mathrm{Aut}(G)| - n}{n - m(G)} d^{-m(G)/2} < 1,$$

then $D(G) \leq d$.

Proof. Let $O_1, O_2, \ldots, O_k$ be the orbits of Aut(G) acting on $V(G)$, with $|O_i| = n_i$ and $n_1 + n_2 + \cdots + n_k = n$. For each vertex x of G, let $H_x = \{\phi \in \mathrm{Aut}(G) : \phi(x) = x\}$. If $x \in O_i$, then $|H_x| = |\mathrm{Aut}(G)|/n_i$; cf. [4, Lemma 2.2.2]. Therefore $\sum_{x \in V} |H_x| = \sum_{i=1}^{k} \sum_{x \in O_i} |H_x| = k|\mathrm{Aut}(G)|$.

For $0 \leq j \leq n - m(G)$, let $\Phi_j = \{\phi \in \mathrm{Aut}(G) : \phi$ fixes exactly j vertices of $G\}$. Let $q_j = |\Phi_j|$. By the definition of $m(G)$, $\cup_{j=0}^{n-m(G)} \Phi_j = \mathrm{Aut}(G) \setminus \{\mathrm{id}_V\}$. Note that id_V fixes n vertices. So both $n + \sum_{j=0}^{n-m(G)} jq_j$ and $\sum_{x \in V(G)} |H_x|$ count the number of pairs (ϕ, x) such that $\phi \in \mathrm{Aut}(G)$, $x \in V(G)$ and $\phi(x) = x$. Therefore

$$n + \sum_{j=0}^{n-m(G)} jq_j = \sum_{x \in V(G)} |H_x| = k|\mathrm{Aut}(G)|.$$

If $\phi \in \Phi_j$, then the probability that a random d-labeling is preserved by ϕ is at most $d^{-(n-j)/2}$. If

$$\sum_{j=0}^{n-m(G)} \sum_{\phi \in \Phi_j} d^{-(n-j)/2} = \sum_{j=0}^{n-m(G)} q_j d^{-(n-j)/2} < 1,$$

then there is a d-labeling ℓ which is not preserved by any automorphism $\phi \neq \mathrm{id}_V$. So to prove Lemma 3.1, it amounts to showing that if $\sum_{j=0}^{n-m(G)} jq_j = k|\mathrm{Aut}(G)| - n$ and $\sum_{j=0}^{n-m(G)} q_j = |\mathrm{Aut}(G)| - 1$, then $\sum_{j=0}^{n-m(G)} q_j d^{-(n-j)/2} < 1$.

For non-negative real numbers $x_0, x_1, \ldots, x_{n-m(G)}$, let

$$f(x_0, x_1, \ldots, x_{n-m(G)}) = \sum_{j=0}^{n-m(G)} x_j d^{-(n-j)/2}.$$

So it suffices to prove that $f(x_0, x_1, \ldots, x_{n-m(G)}) < 1$ for any $(x_0, x_1, \ldots, x_{n-m})$ with

$$\sum_{j=0}^{n-m(G)} jx_j = k|\mathrm{Aut}(G)| - n \qquad \text{and} \qquad \sum_{j=0}^{n-m(G)} x_j = |\mathrm{Aut}(G)| - 1.$$

Suppose there is an index $0 < j^* < n - m(G)$ such that $x_{j^*} > 0$. Define the sequence $(x'_0, x'_1, \ldots, x'_{n-m(G)})$ as follows:

$$x'_0 = x_0 + \frac{n - m(G) - j^*}{n - m(G)} x_{j^*},$$

$$x'_{j^*} = 0,$$
$$x'_{n-m(G)} = x_{n-m} + \frac{j^*}{n-m(G)} x_{j^*},$$
$$x'_j = x_j, \quad \text{otherwise.}$$

Then $\sum_{j=0}^{n-m(G)} jx_j = \sum_{j=0}^{n-m(G)} jx'_j$ and $\sum_{j=0}^{n-m(G)} x_j = \sum_{j=0}^{n-m(G)} x'_j$. Since $d \geq 2$ and $n - m(G) \geq 3$, easy calculation shows that

$$f(x_0, x_1, \ldots, x_{n-m(G)}) < f(x'_0, x'_1, \ldots, x'_{n-m(G)}).$$

Therefore the maximum of $f(x_0, x_1, \ldots, x_{n-m(G)})$ is attained at $x_0 = |\text{Aut}(G)| - \frac{k|\text{Aut}(G)|-n}{n-m(G)} - 1$, $x_j = 0$ for $j = 1, 2, \ldots, n - m(G) - 1$ and $x_{n-m(G)} = \frac{k|\text{Aut}(G)|-n}{n-m(G)}$. That is, for any $(x_0, x_1, \ldots, x_{n-m})$ with

$$\sum_{j=0}^{n-m(G)} jx_j = k|\text{Aut}(G)| - n \qquad \text{and} \qquad \sum_{j=0}^{n-m(G)} x_j = |\text{Aut}(G)| - 1,$$

we have

$$\begin{aligned} f(x_0, x_1, \ldots, x_{n-m(G)}) &\leq \left(|\text{Aut}(G)| - \frac{k|\text{Aut}(G)| - n}{n-m(G)} - 1\right) d^{-n/2} \\ &\quad + \frac{k|\text{Aut}(G)| - n}{n-m(G)} d^{-m(G)/2} \\ &< 1. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Corollary 3.2. *Suppose G is a vertex transitive graph with n vertices and with $n-m(G) \geq 3$. If $d \geq 2$ is an integer and*

$$|\text{Aut}(G)| \leq \frac{(n-m(G))d^{m(G)/2}}{(n-m(G))d^{(m(G)-n)/2} + 1},$$

then $D(G) \leq d$.

Proof. Assume

$$|\text{Aut}(G)| \leq \frac{(n-m(G))d^{m(G)/2}}{(n-m(G))d^{(m(G)-n)/2} + 1}.$$

Then

$$\begin{aligned} 1 &\geq |\text{Aut}(G)|d^{-n/2} + \frac{|\text{Aut}(G)|}{n-m(G)} d^{-m(G)/2} \\ &> \left(|\text{Aut}(G)| - \frac{|\text{Aut}(G)|}{n-m(G)} - 1\right) d^{-n/2} + \frac{|\text{Aut}(G)|}{n-m(G)} d^{-m(G)/2} \\ &> \left(|\text{Aut}(G)| - \frac{|\text{Aut}(G)| - n}{n-m(G)} - 1\right) d^{-n/2} + \frac{|\text{Aut}(G)| - n}{n-m(G)} d^{-m(G)/2}. \end{aligned}$$

Since G is vertex transitive, $\text{Aut}(G)$ acting on $V(G)$ has only one orbit. By Lemma 3.1, $D(G) \leq d$. □

Corollary 3.3. *For any $k \geq 3$ and $r \geq 3$, $D(K_k^r) = 2$.*

Proof. It is obvious that $D(K_k^r) \geq 2$ for any $k \geq 2$ and $r \geq 1$. So we only need to prove that for any $k \geq 3$ and $r \geq 3$, $D(K_k^r) \leq 2$.

It was proved in [1] that for $G = K_k^r$, $|\text{Aut}(G)| = r!(k!)^r$ and $m(G) = 2k^{r-1}$. If $r \geq 4$ or $r = 3$ and $k \geq 5$, then it was shown in [1] that an application of Russell and Sundaram's motion lemma shows that $D(K_k^r) \leq 2$. In the cases $r = 3$ and $k = 3, 4$, the motion lemma cannot yield the desired bound. However, an application of Corollary 3.2 shows that $D(K_k^r) \leq 2$. □

Corollary 3.4. *If G is a connected prime graph with $|V(G)| \geq 3$, then for any integer $r \geq 3$, $D(G^r) = 2$.*

Proof. It was proved by Albertson [1] that for a prime graph G with $|V(G)| = n$, for any integer r, $D(G^r) \leq D(K_n^r)$. □

Theorem 3.5. *Let G be a connected graph. If G has a prime factor of cardinality at least 3, then for any $r \geq 3$, $D(G^r) = 2$.*

Proof. If G is prime, this is Corollary 3.4. Suppose G is not prime, then let $G = G_1^{p_1} \square G_2^{p_2} \square \cdots \square G_k^{p_k}$ be the prime factor decomposition of G, where the G_i's are prime graphs and $p_i \geq 1$. By the theorem's assumption we may assume that G_1 has cardinality at least 3. Hence $D(G_1^{rp_1}) = 2$ by Corollary 3.4. Moreover, $D(G_i^{rp_i}) \leq 3$ for $i = 2, \ldots, k$.

As G is not prime, $k \geq 2$. Then $G^r = G_1^{rp_1} \square \cdots \square G_k^{rp_k}$, and as the G_i's are prime, the factors $G_i^{rp_i}$ are relatively prime. Then by [6, Corollary 4.17], the automorphisms of G^r preserve the layer structure of its factorization $G_1^{rp_1} \square \cdots \square G_k^{rp_k}$. Hence by Corollary 2.2 we conclude that $D(G^r) \leq \max\{D(G_1^{rp_1}), D(G_2^{rp_2}) - 1, \ldots, D(G_k^{rp_k}) - 1\} = 2$. □

Corollary 3.6. *If G is a connected graph and $G \neq K_2$, then $D(G^r) = 2$ for any $r \geq 3$.*

Proof. If G has a factor which is not K_2, then by Theorem 3.5, $D(G^r) = 2$. If each factor of G is K_2, then $G = K_2^p$ for some $p \geq 2$. So again $D(G^r) = D(K_2^{rp}) = 2$. □

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