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Hanoi graphs and some classical numbers

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Abstract

The Hanoi graphs H_p^n model the p-pegs n-discs Tower of Hanoi problem(s). It was previously known that Stirling numbers of the second kind and Stern's diatomic sequence appear naturally in the graphs H_p^n. In this note, second-order Eulerian numbers and Lah numbers are added to this list. Considering a variant of the p-pegs n-discs problem, Catalan numbers are also encountered.

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1. Introduction

All the time since the French number theorist Lucas invented the Tower of Hanoi (TH) in 1883, the puzzle has presented a challenge in mathematics as well as in computer science and psychology. The classical problem consists of three pegs and is by now well-understood (cf. [5,7] and references therein). On the other hand, as soon as there are at least four pegs, the problem turns into a notorious open question (see [1,10,12,21] for recent results).

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The TH problem consists of $p \geqslant 3$ pegs and $n \geqslant 1$ discs of different sizes. A *legal move* is a transfer of the topmost disc from one peg to another peg such that no disc is moved onto a smaller one. Initially, all discs lie on one peg in small-on-large ordering, that is, in a *perfect state*. The objective is to transfer all the discs from one perfect state to another in the minimum number of legal moves. A state (= distribution of discs on pegs) is called *regular* if on every peg the discs lie in the small-on-large ordering.

The natural mathematical object to model the TH problem(s) are the graphs defined as follows. Let $p \geqslant 1$, $n \geqslant 1$, then the *Hanoi graph* H_p^n has all regular states of the p-pegs n-discs problem, two states being adjacent whenever one is obtained from the other by a legal move. Note that for any $n \geqslant 1$, H_1^n is the one-vertex graph K_1. If we have two pegs, only the smallest disc can be moved in any regular state. Hence H_2^n is the disjoint union of 2^{n-1} copies of the complete graph on two vertices K_2.

The graphs H_p^n were studied from the graph theory point of view in [9] where it is proved that they are Hamiltonian and that the only planar Hanoi graphs, besides the trivial cases H_1^n and H_2^n, are H_4^1, H_4^2, and H_3^n, $n \geqslant 1$. In this note we study these graphs from the combinatorial counting point of view. More precisely, we demonstrate that many classical numbers appear naturally in the Hanoi graphs.

For a graph G let $V(G)$ be the set of its vertices and $E(G)$ the set of its edges. It is easy to see that $|V(H_p^n)| = p^n$ and it has been shown in [11] that for $p \geqslant 3$,

$$|E(H_p^n)| = \frac{1}{2}\sum_{k=1}^{p}\left(k\left(p-\frac{1}{2}\right)-\frac{1}{2}k^2\right)\left\{ n \atop k \right\} p^{\underline{k}},$$

where $\left\{ n \atop k \right\}$ denote Stirling numbers of the second kind and $p^{\underline{k}} = p(p-1)\cdots(p-k+1)$. Moreover, it is straightforward to verify that the formula holds for $p = 1$ and $p = 2$ as well. So Stirling numbers of the second kind present the first instance of classical numbers that appear in Hanoi graphs.

In the next section we count some vertex subsets of Hanoi graphs and among others detect the second-order Euler numbers and the Lah numbers. Then, in Section 3, we consider the n-in-a-row TH problem and relate it with the Catalan numbers. We conclude by recalling that Stern's diatomic sequence appears in the Hanoi graphs as well.

2. Vertex subsets of Hanoi graphs

Consider the TH problem with $p \geqslant 2$ pegs numbered $1, 2, \ldots, p$ and $n \geqslant 1$ discs numbered $1, 2, \ldots, n$. We assume that the discs are ordered by size, disc 1 being the smallest one. Suppose that on peg i of a regular state we have discs $j, j+1, \ldots, j+k$ for some $j \geqslant 1$, and $k \geqslant 0$. (Of course, $j+k \leqslant n$.) Then we say that the discs $j, j+1, \ldots, j+k$ form a *superdisc* (on peg i). In addition, we also consider an empty peg as an (empty) superdisc. We call a regular state of the TH problem a *superdisc state* if the discs on every peg form a superdisc. A vertex of H_p^n that corresponds to a superdisc state will be called a *superdisc vertex*.

The term "superdisc" has been coined by Hinz in [6]. In the same paper, the term "presumed minimum solution" was also invented to describe proposed solutions for the

TH problem with more than three pegs. Frame [2] and Stewart [19] were the first to propose such solutions and there are many equivalent approaches to these solutions (cf. [12]). For our point of view it is important to note that a presumed minimum solution of the multi-peg TH problem can be realized by considering some special superdisc states only. Hence, the mentioned notorious open question about the optimality of the presumed minimum solutions can be seen as a problem of understanding superdisc vertices of H_p^n.

Proposition 1. *The number of superdisc vertices of* H_p^n, $n \geqslant 1$, $p \geqslant 1$, *equals*

$$\sum_{k=1}^{p} \binom{n-1}{k-1} p^{\underline{k}}.$$

Proof. Let k be the number of nonempty pegs in a superdisc state, then $1 \leqslant k \leqslant p$. Clearly, we can select k such pegs in $\binom{p}{k}$ ways. Once a selection of k pegs is made, we need to distribute n discs among them. Since on every peg we have at least one disc and since we only have superdiscs, such a distribution is uniquely determined by selecting the smallest disc for each peg. As the disc 1 is always one of them, we need to select $k-1$ discs among $n-1$ discs. After the selection of the top discs, any permutation of them gives a different superdisc state. So the total number of superdisc vertices equals $\sum_{k=1}^{p} \binom{n-1}{k-1} \binom{p}{k} k!$ and the result follows. $\square$

Consider the (very special) graphs H_2^n, $n \geqslant 1$. We have already observed that H_2^n is the disjoint union of 2^{n-1} copies of K_2. By Proposition 1, the number of superdisc vertices of H_2^n equals $2n$. Since there are 2^n regular states, the number of non-superdisc vertices of H_2^n equals $2^n - 2n$. These numbers are known as the *second-order Eulerian numbers* [3], see `A005803` in [18] or M 1838 in [17].

For H_p^n, $n \geqslant 2$, $p \geqslant 3$, the (presumed) minimum solution is based on superdisc states for H_{p-2}^{n-1}. This situation is crucial in the main open problem in the area—the optimality of the four pegs problem. Suppose we are given an optimal solution of the TH problem with $p = 4$ and consider the state in which the largest disc is moved (for the first time). Then we have the following situation. On the source peg lies only the largest disc, the destination peg is empty and the smallest $n-1$ discs are distributed among 2 auxiliary pegs. Hence to deal with this situation we need to understand the related (non-)superdisc states on two pegs only.

We next consider the graphs H_n^n. By $d(u, v)$ we denote the standard shortest path distance between vertices u and v of a given graph, while $I(u, v)$ denotes the interval between u and v, that is, the set of all vertices x such that $\mathrm{d}(u, v) = d(u, x) + d(x, v)$.

Proposition 2. *Let u and v be different perfect states of* H_n^n, $n \geqslant 3$. *Then*

$$|\{x \in H_n^n \mid x \in I(u, v),\ d(x, u) = n\}| = \binom{n-1}{2} (n-2)!.$$

Proof. Let P be a shortest u,v-path in H_n^n. Before we move disc n, at least n moves are required, because we first need to remove discs $1, 2, \ldots, n-1$ from the source peg and also make the destination peg empty. It follows that the length of P is at least $2n+1$. On the other hand, it is straightforward to find a u, v-path of length $2n+1$, hence $d(u, v) = 2n+1$.

Let x be a vertex with $x \in I(u, v)$ and $d(x, u) = n$. Then, by the above, x corresponds to a state in which only the largest disc lies on the source peg, the destination peg is empty and the smallest $n-1$ discs are distributed among $n-2$ auxiliary pegs.

In order to distribute $n-1$ discs among $n-2$ auxiliary pegs in precisely n moves, each of the discs has to move once and one disc has to move a second time (onto another one). The latter disc can therefore only be one of the $n-2$ smallest. For these, there are $(n-2)!$ permutations on the $n-2$ auxiliary pegs. Disc 1 can lie on $n-2$ larger ones, disc 2 on $n-3$ and so on, such that we have

$$\sum_{\ell=1}^{n-2} \ell = \frac{(n-2)(n-1)}{2}$$

alternatives; together we get $\frac{1}{2}(n-2)(n-1)!$. □

The numbers from Proposition 2 are known as the *Lah numbers*; see the sequence A001286 in [18] or M 4225 in [17].

3. Catalan numbers and the adjacent TH problem

Among several variations of the TH problem, one finds the variant in which the discs can only be moved to adjacent pegs, where the pegs are assumed to be linearly ordered. Suppose we have p pegs and we wish to transfer discs from the first to the last peg. Then the variant is called the *p-in-a-row TH problem*. The 3-in-a-row TH problem was first mentioned in [16] and turns out to be a rather trivial puzzle (cf. [20]). Already for four pegs, not really surprisingly, the problem becomes much more involved, (see [20]). Guan [4] established a connection between the 3-in-a-row TH problem with Grey codes, while Sapir [15] obtained optimal algorithms for several restricted versions of the classical TH problem, the 3-in-a-row being one of the cases treated.

Consider the p-in-a-row TH problem with 2 discs: we wish to transfer the two discs from peg 1 to peg p. Suppose that the largest disc is on peg i, and the other disc on peg j; let the ordered pair (i, j) describe this state. Clearly, to move a disc from peg 1 to peg p we need at least $p-1$ moves. Moreover, during the process we must arrive at a state $(i, i+1)$, $2 \leqslant i \leqslant p-1$, such that this state is proceeded with the states (i, i) and $(i, i-1)$. Therefore, the smallest disc needs at least 4 additional moves, so at least $2p+2$ moves are needed. As this is indeed possible, we conclude that the optimal number of moves is $2p+2$.

We next pose a more interesting question: how many optimal solutions exist? More precisely, in how many different ways we can move two discs from peg 1 to peg p in the p-in-a-row TH problem. For this sake, let $A(p, n)$ denote the number of optimal solutions of the p-in-a-row TH problem with n discs, $p \geqslant 3$ and $n \geqslant 1$. Clearly, $A(p, 1) = 1$, for any p.

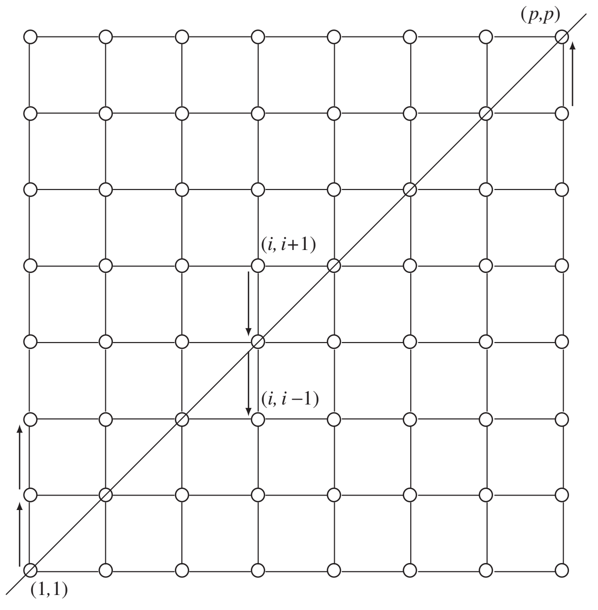

Fig. 1. An optimal solution of the 8-in-a-row TH problem with 2 discs.

Recall that the *Catalan numbers* C_m, $m \geqslant 1$, are defined as $C_m = \frac{1}{m+1}\binom{2m}{m}$ [3]. Now we have:

Theorem 3. *For any* $p \geqslant 3$, $A(p, 2) = C_p - 2C_{p-1}$.

Proof. Consider the states of the p-in-a-row TH problem with 2 discs as the points (i, j), $1 \leqslant i, j \leqslant p$, of the integer grid $\mathbb{Z} \times \mathbb{Z}$. Then, its optimal solutions correspond to the paths in the grid between the points $(1, 1)$ and (p, p) that meet the diagonal points (i, i) for precisely one $i \neq 1, p$. In addition, in such a path P we can only move from (j, k), for any $j \neq i$, to either $(j + 1, k)$ or $(j, k + 1)$, and it is of the form

$$(1, 1) \to (1, 2) \to \cdots \to (i, i + 1) \to (i, i) \to (i, i - 1) \to \cdots \to \times (p, p - 1) \to (p, p)$$

(see Fig. 1).

For P to be optimal, the $P_1 = (1, 2) \to \cdots \to (i, i + 1)$ subpath of P must be the shortest possible, as well as the subpath $P_2 = (i, i - 1) \to \cdots \to (p, p - 1)$. Hence, P_1 must remain inside the triangle T_1 and P_2 inside the triangle T_2 as shown in Fig. 2.

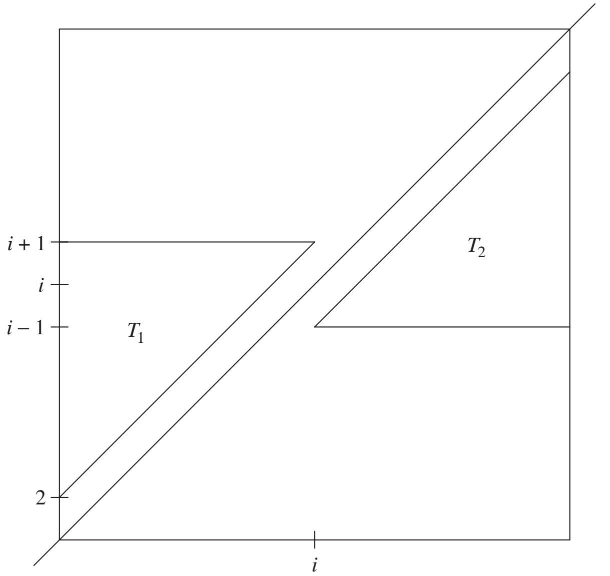

Fig. 2. Triangles T_1 and T_2.

By the well-known interpretation of the Catalan numbers (cf. [14, Exercise 3 on p. 320]), there are C_{i-1} possibilities for P_1 and C_{p-i} possibilities for P_2. Thus,

$$A(p,2) = \sum_{i=2}^{p-1} C_{i-1}C_{p-i} = \sum_{i=1}^{p-2} C_i C_{p-i-1}.$$

Set $C_0 = 1$ and recall [14, p. 318] that $C_p = \sum_{i=0}^{p-1} C_i C_{p-i-1}$. Hence,

$$\begin{aligned} C_p &= \sum_{i=0}^{p-1} C_i C_{p-i-1} = 2C_{p-1} + \sum_{i=1}^{p-2} C_i C_{p-i-1} \\ &= 2C_{p-1} + A(p,2) \end{aligned}$$

and the theorem is proved. □

We next give another interpretation of the numbers $A(p,2)$. Following [13], we introduce the next concepts. A trail P in the plane is called a *mountain range* if from any nonterminal point (x, y) of P we continue to either $(x+1, y+1)$ or to $(x+1, y-1)$ (cf. Fig. 3).

Let m be a positive integer and let $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, where $0 \leqslant x, y \leqslant m$. Let $M_m(x, y)$ be the number of mountain ranges P between (x, y) and $(2m, 0)$ such that for any $(x', y') \in P$ we have $y' \geqslant 0$. Then it is well known that $M_m(0,0) = C_m$ (see [13, pp. 96–98]).

Theorem 4. *For any* $p \geqslant 3$, $A(p,2) = M_p(3,3)$.

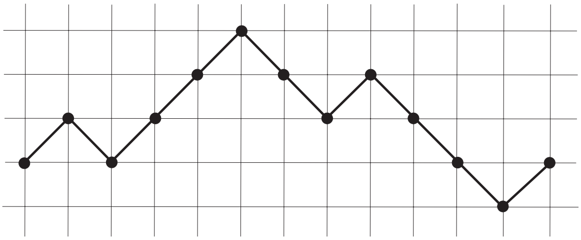

Fig. 3. A mountain range.

Proof. Clearly, $A(3,2) = M_3(3,3) = 1$. For $p \geqslant 4$ we proceed as follows. By Theorem 3,

$$A(p,2) = \frac{1}{p+1}\binom{2p}{p} - \frac{2}{p}\binom{2p-2}{p-1},$$

and from [13, Lemma 3.18] we deduce

$$M_p(3,3) = \binom{2p-3}{p-3} - \binom{2p-3}{p-4} = \binom{2p-3}{p} - \binom{2p-3}{p+1}.$$

An elementary calculation now shows that for any $p \geqslant 2$,

$$\frac{1}{p+1}\binom{2p}{p} - \frac{2}{p}\binom{2p-2}{p-1} - \binom{2p-3}{p} + \binom{2p-3}{p+1} = 0,$$

which completes the argument. □

4. Concluding remarks

Yet some other well-known numbers appear in the Hanoi graphs in a natural way. The graph H_3^n consists of three isomorphic copies of the graph H_3^{n-1}, where the copies are induced by the regular states in which the largest disc is on a fixed peg. These copies are joined by the three edges that correspond to the moves of the largest disc. Call these subgraphs A, B, and C for the time being. Let x be a vertex of H_3^n belonging to A and consider the set of all vertices y from B and C for which there exist precisely two shortest x,y-paths in H_3^n. Then the sizes of these sets (depending on the vertex x chosen) can be expressed in terms of the entries in Stern's diatomic sequence. We refer to [8] for details.

We have considered only some special cases of Hanoi graphs and their subgraphs. We believe that the richness of the structure of Hanoi graphs will lead to more combinatorial results along these lines.

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