

On graphs with largest possible game domination number

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Abstract

Let $\gamma(G)$ and $\gamma_g(G)$ be the domination number and the game domination number of a graph G , respectively. In this paper γ_g -maximal graphs are introduced as the graphs G for which $\gamma_g(G) = 2\gamma(G) - 1$ holds. Large families of γ_g -maximal graphs are constructed among the graphs in which their sets of support vertices are minimum dominating sets. γ_g -maximal graphs are also characterized among the starlike trees, that is, trees which have exactly one vertex of degree at least 3.

Keywords: domination game; γ_g -maximal graph; supportive dominating set; starlike tree

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1 Introduction

If $G = (V(G), E(G))$ is a graph, then a vertex $u \in V(G)$ *dominates* a vertex $v \in V(G)$ if $u = v$ or u is adjacent to v . $S \subseteq V(G)$ is a *dominating set* of G if every vertex in G is dominated by a vertex in S . The size of a smallest dominating set of G is the *domination number* $\gamma(G)$ of G . A smallest dominating set will be briefly called a γ -set.

The domination game is played on a graph G by two players that are usually called *Dominator* and *Staller*. They take turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated until no move is possible. The score of the game is the total number of vertices chosen by them

in this game. The players have opposite goals: Dominator wants to minimize the score and Staller wants to maximize it. A game is called a *D-game* (resp. *S-game*) if Dominator (resp. Staller) has the first move. The *game domination number* $\gamma_g(G)$ of G is the score of a D-game played on G assuming that both players play optimally, the *Staller-start game domination number* $\gamma'_g(G)$ is the score of an optimal S-game.

This game was introduced in [3] and investigated by now in about 30 papers. One of the reasons for this large interest is the 3/5-conjecture due to Kinnersley, West and Zamani asserting that $\gamma_g(G) \leq 3|V(G)|/5$ holds for any isolate-free graph G [15, Conjecture 6.2]. (Related conjectures were stated also for the S-game, as well as for both games played on forests.) Bujtás [6, 7] developed an innovative discharging-like method to attack this conjecture. Using the method, the conjecture was confirmed by Henning and Kinnersley on the class of graphs with minimum degree at least two [11]. Along these lines Schmidt [21] determined a largest known class of trees for which the conjecture holds. Moreover, Marcus and Peleg reported in arXiv [20] that the conjecture holds on all isolate-free forests. Among the other aspects of the domination game we list here: domination game critical graphs [8]; the somehow peculiar behaviour of the game on unions of graphs [10]; graphs with small game domination number [16]; different realizations of the game domination number [17]; a characterization of forests with the game domination number equal to the domination number [19]; bluffing aspects of the domination game [1]; and the PSPACE-completeness of the game domination number [2]. We also mention two related games that were introduced based on the domination game: the total domination game [12] and the disjoint domination game [9].

It was shown in [3, Theorem 1] that $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ holds for any graph G . Moreover, all possible values for γ_g are eventually realizable [3, Theorem 10]. It is hence natural to ask for which graphs G the equalities $\gamma_g(G) = \gamma(G)$ and $\gamma_g(G) = 2\gamma(G) - 1$ hold, respectively. The former problem was solved for the case of trees in [19], where it was also conjectured that if G is a connected graph with $\gamma_g(G) = \gamma(G)$, then G is either a tree or has girth at most 7. The general problem to characterize the graphs G with $\gamma_g(G) = \gamma(G)$ seems highly difficult though. In this paper we consider the other extreme case, that is, which graphs G have the largest possible game domination number $2\gamma(G) - 1$. We will call such graphs *γ_g -maximal*.

In the next section additional concepts needed are introduced, several known results to be used later recalled, and a couple of useful facts deduced. In Section 3 large families of γ_g -maximal graphs are constructed among the graphs in which their sets of support vertices (vertices adjacent to leaves) are γ -sets. In the last two sections we consider trees which have exactly one vertex of degree at least 3, called starlike trees. In Section 4 we characterize γ_g -maximal starlike trees among the starlike trees with at least one 1-arm, while in Section 5 we characterize γ_g -maximal starlike trees among the other starlike trees. In the concluding section we observe that the graphs considered in this paper support the 3/5-conjecture.

2 Preliminaries

We will use the notation $[k] = \{1, \dots, k\}$ for a positive integer k . The maximum degree and the minimum degree in a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex v of G with $\deg_G(v) = 1$ is called a *pendant vertex* (alias *leaf*), the vertex adjacent to v is a *support vertex* (to v). Let $L(G)$ and $\text{Supp}(G)$ denote the set of pendant and support vertices of G , respectively. For a vertex v of G let $L(v) = L(G) \cap N(v)$, where $N(v)$ is the open neighborhood of v . Clearly, $L(v) \neq \emptyset$ if and only if $v \in \text{Supp}(G)$. Note also that $L(K_2) = \text{Supp}(K_2) = V(K_2)$. On the other hand, if G is connected and of order at least 3, then a support vertex is of degree at least 2, so that $L(G) \cap \text{Supp}(G) = \emptyset$.

Suppose that a D-game is played. Then we will denote the sequence of vertices selected by Dominator with d_1, d_2, \dots , and with s_1, s_2, \dots the sequence chosen by Staller. A *partially-dominated graph* is a graph G together with a declaration that some vertices $S \subseteq V(G)$ are already dominated in the sense that they need not be dominated in the rest of the game. It is denoted with $G|S$.

We next recall the following fundamental results to be used later.

Lemma 2.1 (Continuation Principle, [15]) *Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

Theorem 2.2 ([15]) *If F is a forest with $S \subseteq V(F)$, then $\gamma_g(F|S) \leq \gamma'_g(F|S)$.*

Theorem 2.3 ([3, 15]) *If G is any graph, then $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.*

Setting $S = \emptyset$ in Theorem 2.2 and specializing to trees we get:

Corollary 2.4 *If T is a tree, then $\gamma_g(T) \leq \gamma'_g(T)$.*

Denoting with $G \cup H$ the disjoint union of graphs G and H we have the following result that will be useful to us.

Lemma 2.5 ([15, Lemma 5.4]) *If F_1 and F_2 are partially dominated forests, then*

$$\gamma_g(F_1 \cup F_2) \leq \gamma_g(F_1) + \gamma'_g(F_2) \quad \text{and} \quad \gamma'_g(F_1 \cup F_2) \leq \gamma'_g(F_1) + \gamma'_g(F_2).$$

The next result was first proved in the unpublished manuscript [14]. Five years later the first published proof appeared in [18].

Theorem 2.6 ([14, 18]) *If $n \geq 1$, then*

$$(i) \quad \gamma_g(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise.} \end{cases}$$

$$(ii) \quad \gamma'_g(P_n) = \lceil \frac{n}{2} \rceil.$$

Following the notation from [18], let P'_n denote the partially dominated path of order $n + 1$ with one of its leaves dominated. Then from Košmrlj's proof of [18, Theorem 2.6] we extract the following information useful to us.

Lemma 2.7 *If S-game is played on P'_n or a union of some P'_{n_i} s, then some dominated leaf is an optimal move for Staller. Moreover, if $n \geq 1$, then*

$$(i) \gamma_g(P_n) = \gamma_g(P'_n), \quad \text{and}$$

$$(ii) \gamma_g(P_{n+3}) = 1 + \gamma'_g(P'_n).$$

Proof. The first assertion can be found in [18, p. 132]. The assertion (i) follows from the facts that in a D-game, a vertex adjacent to a leaf in P_n is an optimal first move for Dominator, and that choosing a vertex at distance 2 from a dominated leaf in P'_n is always optimal for Dominator, see [18, p. 132] again. The first of these two facts also implies the last assertion of the lemma. \square

3 Graphs with supportive dominating sets

In this section we consider graphs G such that $\text{Supp}(G)$ forms a dominating set of G . In such a case $\text{Supp}(G)$ is called a *supportive dominating set*. Clearly, a supportive dominating set of a graph G must be a γ -set of G .

Ideally we wish to determine $\gamma_g(G)$ for any graph G that contains a supportive dominating set. But this task seems to be quite demanding. For instance, for combs (a k -comb is obtained from P_k by attaching a separate pendant vertex to each of the vertices of P_k), which form a simple class of graphs with supportive dominating sets, the task to determine γ_g turned out to be quite tricky, see [17, Theorem 4.1].

We first establish a large class of graphs with supportive dominating sets that are γ_g -maximal.

Theorem 3.1 *Let G be a connected graph of order at least 3. If G has a supportive dominating set and there are at least $\lceil \log_2 \gamma(G) \rceil + 1$ pendant vertices adjacent to each vertex of $\text{Supp}(G)$, then G is γ_g -maximal.*

Proof. Let $\gamma(G) = t$ and let $\text{Supp}(G) = \{x_1, \dots, x_t\}$. If $t = 1$, then G has a universal vertex and consequently $\gamma_g(G) = 2\gamma(G) - 1 = 1$ holds. Henceforth assume that $t \geq 2$. Set $V_i = \{x_i\} \cup L(x_i)$, $i \in [t]$. By the Continuation Principle (Lemma 2.1) we may assume that during the game in which Dominator is playing optimally, he never chooses a pendant vertex of G . Indeed, if a pendant vertex is a legal move for Dominator, then the support vertex to it is also legal and is at least as effective.

We only need to prove that $\gamma_g(G) \geq 2t - 1$. For this sake consider the following strategy of Staller:

Rule A: When it is Staller's turn, she plays a legal vertex from $L(x_i)$, where i is selected such that the number of vertices played so far in $L(x_i)$ is as small as possible.

When $t = 2^k$ for some positive integer k , after the first 2^{k-1} moves of Dominator, Rule A implies that Staller will play one vertex from distinct sets $L(x_i)$. Inductively, after $2^{k-1} + 2^{k-2} + \dots + 2^{k-j}$ moves, $j \in [k]$, Dominator may have dominated all but 2^{k-j} families $L(x_i)$ of leaves, and, at that stage, Staller will have used at most j leaves in each family. There is still a legal pendant vertex for Staller after Dominator has played $2^k - 1$ vertices, hence Staller plays such a vertex then. At that point of the game there is at least one vertex $x_i \in \text{Supp}(G)$ not yet played by Dominator (nor by Staller), such that in $L(x_i)$ at most $\log_2 t = k$ vertices were played by Staller. Since $|L(x_i)| \geq \lceil \log_2 t \rceil + 1$, Dominator will be forced to play one more move. Then we have $\gamma_g(G) \geq 2(2^k - 1) + 1 = 2t - 1$ as desired.

When $t = 2^k + t_0$, where $0 < t_0 < 2^k$, by a similar argument as above, we find that there is still one pendant vertex which can be played by Staller after Dominator has played $t - 1$ vertices. Moreover, at least one $x_i \in \text{Supp}(G)$ has not yet been played by Dominator, and in $L(x_i)$ at most $\lceil \log_2 t \rceil = k + 1$ vertices were played by Staller. Hence at least one more move is needed which implies that $\gamma_g(G) \geq 2t - 1$.

Hence in any case Rule A guarantees that at least $2\gamma(G) - 1$ moves will be played which completes the argument. \square

Denoting by $c(n)$ the number of connected graphs of order n , Theorem 3.1 yields the following consequence.

Corollary 3.2 *If $n \geq 2$ and $k \geq 1$, then there exist $c(n)$ graphs G of order $n(k + \lceil \log_2 n \rceil + 1)$ that are γ_g -maximal.*

Proof. Let $n \geq 2$, $k \geq 1$, and let G be a connected graph of order n . Let G' be the graph obtained from G by attaching $\lceil \log_2 n \rceil + k$ pendant vertices to each of the vertices of G , respectively. Then $\text{Supp}(G') = V(G)$, so that $\gamma(G') = n = |\text{Supp}(G')|$. The assertion now follows directly from Theorem 3.1. \square

A special case of the construction from Corollary 3.2 (attaching $n + 1$ leaves to each vertex of the complete graph K_n) was earlier applied in [15] in order to show that all the values between γ and $2\gamma - 1$ are possible values for γ_g .

In the rest of the section we restrict ourself to trees and first strengthen Theorem 3.1 in a special case as follows.

Theorem 3.3 *Let T be a tree with $\gamma(T) = t \geq 2$ in which $\text{Supp}(T) = \{x_1, \dots, x_t\}$ forms a γ -set. If the vertices other than the support vertices and their attached leaves induce a subtree of order a , and $|L(x_i)| \geq \lceil \log_2(t - \lfloor \frac{a}{2} \rfloor) \rceil + 1$ for $i \in [t]$, then $\gamma_g(T) = 2t - 1$.*

Proof. Let $V_i = \{x_i\} \cup L(x_i)$, $i \in [t]$. Then by the premise of this theorem the vertices $V(T) \setminus \bigcup_{i=1}^t V_i$ induce a subtree of T ; denote this subtree with T' . It again suffices to prove that $\gamma_g(T) \geq 2t - 1$.

Since T' is a tree, we find that

$$N(u) \cap N(w) \cap \text{Supp}(T) = \emptyset \text{ for any distinct } u, w \in V(T'). \quad (1)$$

This is equivalent to saying that each vertex from $\text{Supp}(T)$ has at most one neighbor in T' . Moreover, since $\text{Supp}(T)$ forms a γ -set of T , we also infer that

$$N(u) \cap \text{Supp}(T) \neq \emptyset \text{ for any } u \in V(T'). \quad (2)$$

By (1) and (2) there exists a matching between $V(T')$ and $\text{Supp}(T)$ that covers all the vertices of T' . Let $V(T') = \{w_1, \dots, w_a\}$, where w_i is the vertex of $V(T')$ matched to x_i . We now claim that the vertices x_1, \dots, x_a form an independent set. Suppose on the contrary that $x_i x_j \in E(T)$ for some $i \neq j$. Let P be the unique w_i, w_j -path in T' . Then P together with the edges $w_i x_i$, $x_i x_j$, and $x_j w_j$ form a cycle of T , a contradiction. In addition, from the same reason, a vertex x_i , $i > a$, is adjacent to at most one vertex among the vertices x_1, \dots, x_a .

The starting strategy of Staller is to reply to the first $\lfloor \frac{a}{2} \rfloor$ moves of Dominator with a vertex in $\text{Supp}(T) \cup V(T')$ in such a way that when this stage of the game is finished, at most $\lfloor \frac{a}{2} \rfloor$ vertices from $\text{Supp}(T)$ are played. By the above described structure this is possible, because whichever move is played by Dominator, he dominates at most one vertex among the vertices x_1, \dots, x_a in each move. More precisely, the strategy of Staller is the following. If Dominator plays a vertex from $\text{Supp}(T)$, she replies with a vertex w_i , such that x_i has not yet been dominated. And if Dominator plays a vertex from T' , then Staller plays a vertex x_i that has not yet been dominated.

After the first part of the game is finished, at most $\lfloor \frac{a}{2} \rfloor$ vertices from $\text{Supp}(T)$ were played. Therefore, there are at least $t - \lfloor \frac{a}{2} \rfloor$ sets V_i such that no vertex of $L(x_i)$ has yet been dominated. Recall that $|L(x_i)| \geq \lceil \log_2(t - \lfloor \frac{a}{2} \rfloor) \rceil + 1$, $i \in [t]$. In the second part of the game Staller now applies the same strategy as she used in the proof of Theorem 3.1. In this way at least $2(t - \lfloor \frac{a}{2} \rfloor) - 1$ additional moves will be played. Hence at least $2(t - \lfloor \frac{a}{2} \rfloor) - 1 + 2\lfloor \frac{a}{2} \rfloor = 2t - 1$ moves will be played all together. \square

Consider the following examples. Let $k \geq 2$ and let $a_i > 0$ for $i \in [k]$. A *comb-like tree* $\text{Co}(a_1, \dots, a_k)$ is a tree obtained from the path $P_k = v_1 \dots v_k$ by attaching a_i pendant vertices to v_i , $i \in [k]$. In particular, if $a_1 = a_2 = \dots = a_k = 1$, then $\text{Co}(a_1, \dots, a_k)$ is exactly an ordinary comb. A *1-generalized comb-like tree* $\text{Co}^{(1)}(a_1, \dots, a_k)$ is a tree obtained by attaching a_i pendant vertices to the i -th leaf, $i \in [k]$, of $\text{Co}(1, \dots, 1)$. In Fig. 1 the 1-generalized comb-like tree $\text{Co}^{(1)}(2, 1, 3, 1, 1, 2, 1, 3)$ is shown. If $a_1 = \dots = a_k = a$, then $\text{Co}^{(1)}(a_1, \dots, a_k)$ will be called a *balanced 1-generalized comb-like tree* and briefly denoted with $\text{Co}^{(1)}(a^{(k)})$.

If $a_i \geq \lceil \log_2 k \rceil + 1$, $i \in [k]$, then Theorem 3.1 implies that $\gamma_g(\text{Co}(a_1, \dots, a_k)) = 2\gamma(\text{Co}(a_1, \dots, a_k)) - 1$. Similarly, if $a_i \geq \lceil \log_2 \lceil \frac{k}{2} \rceil \rceil + 1$, $i \in [k]$, then Theorem 3.3 implies that $\gamma_g(\text{Co}^{(1)}(a_1, \dots, a_k)) = 2\gamma(\text{Co}^{(1)}(a_1, \dots, a_k)) - 1$.

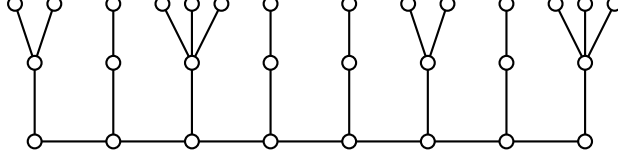


Figure 1: The 1-generalized comb-like tree $\text{Co}^{(1)}(2, 1, 3, 1, 1, 2, 1, 3)$

It was proved in [5] that if T is a tree of order n , then

$$\gamma_g(T) \geq \left\lceil \frac{2n}{\Delta(T) + 3} \right\rceil - 1. \quad (3)$$

Let $k \geq 2$ and $a \geq 4k - 3$, and set $T_k = \text{Co}^{(1)}(a^{(k)})$. Then using (3) we get:

$$\gamma_g(T_k) = 2k - 1 = \left\lceil \frac{2|V(T_k)|}{\Delta(T_k) + 3} \right\rceil - 1. \quad (4)$$

Indeed, $\left\lceil \frac{2|V(T_k)|}{\Delta(T_k) + 3} \right\rceil = \left\lceil \frac{2k(a+2)}{a+4} \right\rceil = 2k$, where the latter equality holds because $\Delta(T) = a + 1 \geq 4k - 2$ and hence $2k \geq \left\lceil \frac{2k(a+2)}{a+4} \right\rceil > 2k - 1$. This proves the second equality of (4). The first equality follows from (3) and the fact that $\gamma(T_k) = k$, so that $\gamma_g(T_k) \leq 2k - 1$.

Clearly, $4k - 3 > \lceil \log_2 \lceil \frac{k}{2} \rceil \rceil + 1$. Hence Theorem 3.3 yields a larger class of balanced 1-generalized comb-like trees with a maximum game domination number than the above argument.

Let T be a tree that fulfils the assumptions of Theorem 3.3 with $|L(x_i)| = \ell + 1$, $i \in [t]$. Does T attain the 3/5 bound from the 3/5-conjecture? If so, then since $|V(T)| = t + t(\ell + 1) + a$, the equality $\gamma_g(T) = 2t - 1 = 3(t + t(\ell + 1) + a)/5$ yields $t = (3a + 5)/(4 - 3\ell)$. Hence $\ell = 1$ must hold and consequently $t = 3a + 5$. But then $\log_2(t - \lfloor \frac{a}{2} \rfloor) \leq 1$ cannot hold. We conclude that T does not attain the 3/5 bound. For trees that do attain the bound see [4] and [13].

4 γ_g -maximal starlike trees with at least one 1-arm

A tree is *starlike* if it contains exactly one vertex of degree at least 3. We will use the notation $T(k_1, \dots, k_t)$ to denote the starlike tree obtained by attaching to an isolated vertex $t \geq 3$ paths of lengths k_1, \dots, k_t . A pendant path of length k_x in $T(k_1, \dots, k_t)$ is called a k_x -arm. Note that $|V(T(k_1, \dots, k_t))| = k_1 + \dots + k_t + 1$.

We first determine the γ_g -maximal paths.

Lemma 4.1 *Let $n \geq 1$. Then $\gamma_g(P_n) = 2\gamma(P_n) - 1$ if and only if $n \in \{1, 2, 3, 5, 6, 9\}$.*

Proof. Combining the well known fact that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ ($n \geq 1$) with Theorem 2.6 it follows that $\gamma_g(P_n) = 2\gamma(P_n) - 1$ if and only if

$$2 \lceil \frac{n}{3} \rceil - 1 = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod{4}, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise.} \end{cases}$$

This equality can hold only for $n \leq 10$, hence the result follows by checking these small values. \square

Lemma 4.2 *If $T = T(k_1, \dots, k_t, 1, \dots, 1)$, where $k_i > 1$, $i \in [t]$, then $\gamma(T) = 1 + \sum_{i=1}^t \lceil \frac{k_i-1}{3} \rceil$.*

Proof. Let v be the unique vertex in T of maximum degree. Since T is a starlike tree with at least one 1-arm, there exists a γ -set D of T with $v \in D$. Since $T \setminus N[v]$ consists of a disjoint union of paths P_{k_i-1} , $i \in [t]$, the result follows from the fact that $\gamma(P_{k_i-1}) = \lceil \frac{k_i-1}{3} \rceil$. \square

Lemma 4.3 *Let $T = T(k_1, \dots, k_s)$ ($s \geq 3$), be a starlike tree with at least one 1-arm. If $\gamma_g(T) = 2\gamma(T) - 1$, then $k_i \in \{1, 3, 4, 7\}$.*

Proof. If T is a star, then the conclusion is clear. Suppose that T has t arms of length $k_i \geq 2$. We may assume without loss of generality that $i = 1, \dots, t$. Suppose that $k_1 \notin \{3, 4, 7\}$. Then by Lemma 4.1,

$$\gamma_g(P_{k_1+2}) \leq 2\gamma(P_{k_1+2}) - 2. \quad (5)$$

Moreover, from Lemma 4.2 we infer that $\gamma(T) = \sum_{i=1}^t \gamma(P_{k_i+2}) - (t-1)$ and consequently

$$2 \sum_{i=1}^t \gamma(P_{k_i+2}) - 2t + 1 = 2\gamma(T) - 1. \quad (6)$$

Let d_1 be the vertex of T of degree at least 3. Then

$$\begin{aligned}
\gamma_g(T) &\leq 1 + \gamma'_g\left(\bigcup_{i=1}^t P'_{k_i-1}\right) \\
&\leq 1 + \sum_{i=1}^t \gamma'_g(P'_{k_i-1}) \quad (\text{by Lemma 2.5}) \\
&= \sum_{i=2}^t \gamma_g(P_{k_i+2}) + \gamma_g(P_{k_1+2}) - (t-1) \quad (\text{by Lemma 2.7(ii)}) \\
&\leq \sum_{i=2}^t \left[2\gamma(P_{k_i+2}) - 1\right] + \left[2\gamma(P_{k_1+2}) - 2\right] - (t-1) \quad (\text{by (5)}) \\
&= \sum_{i=1}^t \left[2\gamma(P_{k_i+2}) - 1\right] - t \\
&= 2 \sum_{i=1}^t \gamma(P_{k_i+2}) - 2t \\
&< 2\gamma(T) - 1 \quad (\text{by (6)}).
\end{aligned}$$

This contradiction proves the lemma. \square

Lemma 4.3 thus asserts that a γ_g -maximal starlike tree T with at least one 1-arm has at most 4 different lengths of arms. To simplify the notation we will write $x^{(j)}$ to briefly denote that there are j arms of length x . For instance, using this convention $T(1, 2, 2, 2, 2, 3, 3)$ is briefly denoted with $T(1, 2^{(4)}, 3^{(2)})$. For convenience, the star $S_n = T(1^{(n-1)})$ will also be written as $T(1^{(n-2)}, 1)$. To formulate the main result of this section we set:

$$\begin{aligned}
\mathcal{ST}^1 &= \{T(1^{(\ell)}, k) : \ell \geq 1, k \in \{1, 3, 4, 7\}\}, \\
\mathcal{ST}^2 &= \{T(1^{(\ell)}, 3, 4), T(1^{(\ell)}, 4^{(2)}), T(1^{(\ell)}, 4^{(3)}), T(1^{(\ell)}, 4, 7) : \ell \geq 1\}, \text{ and} \\
\mathcal{ST}^* &= \mathcal{ST}^1 \cup \mathcal{ST}^2.
\end{aligned}$$

We can now formulate the main result of this section.

Theorem 4.4 *A starlike tree T with at least one 1-arm is γ_g -maximal if and only if $T \in \mathcal{ST}^*$.*

Proof. If all the arms of T are 1-arms, then T is a star and hence $\gamma_g(T) = 1 = 2\gamma(T) - 1$. Assume henceforth that T has $t \geq 1$ arms of length at least 2, let their lengths be k_1, \dots, k_t . Let u be the vertex of T of degree at least 3, and let v_{ij} be the vertex in the k_i -arm ($i \in [t]$) in T at distance j from u .

Using Lemma 4.2 we deduce that $\gamma(T(1^{(\ell)}, 3)) = \gamma(T(1^{(\ell)}, 4)) = 2$, $\gamma(T(1^{(\ell)}, 7)) = \gamma(T(1^{(\ell)}, 3, 4)) = \gamma(T(1^{(\ell)}, 4^{(2)})) = 3$, and $\gamma(T(1^{(\ell)}, 4^{(3)})) = \gamma(T(1^{(\ell)}, 4, 7)) = 4$. For

any tree $T \in \mathcal{ST}^*$ it can be exhaustively checked that $\gamma_g(T) = 2\gamma(T) - 1$ holds. It thus remains to prove the “only if” part of the statement.

Suppose thus that T is a γ_g -maximal starlike tree with at least one 1-arm (and at least one arm longer than 1). By Lemma 4.3 we know that $k_i \in \{3, 4, 7\}$ for $i \in [t]$. If $t = 1$, then using Lemma 4.1 we infer that $k_1 \in \{3, 4, 7\}$. Hence assume henceforth that $t \geq 2$.

Claim 1. $4 \in \{k_1, \dots, k_t\}$.

Proof (of Claim 1). Suppose on the contrary that $4 \notin \{k_1, \dots, k_t\}$. Let $d_1 = u$. Then, by Lemma 2.5,

$$\gamma_g(T) \leq 1 + \gamma'_g\left(\bigcup_{i=1}^t P'_{k_i-1}\right) \leq 1 + \sum_{i=1}^t \gamma'_g(P'_{k_i-1}).$$

By Lemma 2.7, we may without loss of generality assume (by re-indexing the arms if necessary) that $s_1 = v_{11}$ is an optimal reply of Staller to $d_1 = u$. Setting $d_2 = v_{23}$ and applying Lemma 2.7 (ii) we can then estimate as follows:

$$\begin{aligned} \gamma_g(T) &\leq 3 + \sum_{i=3}^t \gamma'_g(P'_{k_i-1}) + \gamma'_g(P'_{k_1-2}) + \gamma'_g(P'_{k_2-4}) \\ &= 3 + \sum_{i=3}^t \gamma_g(P_{k_i+2}) - (t-2) + \gamma_g(P_{k_1+1}) - 1 + \gamma'_g(P'_{k_2-4}) \\ &= 3 + \sum_{i=3}^t \left[2\gamma(P_{k_i+2}) - 1\right] - (t-2) + \gamma_g(P_{k_1+1}) - 1 + \gamma'_g(P'_{k_2-4}) \\ &= 2 \sum_{i=3}^t \gamma(P_{k_i+2}) - 2t + \gamma_g(P_{k_1+1}) + \gamma'_g(P'_{k_2-4}) + 6. \end{aligned}$$

Since $2\gamma(T) - 1 = 2 \sum_{i=1}^t \gamma(P_{k_i+2}) - 2t + 1$ and $\gamma_g(T) < 2\gamma(T) - 1$ holds by Theorem 2.6 for each pair $(k_1, k_2) \in \{(3, 3), (3, 7), (7, 3), (7, 7)\}$, we conclude that the set $[k_t]$ must contain 4. \square (Claim 1)

By the final part of the proof of Claim 1, we have $k_1 = 4$ or $k_2 = 4$ when $d_1 = u$, $s_1 = v_{11}$ and $d_2 = v_{23}$. Moreover, if $k_1 \neq 4$, then $k_2 = 4$. But now by a similar reasoning as that in the proof of Claim 1 we get $\gamma_g(T) \leq 2 \sum_{i=3}^t \gamma(P_{k_i+2}) - 2t + \gamma_g(P_{k_1+1}) + \gamma'_g(P'_{k_2-4}) + 6 < 2\gamma(T) - 1$, where $\gamma'_g(P'_0) = 0$. This is a contradiction. Thus, by Lemma 2.7, s_1 must be on a 4-arm in T as an optimal move. Without loss of generality, assume that $s_1 = v_{11}$ where v_{11} is in a k_1 -arm with $k_1 = 4$ in the following.

If $t = 2$, by Claim 1 and the proof of the “if” part, we deduce that T must belong to the following set $\{T(1^{(\ell)}, 3, 4), T(1^{(\ell)}, 4^{(2)}), T(1^{(\ell)}, 4, 7) : \ell \geq 1\} \subseteq \mathcal{ST}^*$. So we

only need to consider the case $t \geq 3$ in the following. Let $h \geq 1$ be the number of 4-arms in T . Now we can check that $\gamma_g(T) = 2\gamma(T) - 1$ if $T \cong T(1^{(\ell)}, 4^{(3)})$ and $\gamma_g(T) = 2t - 2 < 2t - 1 = 2\gamma(T) - 1$ if $T \cong T(1^{(\ell)}, 4^{(t)})$ with $t \geq 4$. So we only need to prove that $\gamma_g(T) < 2\gamma(T) - 1$ for $t > h \geq 1$ and $t \geq 3$.

First we prove the result for $h = 1$. Note that $s_1 = v_{11}$. Let $d_2 = v_{13}$. Hence we have $\gamma_g(T) \leq 3 + \gamma'_g(\bigcup_{i=2}^t P'_{k_{i-1}})$. Denote by T_1 the subtree of T obtained by deleting all vertices but u of the k_1 -arm of T . Since $h = 1$, we infer that T_1 is a starlike tree without 4-arms. By *the reasoning from* the proof of Claim 1 we get $\gamma_g(T_1) \leq 1 + \gamma'_g(\bigcup_{i=2}^t P'_{k_{i-1}}) < 2\gamma(T_1) - 1$. Then $\gamma_g(T) < 2 + 2\gamma(T_1) - 1 = 2\gamma(T) - 1$.

Assume that $t > h \geq 2$. Recall that $d_1 = u$ and $s_1 = v_{11}$. Also let $d_2 = v_{13}$. Denote by T_2 the subtree obtained by deleting all vertices but u of the k_1 -arm of T . Then T_2 is a starlike tree with $h - 1$ 4-arms. By a similar reasoning iteratively on the number h , we have $\gamma_g(T_2) \leq 1 + \gamma'_g(\bigcup_{i=2}^t P'_{k_{i-1}}) < 2\gamma(T_2) - 1$. Therefore, we have

$$\begin{aligned} \gamma_g(T) &\leq 3 + \gamma'_g(\bigcup_{i=2}^t P'_{k_{i-1}}) \\ &< 2 + 2\gamma(T_2) - 1 \\ &= 2\gamma(T) - 1. \end{aligned}$$

This completes the proof of the theorem. \square

5 γ_g -maximal starlike trees without 1-arms

In this section we characterize the γ_g -maximal starlike trees T without 1-arms. In view of Lemma 4.1, we only need to consider the starlike trees with maximum degree at least 3. Hereafter we denote by P''_k the path of order $k+2$ with both leaves already dominated.

Theorem 5.1 *Let T be a starlike tree without 1-arms. Then T is γ_g -maximal if and only if T is one of the trees $T(4^{(3)})$ and $T(2, 3^{(2)})$.*

Proof. One can check directly (or by computer) that $\gamma(T(4^{(3)})) = 4$ and $\gamma_g(T(4^{(3)})) = 7$, as well as that $\gamma(T(2, 3^{(2)})) = 3$ and $\gamma_g(T(2, 3^{(2)})) = 5$. Hence $T(4^{(3)})$ and $T(2, 3^{(2)})$ are γ_g -maximal. In the rest we thus need to prove that among the starlike trees without 1-arms there is no additional γ_g -maximal tree.

Assume henceforth that T is a starlike tree without 1-arms and with $\gamma_g(T) = 2\gamma(T) - 1$. Let u be the maximum-degree vertex of T . We divide our argument into the following two cases.

Case 1. u lies in some γ -set of T .

Let T have $t \geq 3$ arms of lengths k_1, \dots, k_t with $k_i \geq 2$ for $i \in [t]$. Let T' be

a tree obtained from T by attaching a pendant vertex to u . Then, by the case assumption, $\gamma(T) = \gamma(T')$. Since T has $t \geq 3$ arms of length at least 2, T' can only be isomorphic to $T(1, 4^{(3)})$ among the trees from \mathcal{ST}^* . If $T' \not\cong T(1, 4^{(3)})$, then setting $d_1 = u$ in a game played on T' we have, using a similar reasoning as that in the proof of Theorem 4.4, that $\gamma_g(T') \leq 1 + \gamma'_g(\bigcup_{i=1}^t P'_{k_i-1}) < 2\gamma(T') - 1$. Hence if $T \not\cong T(4^{(3)})$, then by setting $d_1 = u$ in the game played on T , we have $\gamma_g(T) \leq 1 + \gamma'_g(\bigcup_{i=1}^t P'_{k_i-1}) < 2\gamma(T') - 1 = 2\gamma(T) - 1$. By the proof of the ‘‘if’’ part we conclude that $T \cong T(4^{(3)})$.

Case 2. No γ -set of T contains u .

In this case we first assert that there is no arm of length k in T with $k \equiv 1 \pmod{3}$. Indeed, suppose on the contrary that T contains an arm P of length $3k + 1$. If D is a γ -set of T , then $|D \cap P| = k + 1$. But then D can be modified to a γ -set D' of T such that $u \in D'$, a contradiction.

Consider now an arbitrary γ -set D of T . By the case assumption, there is a neighbor v of u such that $v \in D$. Then v dominates one more vertex, call it v' , of the arm in which it lies. Since no γ -set of T contains u , the arm with v on it contains u, v, v' and $3p$ additional vertices, for otherwise we can easily construct a γ -set of T containing u . It follows that the arm on which v lies is of length m , where $m \equiv 2 \pmod{3}$.

Assume that T has a arms of lengths $3t_1 + 2, \dots, 3t_a + 2$ and b arms of lengths $3\ell_1, \dots, 3\ell_b$, where $a \geq 1, b \geq 0$. Thus we have $\gamma(T) = \sum_{i=1}^a (t_i + 1) + \sum_{j=1}^b \ell_j$, that is,

$$2\gamma(T) - 1 = 2 \sum_{i=1}^a t_i + 2 \sum_{j=1}^b \ell_j + 2a - 1. \quad (7)$$

Below we prove three claims.

Claim 1. $b \neq 0$.

Proof (of Claim 1). If not, we have $b = 0$. Then $2\gamma(T) - 1 = 2 \sum_{i=1}^a t_i + 2a - 1$ and $a \geq 3$, since the maximum degree of T is at least 3. Let the first move of Dominator be just u . By Lemma 2.7, without loss of generality, we may assume that s_1 is a neighbor of u on the $(3t_1 + 2)$ -arm in T . By Lemma 2.7 (i) we have

$\gamma_g(P'_{3k_1}) = \gamma_g(P_{3k_1})$. Then, by Lemma 2.5, we have

$$\begin{aligned}
\gamma_g(T) &\leq 2 + \gamma_g(\cup_{i=2}^a P'_{3t_i+1} \cup P'_{3t_1}) \\
&\leq 2 + \gamma_g(P'_{3t_1}) + \sum_{i=2}^a \gamma'_g(P'_{3t_i+1}) \\
&= 2 + \gamma_g(P_{3t_1}) + \sum_{i=2}^a \gamma_g(P_{3t_i+4}) - (a-1) \\
&< 2 + 2\gamma(P_{3t_1}) - 1 + 2 \sum_{i=2}^a \gamma(P_{3t_i+4}) - 2(a-1) \\
&\quad (\text{as } \gamma_g(P_{3t_i+4}) < 2\gamma(P_{3t_i+4}) - 1 \text{ with } t_i \geq 0 \text{ by Lemma 4.1}) \\
&= 2 \sum_{i=2}^a (t_i + 2) + 2t_1 - 2a + 3 \\
&= 2 \sum_{i=1}^a t_i + 2a - 1 \\
&= 2\gamma(T) - 1 \quad (\text{by (7)}).
\end{aligned}$$

This is a contradiction. So $b \geq 1$ holds. □ (Claim 1)

From Claim 1, we have $b > 0$. If $d_1 = u$, then similarly as above we conclude that s_1 must be on a neighbor of u on a $(3\ell_i)$ -arm in T .

Claim 2. $t_i = 0$ for $i \in [a]$.

Proof (of Claim 2). Otherwise assume without loss of generality that $t_1 \geq 1$. Let $d_1 = u$. Then we may without loss of generality assume that s_1 is a neighbor of u on the $(3\ell_1)$ -arm in T . Let d_2 be the vertex of the $(3t_1 + 2)$ -arm at distance 4 to u

in T . Note that $\overline{\gamma}'_g(P''_1) = 1$. Then, by Lemmas 2.5 and 2.7(ii), we have

$$\begin{aligned}
\gamma_g(T) &\leq 3 + \gamma'_g\left(\bigcup_{i=2}^a P'_{3t_i+1} \cup \bigcup_{j=2}^b P'_{3\ell_j-1} \cup P'_{3t_1-5} \cup P'_{3\ell_1-2} \cup P''_1\right) \\
&\leq 3 + \sum_{i=2}^a \gamma'_g(P'_{3t_i+1}) + \sum_{j=2}^b \gamma'_g(P'_{3\ell_j-1}) + \gamma'_g(P'_{3t_1-5}) + \gamma'_g(P'_{3\ell_1-2}) + \gamma'_g(P''_1) \\
&= 4 + \sum_{i=2}^a \gamma_g(P_{3t_i+4}) - (a-1) + \sum_{j=2}^b \gamma_g(P_{3\ell_j+2}) - (b-1) \\
&\quad + \gamma_g(P_{3t_1-2}) - 1 + \gamma_g(P_{3\ell_1+1}) - 1 \\
&\leq 2 + 2 \sum_{i=2}^a \gamma(P_{3t_i+4}) - 2(a-1) + 2 \sum_{j=2}^b \gamma(P_{3\ell_j+2}) - 2(b-1) \\
&\quad + 2\gamma(P_{3t_1-2}) - 1 + 2\gamma(P_{3\ell_1+1}) - 1 \\
&= 2 \sum_{i=2}^a (t_i + 2) - 2a + 2 \sum_{j=2}^b (\ell_j + 1) - 2b + 2t_1 + 2\ell_1 + 4 \\
&= 2 \sum_{i=1}^a t_i + 2 \sum_{j=1}^b \ell_j + 2a - 2 \\
&< 2\gamma(T) - 1.
\end{aligned}$$

A contradiction occurs again. □ (Claim 2)

By Claims 1 and 2 we have $2\gamma(T) - 1 = 2 \sum_{j=1}^b \ell_j + 2a - 1$ with $a \geq 1$, $b \geq 1$ and $\ell_j \geq 1$ for $j \in [b]$. Moreover, if $d_1 = u$, then one optimal vertex s_1 , which is adjacent to u , will be a vertex in a $(3\ell_i)$ -arm as an optimal move of Staller, since she always wants to prolong the game.

Claim 3. $\ell_j = 1$ for $j \in [b]$.

Proof (of Claim 3). Otherwise, we have $\ell_j \geq 2$ for some $j \in [b]$. Similarly as above, let $d_1 = u$. Then we can assume that s_1 lies on a $(3\ell_1)$ -arm as a neighbor of u in T . Next we distinguish the following two subcases.

Subcase 1. $j = 1$.

In this subcase, if $b = 1$, then $a \geq 2$ since T has maximum degree at least 3. Now T has only one $(3\ell_1)$ -arm and a 2-arms with $\gamma(T) = a + \ell_1$. Note that $\gamma'_g(P''_1) = 1$ and $\gamma_g(P'_{3\ell_1-2}) \leq \gamma'_g(P'_{3\ell_1-2})$ from Corollary 2.4. Then, considering that $a \geq 2$ and

by Lemma 2.5, we have

$$\begin{aligned}
\gamma_g(T) &\leq 2 + \gamma_g(\cup_{i=1}^a P'_1 \cup P'_{3\ell_1-2}) \\
&\leq 2 + \sum_{i=1}^a \gamma'_g(P'_1) + \gamma_g(P'_{3\ell_1-2}) \\
&\leq 2 + \sum_{i=1}^a \gamma'_g(P'_1) + \gamma'_g(P'_{3\ell_1-2}) \\
&= a + 2 + \gamma'_g(P'_{3\ell_1-2}) \\
&= a + 3 + \gamma_g(P'_{3\ell_1-3}) \\
&\quad (\text{since } \gamma'_g(P'_{3\ell_1-2}) = 1 + \gamma_g(P_{3\ell_1-3}) \text{ by Lemma 2.7}) \\
&\leq a + 3 + 2\gamma(P_{3\ell_1-3}) - 1 \\
&= 2\ell_1 + a \\
&< 2\gamma(T) - 1.
\end{aligned}$$

If $b \geq 2$, then let d_2 be the vertex of the $(3\ell_2)$ -arm at distance 2 from u . Note that $\gamma'_g(P'_{3\ell_1-2}) = 1 + \gamma_g(P_{3\ell_1-3})$. Similarly as above we now have

$$\begin{aligned}
\gamma_g(T) &\leq 3 + \gamma'_g\left(\cup_{i=1}^a P'_1 \cup \cup_{j=3}^b P'_{3\ell_j-1} \cup P'_{3\ell_1-2} \cup P'_{3\ell_2-3}\right) \\
&\leq 3 + \sum_{i=1}^a \gamma'_g(P'_1) + \sum_{j=3}^b \gamma'_g(P'_{3\ell_j-1}) + \gamma'_g(P'_{3\ell_1-2}) + \gamma'_g(P'_{3\ell_2-3}) \\
&= a + 3 + \sum_{j=3}^b \gamma_g(P_{3\ell_j+2}) - (b-2) + \gamma_g(P_{3\ell_1-3}) + 1 + \gamma_g(P_{3\ell_2}) - 1 \\
&\leq a + 3 + 2 \sum_{j=3}^b \gamma(P_{3\ell_j+2}) - 2(b-2) + 2\gamma(P_{3\ell_1-3}) - 1 + 2\gamma(P_{3\ell_2}) - 1 \\
&= 2 \sum_{j=1}^b \ell_j + a - 1 \\
&< 2\gamma(T) - 1.
\end{aligned}$$

This finishes the proof of Subcase 1.

Subcase 2. $j \neq 1$.

Without loss of generality assume that $j = 2$. Let d_2 be the vertex on the $(3\ell_2)$ -arm in T at distance 4 from u . Note that $\gamma'_g(P''_1) = 1 = \gamma'_g(P'_1)$, $\gamma'_g(P'_{3\ell_1-2}) =$

$1 + \gamma_g(P_{3\ell_1-3})$ and $\gamma'_g(P'_{3\ell_2-5}) = 1 + \gamma_g(P_{3\ell_2-6})$. Then, by Lemma 2.5, we have

$$\begin{aligned}
\gamma_g(T) &\leq 3 + \gamma'_g\left(\bigcup_{i=1}^a P'_1 \cup \bigcup_{j=3}^b P'_{3\ell_j-1} \cup P'_{3\ell_1-2} \cup P'_{3\ell_2-5} \cup P''_1\right) \\
&\leq 3 + \sum_{i=1}^a \gamma'_g(P'_1) + \sum_{j=3}^b \gamma'_g(P'_{3\ell_j-1}) + \gamma'_g(P'_{3\ell_1-2}) + \gamma'_g(P'_{3\ell_2-5}) + \gamma'_g(P''_1) \\
&= a + 4 + \sum_{j=3}^b \gamma_g(P_{3\ell_j+2}) - (b-2) + 1 + \gamma_g(P_{3\ell_1-3}) + 1 + \gamma_g(P_{3\ell_2-6}) \\
&\leq a + 6 + 2 \sum_{j=3}^b \gamma(P_{3\ell_j+2}) - 2(b-2) + 2\gamma(P_{3\ell_1-3}) - 1 + 2\gamma(P_{3\ell_2-6}) - 1 \\
&= a + 4 + 2 \sum_{j=3}^b \ell_j + 2(\ell_1 - 1) + 2(\ell_2 - 2) \\
&= 2 \sum_{j=1}^b \ell_j + a - 2 \\
&< 2\gamma(T) - 1.
\end{aligned}$$

This is also a contradiction. \square (Claim 3)

By Claims 1, 2 and 3, we conclude that $T \cong T(2^{(a)}, 3^{(b)})$ with $n = 2a + 3b + 1$. Note that $a \geq 1$, $b \geq 1$, and $a+b \geq 3$. From the structure of T , we have $\gamma(T) = a+b$, that is, $2\gamma(T) - 1 = 2a + 2b - 1$. Assume that the D -game is played on T . Let d_1 be the vertex with maximum degree in T . Afterwards Dominator can guarantee that the game will be finished in the total of $\lceil \frac{b}{2} \rceil + a + b + 1$ moves. The move d_1 might not be optimal for Dominator, but in any case we have $\gamma_g(T) \leq \lceil \frac{b}{2} \rceil + a + b + 1$. Thus T can only be γ_g -maximal if $\gamma_g(T) = \lceil \frac{b}{2} \rceil + a + b + 1 = 2a + 2b - 1$. From the second equality, we have $(a, b) \in \{(1, 2), (2, 1)\}$. But it can be verified that $\gamma_g(T(2^{(2)}, 3)) = 4 < 2\gamma(T) - 1$. Therefore, T is γ_g -maximal if and only if $T \cong T(2, 3^{(2)})$. This completes the proof of the theorem. \square

Let

$$\mathcal{ST} = \mathcal{ST}^* \cup \left\{ T(4^{(3)}), T(2, 3^{(2)}) \right\}.$$

Then combining Theorems 4.2 and 5.1 we arrive at the following result.

Theorem 5.2 *Let T be a starlike tree. Then T is γ_g -maximal if and only if $T \in \mathcal{ST}$.*

6 Concluding remarks

As already mentioned in the introduction, the 3/5-conjecture was confirmed by Henning and Kinnersley on the class of graphs with minimum degree at least two [11].

Hence the conjecture remains to be verified (or disproved) on the class of graphs that contain pendant vertices. In Theorem 3.1 we have determined the game domination number for such a class of graphs. Let us check that these graphs support the conjecture.

Let G be a connected graph of order at least 3 and with a supportive dominating set of order t such that there are at least $\lceil \log_2 t \rceil + 1$ pendant vertices adjacent to each vertex of the supportive dominating set. Then Theorem 3.1 asserts that $\gamma_g(G) = 2t - 1$. Since $|V(G)| \geq t + t(\lceil \log_2 t \rceil + 1)$, to verify the 3/5-conjecture for G it suffices to show that $2t - 1 \leq \frac{3}{5}(t + t(\lceil \log_2 t \rceil + 1))$. This inequality reduces to $4t \leq 3\lceil \log_2 t \rceil + 5$ which holds for any $t \geq 1$. Hence the 3/5-conjecture holds for G .

In the introduction we have also mentioned that in [20] it is reported that the 3/5-conjecture holds for trees. Hence let us just state that similarly as above one can check that the starlike trees from Sections 4 and 5 also support the conjecture. Therefore, the remaining task for the 3/5-conjecture is to check (or disprove) it for the graphs with at least one cycle and minimum degree 1.

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