Convex excess in partial cubes

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Abstract

The convex excess $ce(G)$ of a graph G is introduced as $\sum(|C|-4)/2$ where the summation goes over all convex cycles of G. It is proved that for a partial cube G with n vertices, m edges, and isometric dimension $i(G)$, inequality $2n-m-i(G)-ce(G)\leq 2$ holds. Moreover, the equality holds if and only if the so-called zone graphs of G are trees. This answers the question from [9] whether partial cubes admit this kind of inequalities. It is also shown that a suggested inequality from [9] does not hold.

Key words: partial cube; hypercube; convex excess; zone graph

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1 Introduction

Partial cubes present one of the central and most studied classes of graphs in metric graph theory. They were introduced by Graham and Pollak [27] as a model for interconnection networks but found many additional applications afterwards. For instance, they form the key graph class in media theory, see the recent book [24]. For classical results on partial cubes we refer to the books [18, 29] and the semi-survey [39].

Many important classes of graphs are partial cubes. First of all, hypercubes, trees, and median graphs form very special, but at the same time very striking examples. The list of other significant classes of partial cubes include: tope graphs of oriented matroids (in particular graphs of regions of hyperplane arrangements) [8, 25]; benzenoid graphs [14, 16, 31] and more generally bipartite $(6, 3)$-graphs [5, 15]; tiled partial cubes and several related classes [9, 28]; netlike partial cubes, a class that forms a common generalization of median graphs and even cycles [41, 42, 43]; flip graphs of point sets that have no empty pentagons [22]; and special subdivisions of graphs [1, 7, 17]. For more information on several of these classes of partial cubes and related classes of graphs (as well as on more aspects of metric graph theory) see the survey [5].

Among numerous recent results on partial cubes we mention the following. A lot of effort was made in order to classify cubic partial cubes, see [21, 30, 35]. Several types of graphs derived from partial cubes were studied, see, for instance, [11, 20, 32, 33]. For two additional recent aspects of partial cubes see [26, 40, 47]. Eppstein [23] gives a quadratic algorithm for recognizing partial cubes.

Among subclasses of partial cubes, median graphs deserve special attention. (In fact, they were studied as much as general partial cubes.) It was proved in [34] that for a median graph G with n vertices and m edges,

$$2n - m - i(G) \leq 2\,, \tag{1}$$

where $i(G)$ stands for the isometric dimension of G (i.e., the number of the classes of the Djoković-Winkler relation Θ). Moreover, the equality holds if and only if G is a cube-free median graph. This theorem was extended in [9] to those partial cubes that can be obtained by means of a connected expansion procedure. A similar result was established by Bandelt and Chepoi for another class of partial cubes—bipartite cellular graphs (graphs obtainable from a collection of single edges and even cycles by successive gated amalgamations). In [4] they proved that for these graphs the equality

$$n - m + g = 1$$

holds, where g is the number of gated cycles. In addition, Soltan and Chepoi [45, Theorem 4.2.(6)] proved that for a median graph G the following Euler-type formula holds:

$$\sum_{i\geq 0} (-1)^i \, \alpha_i(G) = 1\,,$$

where $\alpha_i(G)$ denotes the number of induced i-cubes of G. We refer to [6] for a generalization of this formula to the graphs of lopsided sets and to [12] for a general approach for obtaining such equalities. The above equality was independently obtained in [44]. In the same paper it is also proved that for a median graph G,

$$k = -\sum_{i\geq 0} (-1)^i \, i \, \alpha_i(G)\,,$$

where $k = i(G)$ is the isometric dimension of G.

2 Main result and related concepts

To answer the above question, we introduce the following concepts. For a graph G let

$$\mathcal{C}(G) = \{C \mid C \text{ is a convex cycle of } G\}$$

and set

$$ce(G) = \sum_{C\in\mathcal{C}(G)} \frac{|C|-4}{2}\,.$$

We call $ce(G)$ the *convex excess* of G. Let F be a Θ-class of a partial cube G. Then the *F-zone graph*, Z_F, is the graph with $V(Z_F) = F$, vertices f and f' being adjacent in Z_F if they belong to a common convex cycle of G. Let us call partial cubes whose all zone graphs are trees *tree-zone partial cubes.*

Now we can state the main result of this paper:

Theorem 2.1 *For a partial cube G with n vertices and m edges,*

$$2n - m - i(G) - ce(G) \leq 2\,. \tag{2}$$

Moreover the equality holds if and only if G is a tree-zone partial cube.

Let $C = C_{2r}$ be an isometric cycle of a median graph G. Then, as proved by Bandelt [3], the convex closure of C is Q_r. Therefore all convex cycles of median graphs have length four and consequently $ce(G) = 0$. Thus Theorem 2.1 immediately implies (1). The cubes Q_d also show that no lower bound on $2n - m - i(G) - ce(G)$ is possible. Indeed, since $i(Q_d) = d$ and $ce(Q_d) = 0$, we have

$$2n - m - i(Q_d) - ce(Q_d) = 2^{d+1} - d2^{d-1} - d = 2^{d-1}(4-d) - d$$

which tends to $-\infty$ when $d \to \infty$.

The key to the inequality part of Theorem 2.1 is the fact that during an expansion step convex cycles lift to convex cycles, cf. Proposition 4.2. This property is not true for isometric cycles, see Fig. 1 for an example. This fact might be one of the reasons why a desired inequality for partial cubes was so elusive.

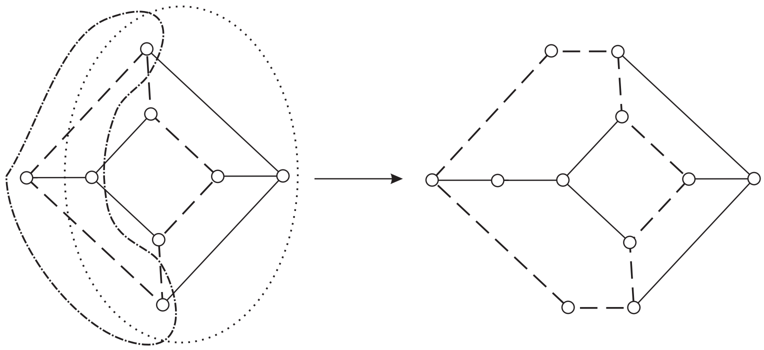

Figure 1: An isometric cycle need not expand to an isometric cycle

The equality in (2) is achieved precisely on tree-zone partial cubes. Examples of such graphs are bipartite cellular graphs and bipartite plane graphs in which all inner faces have lengths at least 6 and all inner vertices have degrees at least 3, cf. [5]. Along the way we will prove that the zone graphs of partial cubes are always connected. It seems an interesting project to further study these graphs and their relation to partial cubes.

It was further asked in [9] if it is true that $2n - m - 2i(G) \leq 0$ holds for partial cubes with more than two vertices. We next show that the answer is negative. With Theorem 2.1 in hand it seems plausible to search for possible counterexamples among graphs with convex excess bigger than their isometric dimension. Such examples can indeed be constructed, for instance, as follows. Let $P(r, s)$, $1 \leq s \leq r$, be the parallelogram hexagonal graph, see Fig. 2 for $P(5, 3)$.

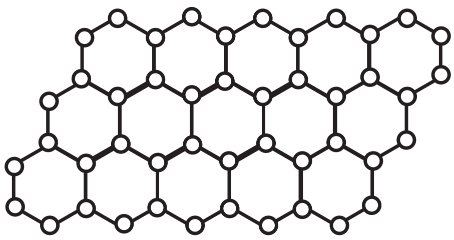

Figure 2: The parallelogram hexagonal graph $P(5, 3)$

Then $P(r, s)$ has $n = (r+1)(2s+2)-2$ vertices, $m = (r+1)(2s+1)-2+r(s+1)$

edges, while $i(P(r,s)) = 2r + 2s - 1$. Consequently,

$$2n - m - 2i(P(r,s)) = rs - 2(r+s) + 3\,.$$

In particular, for $r = s$ this reduces to $r^2 - 4r + 3$ which is strictly positive for any $r \geq 4$ and in fact arbitrarily large.

In the next section we define the concepts and introduce the techniques needed in this paper. In Section 4 we consider the position of convex cycles in partial cubes and in particular prove that they always expand to convex cycles. In Section 5 we prove the inequality part of Theorem 2.1 while the equality part of the theorem is demonstrated in the concluding section.

3 Preliminaries

The graph distance $d_G(u,v)$ between vertices u and v of a connected graph G is the usual shortest path distance. If H and H' are subgraphs of G, the distance between the subgraphs is defined as $d(H,H') = \min\{d(u,u') \mid u \in H, u' \in H'\}$. A shortest path between vertices u and v will be briefly called a (u,v)-*geodesic*.

A subgraph H of G is called *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$. *Partial cubes* are isometric subgraphs of hypercubes. H is *convex*, if for all u, $v \in V(H)$, all shortest (u,v)-paths from G belong to H. A convex subgraph is isometric but the converse need not be true. The *convex closure* of H is the smallest convex subgraph of G containing H.

The *interval* $I_G(u,v)$ between vertices u and v of a graph G consists of all vertices on shortest paths between u and v. In partial cubes, intervals induce convex subgraphs, we will simply say that intervals are convex. To see this, note first that cubes have convex intervals and then use the definition of partial cubes. On the other hand, a bipartite graph with convex intervals need not be a partial cubes [10].

A graph G is a *median graph* [2, 36, 38] if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest (u,v)-path, shortest (u,w)-path, and shortest (w,v)-path.

An important concept that yields a characterization of partial cubes is the Djoković-Winkler relation Θ defined on the edge set of a connected graph G as follows. Two edges $e = xy$ and $f = uv$ of G are in the relation Θ if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. Relation Θ is reflexive and symmetric. Moreover, a connected bipartite graph is a partial cube if and only if Θ is also transitive [19, 46]. Hence Θ is an equivalence relation on the edge set $E(G)$ of a partial cube G. It partitions $E(G)$ into the so-called Θ-*classes*. The *isometric dimension* $i(G)$ of a partial cube G is the number of its Θ-classes. Equivalently, $i(G)$ equals the dimension of the smallest hypercube into which G embeds isometrically.

We say that two nonempty isometric subgraphs G_1 and G_2 form an *isometric cover* of a graph G provided that $G = G_1 \cup G_2$, by which we mean that $V(G) =$

$V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If G is connected then $G_1 \cap G_2 \neq \emptyset$ for every isometric cover G_1, G_2.

Suppose G_1, G_2 is an isometric cover of G. For $i = 1, 2$, let $\widetilde{G}_i$ be an isomorphic copy of G_i, and for a vertex $u \in G_1 \cap G_2$, let u_i be the corresponding vertex in $\widetilde{G}_i$. The *expansion* of G with respect to G_1, G_2 is the graph $\widetilde{G}$ obtained from the disjoint union of $\widetilde{G}_1$ and $\widetilde{G}_2$, where for each $u \in G_1 \cap G_2$ the vertices u_1 and u_2 are joined by a new edge in $\widetilde{G}$. We say that the expansion is *connected* provided that $G_1 \cap G_2$ is a connected graph. Chepoi [13] proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions.

The *prism* over a graph G is the Cartesian product $G \,\Box\, K_2$. Note that in the expansion of G with respect to G_1, G_2, $G_1 \cap G_2$ expands to $(G_1 \cap G_2) \,\Box\, K_2$ and in particular each cycle C from $G_1 \cap G_2$ expands to the prism $C \,\Box\, K_2$.

The concept of expansion is due to Mulder [36, 37] in the context of median graphs and Chepoi [13] in the context of partial cubes (and partial Hamming graphs).

The reverse operation to the expansion is defined as follows. Let G be a partial cube and F its Θ-class. Contracting every edge of F, a partial cube is obtained that is called the *contraction* of G (with respect to F).

We will need the following well-known facts about partial cubes.

Lemma 3.1 *Let C be an isometric cycle of a partial cube G and e an edge of C. Then e is in relation Θ to exactly one edge of C, namely, its antipodal edge.*

Lemma 3.2 *Let G be a partial cube and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. Then the edges between $\widetilde{G}_1$ and $\widetilde{G}_2$ form a Θ-class of $\widetilde{G}$, while the other Θ-classes of $\widetilde{G}$ are induced by those of G. In particular, $i(\widetilde{G}) = i(G) + 1$.*

4 Convex cycles in partial cubes

In this section we have a closer look at convex cycles in partial cubes. We particularly address the question how such cycles are transformed when expanding or contracting a partial cube. We begin with the following general property.

Proposition 4.1 *Let G_1, G_2 be an isometric cover of a partial cube G and let C be a convex cycle of G. Then C is either completely contained in at least one of the G_i's or C meets $G_1 \cap G_2$ in two antipodal vertices.*

Proof. Let $G_0 = G_1 \cap G_2$. Suppose C meets $G_1 - G_2$ as well as $G_2 - G_1$. Let y be a vertex of C that lies in $G_1 - G_2$ and z a vertex of C from $G_2 - G_1$. Let u be the first vertex from $C \cap G_0$ on one of the two (y, z)-paths along C and let v be the first vertex from $C \cap G_0$ on the other (y, z)-path. Let P be the (u, v)-path that goes along C and contains y. Let Q be the other (u, v)-path along C, that is, the one that contains z. Since C is convex and hence $d(u,v) = d_C(u,v)$, at least one of P and Q is a geodesic. Furthermore, since G_1 and G_2 are isometric and hence $k = d_{G_1}(u,v) = d_{G_2}(u,v)$, and because C is convex, we infer that $|P| = |Q| = k$. Therefore u and v are antipodal vertices of C. Suppose there is a vertex $w \in C \cap G_0$, $w \neq u, v$. Then $w \in Q$ and $d(w,v) < d$. Since $z \notin G_1$ and G_1 is isometric, there must be a (v,w)-geodesic contained in G_1. But this is not possible as C is convex. Hence we conclude that $C \cap G_0 = \{u,v\}$. □

According to Proposition 4.1 we classify convex cycles of G with respect to the isometric cover G_1, G_2 into the following three types.

(i) A convex cycle that meets $G_1 \cap G_2$ in two antipodal vertices will be called a *cross cycle* (with respect to G_1, G_2).

(ii) A convex cycle that lies in $G_1 \cap G_2$ will be called an *intersection cycle* (with respect to G_1, G_2).

(iii) The remaining convex cycles will be called *lateral cycles* (with respect to G_1, G_2). Such a cycle either contains a vertex from $G_1 - G_2$ in which case it lies completely in G_1, or contains a vertex from $G_2 - G_1$ in which case it lies in G_2.

Let $\widetilde{G}$ be the expansion of the partial cube G with respect to the isometric cover G_1, G_2. If C is a lateral cycle contained in G_i then it lifts to a cycle $\widetilde{C}$ in $\widetilde{G}_i$, of the same length as C. We call $\widetilde{C}$ the *expansion of* C. Suppose C is a cross cycle and let y and z be its (antipodal) vertices from $G_1 \cap G_2$. Then C naturally expands to a cycle $\widetilde{C}$ of $\widetilde{G}$ that consists of the copy of the (y,z)-subpath of C in $\widetilde{G}_1$, the copy of the (z,y)-subpath of C in $\widetilde{G}_2$, and the edges y_1y_2 and z_1z_2. Note that $|\widetilde{C}| = |C| + 2$. Again we refer to $\widetilde{C}$ as to the expansion of C. Finally, let C be an intersection cycle. Then the expansion of C is a prism $C \,\Box\, K_2$, where the C-layers of this prism are cycles C_1 and C_2 of the same length as C and lie in $\widetilde{G}_1$ and $\widetilde{G}_2$, respectively.

Proposition 4.2 *Let G be a partial cube and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. If C is a lateral or cross cycle of G, then its expansion $\widetilde{C}$ is a convex cycle of $\widetilde{G}$. If C is an intersection cycle of G, then the cycles C_1 and C_2 of its expansion are both convex.*

Proof. Suppose that C is a lateral cycle and assume without loss of generality that C lies in G_1. Then $\widetilde{C}$ lies in $\widetilde{G}_1$. As $\widetilde{G}_1$ is convex in $\widetilde{G}$, the expansion $\widetilde{C}$ of C is convex, for otherwise C would not be convex in G. By the same argument, the cycles C_1 and C_2 of the prism $C \,\Box\, K_2$ that is the expansion of an intersection cycle C are convex.

Suppose that C is a cross cycle. Let C meet $G_0 = G_1 \cap G_2$ in antipodal vertices y and z and suppose $\widetilde{C}$ is not convex. Then there are vertices $u, v \in \widetilde{C}$ and a (u,v)-geodesic P such that P is not contained in $\widetilde{C}$. Assume that $d(u,v)$ is as small as possible, so that P meets $\widetilde{C}$ only in u and v. Suppose $u, v \in \widetilde{G}_i$ for some i. Since the distance function d_{G_i} coincides with the distance function $d_{\widetilde{G}_i}$ we infer that C is not convex. Hence we may assume that $u \in \widetilde{G}_1$ and $v \in \widetilde{G}_2$. But now P necessarily contains an edge x_1x_2 where $x_1 \in \widetilde{G}_1$ and $x_2 \in \widetilde{G}_2$. Since the contraction shorten the length of P we infer again that C is not convex. $\square$

Thus convex cycles expand into convex cycles or into pairs of convex cycles forming prisms. On the other hand, it need not be the case that a convex cycle contracts to a convex cycle. To see this, let G be an arbitrary partial cube that contains an isometric but not convex cycle C of length $2n$, $n \geq 3$ (for instance, G can be Q_n). Let $\widetilde{G}$ be the expansion of G with respect to the isometric cover $G_1 = G, G_2 = C$. Then $\widetilde{C} = \widetilde{G}_2$ is a convex cycle of $\widetilde{G}$, while its contraction C is not convex.

5 Proof of the inequality

To prove the inequality part of Theorem 2.1 we will make use of the following general statement that might be of independent interest.

Proposition 5.1 *The zone graphs of partial cubes are connected.*

Proof. Let G be a partial cube, F its Θ-class and let G_1, G_2 be the connected components of $G \setminus F$. We recall that G_1 and G_2 are convex subgraphs of G. For any edge $x \in F$ we will denote its end vertices with x_i, $i = 1, 2$, where $x_i \in G_i$.

Suppose that Z_F is not connected. Select two vertices e and f (that is, two edges of F) from Z_F from different connected components of the zone graph, such that e and f are in G as close as possible. Clearly, e and f do not lie in a square. Let P_1 be a shortest (e_1, f_1)-path in G_1 and P_2 a shortest (e_2, f_2)-path in G_2. Since G_1 and G_2 are convex, P_1 and P_2 lie in G_1 and G_2, respectively. We claim that the cycle C consisting of e, f, and the paths P_1 and P_2 is convex.

Suppose C is not convex. Consider a shortest path P between vertices u and v of C that is not contained in C. Select u and v to be as close as possible, so that only the end points u and v of P lie on C.

Let us first assume that $u \in P_1$ and $v \in P_2$. Then there is an edge $g \in F$, $g \neq e, f$, that lies on P. We observe that C fully lies inside $I = I(e_1, f_2)$, which is convex. Hence P is also contained in I, thus g lies in I. In particular, $d(e, g) = d(e_1, g) < d(e_1, f) = d(e, f)$. By our choice of e and f this means that e and g lie in the same component of Z_F. Symmetrically, also f and g are in the same component, but then e and f are in the same component of Z_F, a contradiction.

Note that the above argument already implies that C is isometric.

Now assume that u and v are in the same G_i, say in G_1. The path P gives us a second geodesic P_1' from e_1 to f_1. Form a second cycle C' combining e, f, P_1', and P_2. By the above, C' is also isometric. If a and b are the first different edges (incident to u, say) on C and C', then these edges are opposite (in C and

C', respectively) to the same edge and hence they belong to the same Θ-class by Lemma 3.1, a contradiction.

We have proved that C is convex. Hence C is an edge of Z_F that connects components containing e and f, the final contradiction. $\square$

With Proposition 5.1 in hand we can now prove:

Lemma 5.2 *For a partial cube G with n vertices and m edges,*

$$2n - m - i(G) - ce(G) \leq 2\,.$$

Proof. The result clearly holds for the smallest partial cubes K_1 and K_2. Suppose the inequality holds for a partial cube G and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. Let F be the newly created Θ-class. By assumption we have $2n-m-i(G)-ce(G) \leq 2$. Let $G_0 = G_1 \cap G_2$, $n_0 = |V(G_0)|$, and $m_0 = |E(G_0)|$. Setting $\widetilde{n} = |V(\widetilde{G})|$ and $\widetilde{m} = |E(\widetilde{G})|$ we have

$$\widetilde{n} = n + n_0 \quad \text{and} \quad \widetilde{m} = m + n_0 + m_0\,.$$

From Lemma 3.2 we also know that

$$i(\widetilde{G}) = i(G) + 1\,.$$

By Proposition 4.2, any convex lateral cycle of G expands to an identical convex cycle in $\widetilde{G}$ and any convex intersection cycle of G expands to two convex cycles of the same length in $\widetilde{G}$. Since Z_F is connected by Proposition 5.1, $\widetilde{G}$ contains at least $t-1$ convex cross cycles (with respect to G_1, G_2) of length at least six, where t is the number of connected components of G_0. Let $\widetilde{C}$ be such a cycle of $\widetilde{G}$, that is, a cross cycle of length at least six, and let C be its contraction. If C is convex, then since $|\widetilde{C}| = |C|+2$, the contribution of $\widetilde{C}$ to $ce(\widetilde{G})$ is one more than the contribution of C to $ce(G)$. If C is not convex, the contribution of C to $ce(G)$ is zero, while the contribution of $\widetilde{C}$ to $ce(\widetilde{G})$ is at least one. Therefore,

$$ce(\widetilde{G}) \geq ce(G) + t - 1\,.$$

Note also that $m_0 \geq n_0 - t$. Having all these relations in mind we obtain:

$$\begin{aligned}
&2\widetilde{n} - \widetilde{m} - i(\widetilde{G}) - ce(\widetilde{G}) \\
&\leq 2(n + n_0) - (m + n_0 + m_0) - (i(G) + 1) - (ce(G) + t - 1) \\
&= (2n - m - i(G) - ce(G)) + (n_0 - m_0 - t) \\
&\leq 2 + (n_0 - (n_0 - t) - t) \\
&= 2\,.
\end{aligned}$$

$\square$

We remark that the above proof shows that for the equality to hold we must have the following three statements at the same time:

1. $2n - m - i(G) - ce(G) = 2$, that is, the contraction G satisfies the equality;
2. $m_0 = n_0 - t$, that is, G_0 must be a forest; and
3. $ce(\widetilde{G}) = ce(G) + t - 1$, that is, among the edges of the zone graph Z_F there are exactly $t - 1$ cycles of length at least six and, furthermore, every convex cycle of $\widetilde{G}$ contracts to a convex cycle in G.

Conditions 2 and 3 together imply that Z_F is a tree. Since this can be said about each Θ-class we conclude that for the equality to hold G must be a tree-zone partial cube. In Section 6 we establish that this is also sufficient.

Notice also that Condition 3 includes the following interesting property.

Corollary 5.3 *Let G be a partial cube that satisfies equality in* (2)*. Then a contraction of a convex cycle of G of length at least six is a convex cycle.*

Recall from the end of Section 4 that this property is not shared by all partial cubes.

6 Proof of the equality part

It remains to prove that when G is a tree-zone partial cube then (2) holds with equality. For this same we first state another general property of zone graphs.

Proposition 6.1 *The zone graphs of partial cubes have no multiple edges.*

Proof. Let G be a partial cube, F its Θ-class and let G_1, G_2 be the connected components of $G \backslash F$. Let $e, f \in F$ and suppose that Z_F contain two edges connecting e with f. Let C and C' be the corresponding convex cycles of G containing e and f. Clearly, C and C' are of length at least 6. Let $R_i = C \cap G_i$ and $R'_i = C' \cap G_i$, $i = 1, 2$. Since $C \neq C'$ we may without loss of generality assume that $R_1 \neq R'_1$. As G_1, C and C' are all convex, $|R_1| = |R'_1|$. But then neither C nor C' is convex, a contradiction. $\square$

We continue with some general properties of how zone graphs intersect.

Lemma 6.2 *Let F and F' be different Θ-classes of a partial cube G and let G_1, G_2 be the connected components of $G \setminus F$. Then $F' \cap G_1$ is either empty or a Θ-class of the partial cube G_1.*

Proof. Follows since G_1 is convex in G and hence all the distances can be computed within G_1. $\square$

Note that when we remove edges from F, then in the zone graph $Z_{F'}$ all vertices are preserved but some edges are removed, namely the edges of $Z_{F'}$ which correspond to the convex cycles of G sharing edges with F and F'. We call this operation the *F-splitting of $Z_{F'}$*.

Lemma 6.3 *Let F and F' be different Θ-classes of a partial cube G. Either the F-splitting of $Z_{F'}$ leaves $Z_{F'}$ unchanged or it splits it into exactly two connected components.*

Proof. Let G_1, G_2 be the connected components of $G \setminus F$. Suppose that the F-splitting does remove at least one convex cycle C, that is an edge of $Z_{F'}$. Then C is a cross cycle with respect to G_1, G_2, and hence the edges of F' can be found in both G_1 and G_2. Hence by Lemma 6.2, $F' \cap G_1$ is a Θ-class of G_1 and $F' \cap G_2$ is a Θ-class of G_2. The assertion now follows since $Z_{F' \cap G_1}$ and $Z_{F' \cap G_2}$ are connected graphs by Proposition 5.1. $\square$

For our proof the following observation is essential.

Lemma 6.4 *Suppose G is a tree-zone partial cube. Then for any two different Θ-classes F and F' there is at most one convex cycle such that it is an edge in both Z_F and $Z_{F'}$.*

Proof. Indeed, removing two edges from a tree leaves more than two connected components. $\square$

Corollary 6.5 *Suppose G is a tree-zone partial cube. Let C and C' be different convex cycles that are edges of Z_F. Then these cycles share no edges outside F.*

Proof. If C and C' share an edge e which is not in F, then setting F' to be the Θ-class containing e we would have a contradiction with Lemma 6.4. $\square$

Proposition 6.6 *Let G be a partial cube with n vertices and m edges. Then $2n - m - i(G) - ce(G) = 2$ if and only if G is a tree-zone partial cube.*

Proof. We have already shown that if for a partial cube the equality holds in (2) then it is a tree-zone partial cube.

Suppose now that G is a tree-zone partial cube. We are going to prove that $2n - m - i(G) - ce(G) = 2$ by induction on $i(G)$. The statement is clearly true for $i(G) \leq 2$.

Select an arbitrary Θ-class F of G and let G_1 and G_2 be the connected components of $G \setminus F$. Then G_1 and G_2 are partial cubes of lower isometric dimension and furthermore their zone graphs are subgraphs of the corresponding zone graphs of G (cf. Lemma 6.2) which are trees, hence the zone graphs of G_1 and G_2 are also trees. By induction we have $2n_1 - m_1 - i(G_1) - ce(G_1) = 2$ and $2n_2 - m_2 - i(G_2) - ce(G_2) = 2$, where n_1, n_2, m_1, m_2 are the number of vertices and edges of G_1 and G_2. Let G_{10} be the subgraph of G_1 induced on the vertices that have a neighbor in G_2 and let G_{20} be the corresponding (isomorphic) subgraph of G_2. Let $n_0 = |V(G_{10})| = |V(G_{20})|$.

Note that G_{10} is a forest because it is isomorphic to a subgraph of Z_F.

Let t be the number of connected components of G_{10} (and of G_{20}). Since Z_F is a tree, Proposition 6.1 implies that there exist exactly $t-1$ convex cycles $C^{(1)}, \ldots, C^{(t-1)}$ of length at least six that are edges of Z_F. Note that each of these cycles contains two edges of F while its other edges lie in G_1 or in G_2 but not in G_{10} or G_{20}. Clearly, we have:

$$n = n_1 + n_2 \quad \text{and} \quad m = m_1 + m_2 + n_0 \,.$$

Also

$$i(G) = 1 + i(G_1) + i(G_2) - (n_0 - 1) - \sum_{j=1}^{t-1} ce(C^{(j)}) \,.$$

Indeed, every Θ-class of G is either F, or is fully contained in one of the G_1 or G_2, or intersects both G_1 and G_2 in one of their Θ-classes (cf. Lemma 6.3). Furthermore, by Lemma 6.4 and Corollary 6.5 each of the latter classes corresponds to a unique pair of opposite edges on a convex cycle that is an edge of Z_F. Thus the number of such classes is equal to

$$\sum_{C \in E(Z_F)} \frac{|C| - 2}{2} = \sum_{C \in E(Z_F)} (ce(C) + 1) = n_0 - 1 + \sum_{j=1}^{t-1} ce(C^{(j)}) \,.$$

The last equality holds because the number of edges in Z_F is $n_0 - 1$ and $ce(C_4) = 0$, hence only the cycles of length more than four count.

Again, using that every convex cycle in G lies either fully in G_1, or fully in G_2, or is an edge in Z_F, we get that

$$ce(G) = ce(G_1) + ce(G_2) + \sum_{j=1}^{t-1} (ce(C^{(j)})) \,,$$

because in the sum the 4-cycles can be ignored. Using these equalities, we obtain for $u = 2n - m - i(G) - ce(G)$ the following:

$$\begin{aligned}
u &= 2(n_1 + n_2) - (m_1 + m_2 + n_0) \\
&\quad - \Big(1 + i(G_1) + i(G_2) - (n_0 - 1) - \sum_{j=1}^{t-1} ce(C^{(j)})\Big) \\
&\quad - \Big(ce(G_1) + ce(G_2) + \sum_{j=1}^{t-1} ce(C^{(j)})\Big) \\
&= (2n_1 - m_1 - i(G_1) - ce(G_1)) + (2n_2 - m_2 - i(G_2) - ce(G_2)) - 2 \\
&= 2 + 2 - 2 = 2 \,.
\end{aligned}$$

□

This concludes the proof of Theorem 2.1.

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