On Even and Harmonic-Even Partial Cubes

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Abstract

Fukuda and Handa [7] asked whether every even partial cube G is harmonic-even. It is shown that the answer is positive if the isometric dimension of G equals its diameter which is in turn true for partial cubes with isometric dimension at most 6. Under an additional technical condition it is proved that an even partial cube G is harmonic-even or has two adjacent vertices whose diametrical vertices are at distance at least 4. Some related open problems are posed.

Key words: Even graph; Harmonic-even graph; Partial cube; Isometric dimension

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1 Introduction

A connected graph is called *even* if, for any vertex $v \in V(G)$, there exists a unique vertex $\overline{v} \in V(G)$ such that $d(v, \overline{v}) = \mathrm{diam}(G)$, the diameter of

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G. We will call $\overline{v}$ the *diametrical vertex of* v. Clearly, hypercubes Q_d and even cycles are even graphs. It is also easy to observe that the Cartesian product of even graphs is again even. Q_d, $1 \leq d \leq 3$, C_6, C_8, $\overline{3K_2}$, and $\overline{4K_2}$ are the only even graphs on at most 8 vertices [8], where $\overline{G}$ denotes the complement of G.

Even graphs were first considered by Mulder under the name *diametrical graphs*. In [15] he proved that a graph G is Q_d if and only if G is a diametrical median graph of diameter d. Even graphs were further studied by Parthasarathy and Nandakumar in [16] under the name *self-centered unique eccentric point graphs*. This class of graphs has been named even graphs by Göbel and Veldman [8]; afterwards the name "even" seems to be accepted.

For $u, v \in V(G)$, let $d_G(u, v)$, or $d(u, v)$ for short, denote the length of a shortest path in G from u to v. A subgraph H of a graph G is an *isometric* subgraph if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. The *d-cube* Q_d is the graph with $V(Q_d) = \{0, 1\}^d$, vertices $u_1u_2 \ldots u_d$ and $v_1v_2 \ldots v_d$ being adjacent whenever they differ in precisely one coordinate. The class of *partial cubes* consists of all isometric subgraphs of hypercubes. This class of graphs has been extensively investigated, see [5, 11, 17] and recent references [1–4,6,13]. The *isometric dimension* $\mathrm{idim}(G)$ of a partial cube G is the least d for which G is an isometric subgraph of Q_d.

Even graphs play an important role in the theory of oriented matroids and partial cubes. Call an even graph *harmonic-even* if $\overline{u}\,\overline{v} \in E(G)$ whenever $uv \in E(G)$ for all $u, v \in V(G)$. Then Fukuda and Handa proved in [7] that a graph G is the tope graph of an acycloid if and only if G is a harmonic-even partial cube. Moreover, G is the tope graph of an oriented matroid of rank at most 3 if and only if G is a harmonic-even planar partial cube. In this context we wish to add that $P(10, 3)$ is the only (nontrivial) generalized Petersen graph that is a tope graph of an acycloid, see [13].

Fukuda and Handa finish their paper [7] with the following question: is every even partial cube harmonic-even? Mulder [15, Corollary 5] showed that the only harmonic-even graphs among median graphs are hypercubes, a fact that also easily follows from [11, Lemma 2.41]. Hence in this case the answer is clearly affirmative. However, the variety of harmonic-even partial cubes is much richer and also seems elusive.

In the next section we introduce the notations and concepts needed later. In Section 3 we observe that every even partial cube with equal isometric dimension and diameter is harmonic-even. Then we prove that if G is an even partial cube fulfilling an additional technical condition then either G is harmonic-even or there is an edge uv of G such that $d(\overline{u}, \overline{v}) \geq 4$. In Section 4 we use the approach using isometric dimension and diameter to prove that even partial cubes of isometric dimension at most 6 are harmonic-even. We conclude the paper with three related problems.

2 Preliminaries

The *Cartesian product* $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \,\square\, H)$ whenever either $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. It is well-known that the d-cube Q_d can be represented as the Cartesian product of d copies of the complete graph on two vertices K_2.

Let G be a connected graph. Then for any edge ab of G we write

$$\begin{aligned} W_{ab} &= \{w \in V(G) \mid d_G(a,w) < d_G(b,w)\}, \\ U_{ab} &= \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}, \\ F_{ab} &= \{e \in E(G) \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}. \end{aligned}$$

Note that if G is bipartite then for any edge ab, $V = W_{ab} \cup W_{ba}$. Djoković [5] characterized partial cubes as the connected bipartite graphs in which all subgraphs W_{ab} are convex.

The Djoković-Winkler relation Θ [5, 17] is defined on the edge set of a graph in the following way. Edges $e = xy$ and $f = uv$ of a graph G are in relation Θ if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)\,.$$

Winkler [17] proved that among bipartite graphs Θ is transitive precisely for partial cubes. The following lemma first explicitly stated in [10], see also [11], will be used several times.

Lemma 2.1 *Let G be a bipartite graph and $e = uv$, $f = xy$ be two edges of G with $e\Theta f$. Then the notation can be chosen such that*

$$d(u,x) = d(v,y) = d(u,y) - 1 = d(v,x) - 1.$$

Let G be a partial cube and $ab \in E(G)$. Then the Θ-equivalence class of ab coincides with the set F_{ab}. Hence the notation of Lemma 2.1 is selected in such a way that $x \in U_{uv}$ and $y \in U_{vu}$.

Göbel and Veldman proved in [8] the following connectivity result for even graphs.

Theorem 2.2 *Let G be an even graph on at least three vertices. Then G is 2-connected. Moreover, if the smallest degree of G is at least 3, then G is 3-connected.*

Handa [9] followed with a related result asserting that a harmonic-even partial cube G with more than two vertices is 3-connected unless G is an even cycle. Finally, for later use we recall from [8] the following useful fact.

Proposition 2.3 *If u and v are adjacent vertices of an even graph G of diameter d, then $d(v,\overline{u}) = d - 1$.*

3 Two sufficient conditions

We begin with the following observation.

Proposition 3.1 *Let G be an even partial cube with* $\mathrm{idim}(G) = \mathrm{diam}(G)$. *Then G is harmonic-even.*

Proof. Let G be an even partial cube with $\mathrm{idim}(G) = d$ and consider G as an isometric subgraph of Q_d. Let uv be an arbitrary edge of G, then we need to show that $\overline{u}\,\overline{v} \in E(G)$. As hypercubes are vertex-transitive, we can assume that $u = 00\ldots0$. Since $\mathrm{diam}(G) = d$, we have $\overline{u} = 11\ldots1$. As v is adjacent to u it has exactly one coordinate equal 1, and hence $\overline{v}$ must have exactly $k-1$ coordinates equal 1. We conclude that $\overline{v}$ is adjacent to $\overline{u}$. $\square$

We know of no even partial cube G for which $\mathrm{idim}(G) > \mathrm{diam}(G)$. Hence Proposition 3.1 shows a possible way to attack the general case of the Fukuda-Handa question. We will further explore this idea in the next section. Here we follow with another result which shows that under an additional technical condition an even partial cube is either harmonic-even or it is "far away" from being such. For this sake we need the following definition.

Let ab be an edge of a partial cube G. Let us call the set U_{ab} *even* if for any vertex u of U_{ab} there exists a unique vertex v of U_{ab} such that $d(u,v) = \max_{w \in U_{ab}} d(u,w)$.

Theorem 3.2 *Let G be an even partial cube with even U_{ab}'s. Then either G is harmonic-even or there is an edge uv of G such that $d(\overline{u},\overline{v}) \geq 4$.*

Proof. Suppose that G is even but not harmonic-even and let $k = \mathrm{diam}(G)$. Then for some edge uv of G we have $d(\overline{u},\overline{v}) \geq 2$.

Suppose $\overline{u} \in U_{vu}$. Let $w \in U_{uv}$ be such that $d(\overline{u}, w) = 1$. Then $d(u,w) = k-1$ by Proposition 2.3, and hence $d(v,w) = k$ by Lemma 2.1. Therefore $\overline{v} = w$, a contradiction with the assumption $d(\overline{u},\overline{v}) \geq 2$.

Hence $\overline{u} \in W_{vu} \backslash U_{vu}$ and by the symmetry, $\overline{v} \in W_{uv} \backslash U_{uv}$. If $d(\overline{u}, U_{vu}) \geq 2$ or $d(\overline{v}, U_{uv}) \geq 2$ we have $d(\overline{u},\overline{v}) \geq 4$. Therefore $d(\overline{u}, U_{vu}) = d(\overline{v}, U_{uv}) = 1$.

Let x be a neighbor of $\overline{u}$ in U_{vu} and let y be the neighbor of x in U_{uv}. By Proposition 2.3, $d(v,\overline{u}) = k-1$. Moreover, as G is bipartite, and x is not the diametrical vertex of v, we have $d(v,x) = k-2$ $(= d(u,y))$. If for some vertex w of U_{uv} we would have $d(u,w) = k-1$, the neighbor of w in U_{vu} would be at distance k from u which is not possible as $\overline{u} \notin U_{vu}$. It follows, having in mind that U_{uv} is even, that y is the unique vertex from U_{uv} at distance $k-2$ from u. Analogously, x is the unique vertex in U_{vu} with $d(v,x) = k-2$.

Consider next $\overline{v}$. We know already that $\overline{v} \in W_{uv} \setminus U_{uv}$ and let w be a neighbor of $\overline{v}$ in U_{uv}. Since $d(u, \overline{v}) = k - 1$, we have $d(u, w) = k - 2$. Since U_{uv} is even it follows that $w = y$, that is, $\overline{v}$ is adjacent to y.

Since U_{uv} and U_{vu} are both even, $\overline{x} \in W_{uv} \setminus U_{uv}$ and $\overline{y} \in W_{vu} \setminus U_{vu}$. Moreover, $d(\overline{x}, U_{uv}) = 1$ and $d(\overline{y}, U_{vu}) = 1$ for otherwise we are done by considering the edges xy and $\overline{x}\,\overline{y}$. By an analogous argument as above we infer that $\overline{x}$ is adjacent to u and $\overline{y}$ to v, see Fig. 1.

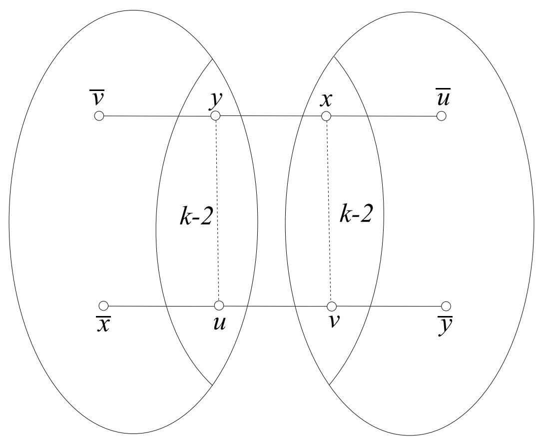

Figure 1: Diametrical vertices from the proof

Observe next that $d(\overline{x}, \overline{v}) = k-2$. Clearly, $d(\overline{x}, \overline{v}) \geq k-2$ since $d(v, \overline{v}) = k$. Now, $d(\overline{x}, \overline{v})$ cannot be k, for otherwise $\overline{x}$ would have two diametrical vertices and it is also not $k - 1$ because G is bipartite. Analogously we conclude that $d(\overline{u}, \overline{y}) = k-2$. It follows that $\overline{x}u\Theta\overline{v}y$ and $\overline{y}v\Theta\overline{u}x$. Moreover, $d(\overline{x}, x) = d(\overline{u}, u) = k$ and by Proposition 2.3, $d(u, x) = d(\overline{u}, \overline{x}) = k - 1$. Therefore Lemma 2.1 implies $\overline{x}u\Theta\overline{u}x$. Transitivity of Θ thus implies $\overline{v}y\Theta\overline{u}x$. But then $\overline{v}$ must be adjacent to $\overline{u}$, the final contradiction. $\square$

In view of Theorem 3.2 we ask the following question: given an even partial cube G, is it always the case that the U_{ab}'s are even?

Note that if a partial cube G is harmonic-even, then $\overline{a} \in U_{ba}$, for otherwise $\overline{a}$ would not be adjacent to $\overline{b}$. Hence $\overline{b}$ is a unique vertex in U_{ab} such that $d(a, \overline{b}) = \mathrm{diam}(G) - 1$. So harmonic-even partial cubes have even U_{ab}'s.

4 Even partial cubes of small dimension

In this section we use the idea of Proposition 3.1 to show that even partial cubes of small isometric dimension are harmonic-even.

Theorem 4.1 *Let G be an even partial cube with* $\mathrm{idim}(G) \leq 6$*. Then G is harmonic-even.*

Proof. We will apply Proposition 3.1. More precisely, let G be an even partial cube with $\mathrm{idim}(G) = k$, $k \leq 6$. Then we will prove that $\mathrm{diam}(G) = k$.

It is easy to verify that the only even partial cubes of idim at most 3 are Q_1, Q_2, Q_3, and C_6. So let $4 \leq k \leq 6$ and let uv be an arbitrary edge of G. As hypercubes are vertex-transitive, we can assume that $u = 00\ldots0$. Suppose on the contrary that $\mathrm{diam}(G) < k$, then $\overline{u} \neq 11\ldots1$. We may assume that the last coordinate of $\overline{u}$ equals 0. We will show that then in all cases any vertex of G has the last coordinate 0 as well. But then G could be isometrically embedded into Q_{k-1} just by removing the last coordinates of vertices, hence $\mathrm{idim}(G)$ would be at most $k-1$.

For a vertex w of G let $s(w)$ be the number of 1s in w. We have assumed that $\overline{u}$ contains at least one 0 and we may assume that $\overline{u} = 1\ldots10\ldots0$. Note that $\mathrm{diam}(G) = s(\overline{u})$.

Case 1. $k = 4$.
The cases $s(\overline{u}) = 1$ and $s(\overline{u}) = 2$ are easily seen not to be possible. Indeed, suppose $s(\overline{u}) = 2$ and let without loss of generality $\overline{u} = 1100$. Let w be an arbitrary vertex of G with the last coordinate 1. (Such a vertex exists since $\mathrm{idim}(G) = 4$.) From $d(u, \overline{u}) = 2$ it follows $d(w, u) = 1$. But then $w = 0001$ and thus $d(w, \overline{u}) = 3$, a contradiction.

So let $s(\overline{u}) = 3$, that is, $\overline{u} = 1110$. Suppose $w = w_1w_2w_31 \in G$. Then $w \neq 0001, 0011, 0101, 1001$ since any of these vertices is at distance at least 3 from $\overline{u}$. If $s(w) = 3$, then $d(u, w) \geq 3$, which is again not possible. Hence the last coordinate of an arbitrary vertex of G is 0. Therefore, $\mathrm{idim}(G) \leq 3$, a contradiction.

Case 2. $k = 5$.
Suppose that $w = w_1w_2w_3w_41 \in G$. Let $\overline{u} = 11100$. Then if $s(w) \geq 3$ we have $d(w, u) \geq 3$, and if $s(w) = 2$, then $d(w, \overline{u}) \geq 3$. It follows that $\overline{u} = 11110$. If $s(w) = 2$ then $d(w, \overline{u}) = 4$. Suppose next $s(w) = 3$ and assume without loss of generality that $w = 00111$. Then 00110 is the unique neighbor of w in G, since for any other possible neighbor y of w we infer $d(y, \overline{u}) = 4$. So the degree of w is 1, which is not possible as G is 2-connected by Theorem 2.2. Finally, the case $s(w) = 4$ is also impossible as then $d(w, u) = 4$.

Case 3. $k = 6$.
Suppose $w = w_1w_2w_3w_4w_51 \in G$. If $\overline{u} = 111000$, then $s(w) \leq 2$. However, $d(w, \overline{u}) \geq 3$ in this case. Suppose next $\overline{u} = 111100$. Then $s(w) \leq 3$. If $s(w) = 2$, then $d(w, \overline{u}) \geq 4$. Let $s(w) = 3$. If $w_5 = 1$, then $d(w, \overline{u}) = 5$. Let $w_5 = 0$ and without loss of generality consider the vertex $w = 110001$. Then the only possible neighbor of w is 110000, but then G is not 2-connected.

Suppose finally $\overline{u} = 111110$. Then $s(w) \leq 4$. Moreover, $s(w) = 2$ is not possible for otherwise $d(w, \overline{u}) = 5$. Assume $s(w) = 4$ and assume without loss of generality that $w = 001111$. Vertex $\overline{u}$ has at most 5 neighbors in G: $a^{(1)} = 011110$, $a^{(2)} = 101110$, $a^{(3)} = 110110$, $a^{(4)} = 111010$, and $a^{(5)} = 111100$.

Because $d(w, \overline{u}) = 3$ it follows that $b = 001110 \in V(G)$. Indeed, the other two possible neighbors of w that are at distance 2 from $\overline{u}$ are both at distance 5 from u. Moreover, at least one of the vertices $a^{(1)} = 011110$ and $a^{(2)} = 101110$ must belong to G.

Since G is 2-connected at least one of three possible neighbors of w $x^{(1)} = 000111$, $x^{(2)} = 001011$, $x^{(3)} = 001101$ belongs to G. Next, $d(u, x^{(1)}) = d(u, x^{(2)}) = d(u, x^{(3)}) = 3$. Since any neighbor of $x^{(1)}$, $x^{(2)}$, or $x^{(3)}$ on a shortest $x^{(i)}, u$-path has the last coordinate equal 0 (otherwise its distance to $\overline{u}$ is 5), at least one of the vertices $y^{(1)} = 000110$, $y^{(2)} = 001010$, and $y^{(3)} = 001100$ belongs to G.

Assume first that $x^{(1)} \in V(G)$. Then also $y^{(1)} \in V(G)$. In Table 1 we have collected all possible diametrical vertices for $x^{(1)}$ and $y^{(1)}$. Next to each of the possible diametrical vertices we give a vertex (if such a vertex exists) at distance 5 from it or a vertex at distance 6 from it.

$x^{(1)} = 000111$		$y^{(1)} = 000110$	
011000		011001	$a^{(2)}, a^{(3)}$
101000		101001	$a^{(1)}, a^{(3)}$
110000	w	110001	w
$111100 = a^{(5)}$	$x^{(2)}$	111101	u
$111010 = a^{(4)}$	$x^{(3)}$	111011	u
111001	$y^{(1)}$	111000	$x^{(1)}$

Table 1: Vertices $x^{(1)}, y^{(1)}$ with all possible diametrical vertices

From Table 1 we infer that only one of the top two vertices can be the diametrical vertex of $y^{(1)}$. The vertex 011001 is the diametrical vertex of $a^{(2)}$ and $a^{(3)}$. The vertex 101001 is the diametrical vertex of $a^{(1)}$ and $a^{(3)}$. Hence $a^{(3)} \notin V(G)$, for otherwise 101001 or 011001 would be the diametrical vertex of $y^{(1)}$ and $a^{(3)}$. Similarly, only one of the vertices $a^{(1)}$ and $a^{(2)}$ belongs to G.

Assume first that $a^{(1)}$ belongs to G. It follows that $\overline{y^{(1)}} = 011001$. Moreover since 011001 is at distance 3 from u (and we already know that the vertices 010001 and 001001 do not belong to G) the vertex 011000 belongs to G as well since $d(011001, u) = 3$. Hence we conclude that $\overline{x^{(1)}} = 011000$ and vertices $a^{(4)}$ and $a^{(5)}$ do not belong to G since their distance to $x^{(1)}$ is 5. But this means that the only neighbor of $\overline{u}$ is $a^{(1)}$ a contradiction with Theorem 2.2. The subcase when $a^{(2)}$ belongs to G is treated similarly, only the roles of $a^{(3)}$, $a^{(4)}$, and $a^{(5)}$ are interchanged.

The cases when $x^{(2)} \in V(G)$ or $x^{(3)} \in V(G)$ are done analogously as the case when $x^{(1)} \in V(G)$, and are left to the reader.

Suppose at the very end that $s(w) = 3$. We may assume $w = 000111$. Among the vertices with four zeros the only possible neighbor of w is the vertex 000110. Since by the above w has no neighbor with four 1s, we conclude that the degree of w is 1, and so G is not 2-connected. Hence $\mathrm{idim}(G) \leq 5$, the final contradiction. $\square$

Note that every even partial cube with idim(G) ≥ 7 has diameter at least 5. Hence Theorem 4.1 implies that a possible even partial cube that is not harmonic-even has diameter at least 5.

5 Three problems

Call a graph G *distance-balanced* if $|W_{ab}| = |W_{ba}|$ holds for any edge ab of G, see [12, 14]. Handa [9] observed that harmonic-even graphs are distance-balanced. The converse is not always true, even restricted to partial cubes, as an example from [9, Fig.2] asserts. It is a distance-balanced partial cube with idim=5 that is not even. So the following question naturally appears.

Problem 5.1 *Characterize even distance-balanced partial cubes.*

All regular partial cubes from [1, 3] that are also distance-balanced are harmonic-even.

Problem 5.2 *Is every regular distance-balanced partial cube harmonic-even?*

Let G be a harmonic-even partial cube. Then $u \in W_{ab}$ if and only if $\overline{u} \in W_{ba}$. Let α be an automorphism of G defined with $\alpha(u) = \overline{u}$. Since G is harmonic-even, α induces an isomorphism between the graphs induced by W_{ab} and W_{ba}. Is the converse true as well? That is:

Problem 5.3 *Is G a harmonic-even partial cube if and only if for every $ab \in E(G)$, W_{ab} is isomorphic to W_{ba}?*

References

[1] C. P. Bonnington, S. Klavžar, and A. Lipovec, On cubic and edge-critical isometric subgraphs of hypercubes, Australas. J. Combin. 28 (2003) 217–224.

[2] B. Brešar and S. Klavžar, Θ-graceful labelings of partial cubes, Discrete Math. 306 (2006) 1264–1271.

[3] B. Brešar, S. Klavžar, A. Lipovec, and B. Mohar, Cubic inflation, mirror graphs, regular maps, and partial cubes, European J. Combin. 25 (2004) 55–64.

[4] M. Deza, M. Dutour-Sikiric, and S. Shpectorov, Graphs 4_n that are isometrically embeddable in hypercubes, Southeast Asian Bull. Math. 29 (2005) 469–484.

[5] D. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.

[6] D. Eppstein, The lattice dimension of a graph, European J. Combin. 26 (2005) 585–592.

[7] K. Fukuda and K. Handa, Antipodal graphs and oriented matroids, Discrete Math. 111 (1993) 245–256.

[8] F. Göbel and H. J. Veldman, Even graphs, J. Graph Theory 10 (1986) 225–239.

[9] K. Handa, Bipartite graphs with balanced (a, b)-partitions, Ars Combin. 51 (1999) 113–119.

[10] W. Imrich and S. Klavžar, A simple $O(mn)$ algorithm for recognizing Hamming graphs. Bull. Inst. Combin. Appl. 9 (1993) 45–56.

[11] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.

[12] J. Jerebic, S. Klavžar, and D.F. Rall, Distance-balanced graphs, manuscript, 2005.

[13] S. Klavžar and A. Lipovec, Partial cubes as subdivision graphs and as generalized Petersen graphs, Discrete Math. 263 (2003) 157–165.

[14] K. Kutnar, A. Malnič, D. Marušič, and Š. Miklavič, Distance-balanced graphs: symmetry conditions, manuscript, 2005.

[15] H. M. Mulder, n-cubes and median graphs, J. Graph Theory 4 (1980) 107–110.

[16] K. R. Parthasarathy and R. Nandakumar, Unique eccentric point graphs, Discrete Math. 46 (1983) 69–74.

[17] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.